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FOR WHAT TOPOLOGICAL PROPERTY P DOES (P AND T_0) EXIST?

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Abstract

Within this paper, recent discoveries in the study of topology are used to precisely determine all topological properties P for which (P and T_0) exist or for which (P and "not-T_0") exist.

1. Introduction and Preliminaries

Within classical topology, it is known that for many topological properties P, (P and T_0) exists. Thus a natural and logical question to ask is: "For precisely what topological properties P does (P and T_0) exist?" The question is one of many very natural questions to ask that are unresolved in classical topology. Because of the vastness of the long-known topological properties and the continual addition of new

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topological properties, the topological tools needed to resolve the question were not available within classical topology, leaving the question unresolved until, and if, the needed tools became available. In this paper, properties and tools discovered in the continued study of T_0-identification spaces are used to not only resolve the above question, but, also, resolve the dual question: "For what topological properties *P* does (*P* and "not-T_0") exist?"

T_0-identification spaces were introduced in 1936 [10].

Definition 1.1. Let (X, T) be a space, let *R* be the equivalence relation on *X* defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of *R* equivalence classes of *X*, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map *N*. Then $(X_0, Q(X, Y))$ is the T_0-identification space of (X, T).

T_0-identification spaces were a cleaver creation that for a space (X, T) gave a strongly (X, T) related T_0-identification space with the T_0 axiom added, which was used in the 1936 paper [10] to jointly characterize pseudometrizable and metrizable: A space is pseudometrizable iff its T_0-identification space is metrizable.

In 1975 [9], T_0-identification spaces were used to jointly characterize R_1 and T_2: A space is R_1 iff its T_0-identification space is T_2. A space (X, T) is R_1 iff for *x* and *y* in *X* such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets *U* and *V* such that $x \in U$ and $y \in V$ [1].

Then in 2015 [2], using the characterizations of pseudometrizable and R_1 given above as motivation and model, weakly Po spaces and properties were introduced.

Definition 1.2. Let *P* be a topological property such that $P\text{o} = (P \text{ and } T_0)$ exists. Then a space (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property *P*. A topological property Po for which weakly Po exists is called a weakly Po property.

Because T_0-identification spaces were cleverly created to add T_0 to the T_0-identification space of each space, T_0 has a special role in weakly Po spaces and properties: To be a weakly Po property, *P* and (*P* and T_0) must be equivalent in the T_0-identification space requiring (*P* and T_0) to exist. Thus $\{Q \mid Q$ is weakly Po$\} \subseteq \{Q \mid Q$ is a topological property and $(Q$ and $T_0)$ exists$\}$ and a partial solution is obtained.

Within the introductory weakly Po paper [2], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo, which led to the introduction and investigation of T_0-identification *P* properties [3].

Definition 1.5. Let *S* be a topological property. Then *S* is a T_0-identification *P* property iff both a space and its T_0-identification space simultaneously shares property *S*.

Also, in the introductory weakly Po property paper [2], it was shown that weakly Po is neither T_0 nor "not-T_0", where "not-T_0" is the negation of T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful topological property and tool, motivating the inclusion of the long-neglected properties "not-*P*", where *P* is a topological property for which "not-*P*" exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties have been discovered, expanding and changing the study of topology forever.

As one example, prior to the study of weakly Po spaces and properties, where an important role for "not-*P*" in the study of topology was revealed, the existence of the least of all topological properties was not even imagined, but, in the paper [4], $L = (T_0$ or "not-T_0") = (*P* or "not-*P*"), where *P* is a topological property for which "not-*P*" exists, was shown to be the least of all topological properties. As expected, the existence of the never before even imagined least topological property L required corrections in classical topology, including product properties [5] and subspace properties [6], changing the study of topology forever.

In past studies of weakly Po spaces and properties, for a classical topological property Qo, a special topological property W was sought such that for a space with property W, its T_0-identification space has property Qo, which then implied the initial space has property W, all with no assurance that a topological property W exists. As given above, the study of weakly Po spaces and related properties has been a productive study, but, if the past process was continued, the study of weakly Po spaces and properties would continue to be uncertain, tedious, and never ending. Thus, the question of whether there is a shortcut for the weakly Po space and property search process arose, which was resolved in a recent paper [7].

Answer 1.1. Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly Po and $Q =$ weakly Qo = (Qo or (Q and "not-T_0")) [7].

Answer 1.2. $\{Q$o$\mid Q$ is a T_0-identification P property$\}$ $= \{Q$o$\mid Q$o is a weakly Po property$\}$ $= \{Q$o$\mid Q$ is a topological property and Qo exists$\}$ [7].

Answer 1.3. $\{Q \mid Q$ is a T_0-identification P property$\}$ $= \{Q \mid Q$ is weakly Po$\} = \{Q \mid Q$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$ [7].

Thus, major progress was achieved in the study of weakly Po and related properties. If Q is a topological property for which both Qo and (Q and "not-T_0") exist, Answer 1.1 quickly and easily gives the shortcut. If Q is a topological property for which $Q = Q$o, then $Q = Q$o is a weakly Po property, but $Q = Q$o is not a T_0-identification P property or weakly Po. Within the recent paper [7], a topological property W that can be both T_0 and "not-T_0" was given that is a T_0-identification P property that is weakly Po such that $W =$ weakly Qo, again making the search process certain. However, the definition and construction of the topological property *W* did not precisely describe *W*, raising the question of whether the known information could somehow be used to more precisely determine *W* for a fixed *Q*o. The investigation of that question led to an internalization of the T_0-identification and weakly *P*o processes [8], which can, and has been, used to more precisely determine the needed *W* for a given *Q*o.

Below *L* is used along with the results above to resolve the questions above concerning the existence of (*P* and T_0) and (*P* and "not-T_0").

2. Resolution of the Questions Above

Theorem 2.1. *Let P be a topological property. Then the following are equivalent*: (a) (*P and* T_0) *does not exist*, (b) $P = (P$ *and* "*not*-T_0"), *and* (c) *P implies* "*not*-T_0".

Proof. (a) implies (b): Since *L* is the least topological property, then $P = (P$ and $L) = P$ and (T_0 or "not-T_0")) = ((*P* and T_0) or (*P* and "not-T_0")) and, since (*P* and T_0) does not exist, then $P = (P$ and "not-T_0").

(b) implies (c): Since $P = (P$ and "not-T_0") exists and (*P* and "not-T_0") implies "not-T_0", then *P* implies "not-T_0".

(c) implies (a): Since *P* implies "not-T_0", then (*P* and "not-T_0") $= P$, and (*P* and T_0) = ((*P* and "not-T_0") and T_0) = (*P* and (("not-T_0") and T_0)), which does not exist.

Theorem 2.2. *Let P be a topological property. Then the following are equivalent:* (a) (*P and* "*not*-T_0") *does not exist*, (b) $P = (P$ *and* T_0), *and* (c) *P implies* T_0.

Proof. (a) implies (b): As above, by the use of *L*, $P = ((P$ and $T_0)$ or (*P* and "not-T_0")), and, since (P and "not-T_0") does not exist, $P = (P$ and $T_0)$.

(b) implies (c): Since $P = (P$ and $T_0)$ exists and (P and T_0) implies T_0, then P implies T_0.

(c) implies (a): Since P exists and P implies T_0, then $(P$ and $T_0) = P$, and $(P$ and "not-T_0") $= ((P$ and $T_0)$ and "not-T_0") $= P$ and (T_0 and "not-T_0")), which does not exist.

Theorem 2.3. *$A = \{Q \mid Q$ is a topological property and $(Q$ and $T_0)$ exists$\}$ $= B = \{Q \mid Q$ is a topological property and $Q = (Q$ and $T_0)$ or $Q =$ weakly Qo$\} = C = \{Q \mid Q$ is a topological property and $Q = (Q$ and $T_0)$* or *Q is a T_0-identification P property$\}$ $= D = \{Q \mid Q$ is a topological property and Q implies T_0 or $Q =$ weakly Qo$\} = E = \{Q \mid Q$ is a topological property and Q implies T_0 or Q is a T_0-identification P property$\}$.*

Proof. Let $Q \in A$. Then $Q = (Q$ and $L) = (Q$ and (T_0 or "not-T_0")) $= ((Q$ and $T_0)$ or (Q and "not-T_0")). Since (Q and T_0) exists, $Q = (Q$ and $T_0)$ or both $(Q$ and $T_0)$ and (Q and "not-T_0") exist. If $Q = (Q$ and $T_0)$, then $Q \in B$. Thus, consider the case that both $(Q$ and $T_0)$ and (Q and "not-T_0") exist. Then, by Answer 1.1 above, $Q =$ weakly Qo and $Q \in B$. Thus, $A \subseteq B$.

Let $P \in B$. Since Q is a topological property, Q exists. If $Q = (Q$ and $T_0)$, then $Q \in C$. Thus, consider the case that $Q =$ weakly Qo. Since weakly Qo $= ((Q$ and $T_0)$ or (weakly Qo and "not-T_0")), where both $(Q$ and $T_0)$ and (weakly Qo and "not-T_0") exist [4] and $Q =$ weakly Qo, then, by substitution, both $(Q$ and $T_0)$ and (Q and "not-T_0") exist, and by Answer 1.3 above, Q is a T_0-identification P property. Hence $B \subseteq C$.

Let $Q \in C$. If $Q = (Q$ and $T_0)$, then $Q \in A$. Thus consider the case that Q is a T_0-identification P property. Then, by Answer 1.3, both $(Q$ and $T_0)$ and (Q and "not-T_0") exist and $Q \in A$. Therefore $A = B = C$.

By Theorem 2.2, $B = D$ and $C = E$. Hence $A = B = C = D = E$.

Theorem 2.4. *$F = \{Q \mid Q$ is a topological property and (Q and "not-T_0") exists$\} = G = \{Q \mid Q$ is a topological property and $Q =$ (Q and "not-T_0") or $Q =$ weakly Qo$\} = H = \{Q \mid Q$ is a topological property and $Q =$ (Q and "not-T_0") or Q is a T_0-identification P property$\} = I = \{Q \mid Q$ is a topological property and Q implies "not-T_0" or $Q =$ weakly Qo$\} = J = \{Q \mid Q$ is a topological property and Q implies "not-T_0" or Q is a T_0-identification P property$\}$.*

The proof is similar to that of Theorem 2.3 and is omitted.

Thus, the continued study of T_0-identification spaces with the introduction and investigation of weakly Po spaces and properties has not only led to the discovery of new, basic, foundational, and never before imagined topological properties changing the study of topology forever, but, also, provided new, needed tools and classifications to further expand knowledge in the study of topology, as seen above.

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