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FIXED POINT THEOREMS FOR $(\delta, 1-\delta)$ WEAK CONTRACTIONS IN THE SENSE OF AMPADU ON PARTIAL METRIC SPACES

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Abstract

In this paper, we obtain some fixed point theorems in partial metric spaces using the concept of $(\delta, 1-\delta)$-weak contraction introduced in Ampadu [1]. Further taking inspiration from Berinde [2], we introduce a nonlinear type $(\delta, 1-\delta)$-weak contraction and give a fixed point result in partial metric spaces using this new concept.

1. Introduction

Recall that the concept of $(\delta, 1-\delta)$-weak contraction was introduced as follows:

Definition 1.1 (Ampadu [1]). Let (X, d) be a metric space. A map

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$T : X \mapsto X$ is called a $(\delta, 1-\delta)$-weak contraction, if there exists $\delta \in (0, 1)$ such that the following holds for all $x, y \in X$

$$d(Tx, Ty) \leq \delta d(x, y) + (1-\delta)d(y, Tx).$$

A uniqueness theorem related to $(\delta, 1-\delta)$-weak contraction was obtained as follows:

Theorem 1.2 (Ampadu [1]). *Let (X, d) be a metric space, and $T : X \mapsto X$ be a $(\delta, 1-\delta)$-weak contraction. T has a unique fixed point, provided (X, d) is complete.*

Definition 1.3 (Altun and Acar [3]). A map $\varphi : [0, \infty) \mapsto [0, \infty)$ is called a comparison function if it satisfies the following:

(a) φ is monotone increasing,

(b) $\lim_{n\to\infty} \varphi^n t = 0$ for all $t \in [0, \infty)$.

Definition 1.4 (Altun and Acar [3]). A map $\varphi : [0, \infty) \mapsto [0, \infty)$ is called a (c)-comparison function if it satisfies the following:

(a) φ is monotone increasing,

(b) $\sum_{n=0}^{\infty} \varphi^n(t)$ is convergent for all $t \in [0, \infty)$.

Remark 1.5. Properties and examples of comparison and (c)-comparison functions can be found in Berinde [4].

Definition 1.6 (Berinde [2]). Let (X, d) be a metric space, and T be a self-map of X. T is called a (φ, L)-weak contraction, if there exists a comparison function φ and some $L \geq 0$ such that for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(y, Tx).$$

Remark 1.7. In order to check the weak φ-contractiveness of a mapping *T*, it is necessary to check the inequality in the previous definition, and the following dual one for all $x, y \in X$

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(x, Ty).$$

Related to the above definition, the following existence theorem was obtained:

Theorem 1.8 (Berinde [2]). *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ be a (φ, L)*-weak contraction with* φ, *a* (c)*-comparison function*, *then T has a fixed point*.

Definition 1.9 (Matthews [5]). A partial metric on a nonempty set *X* is a function $\rho : X \times X \mapsto \mathbb{R}^+$ such that for all $x, y, z \in X$ the following holds:

(a) $x = y \Leftrightarrow \rho(x, x) = \rho(x, y) = \rho(y, y)$,

(b) $\rho(x, x) \leq \rho(x, y)$,

(c) $\rho(x, y) = \rho(y, x)$,

(d) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$.

Moreover, the pair (X, ρ) is called a partial metric space, where *X* is a nonempty set and ρ satisfies the above axioms.

It is well known if $\rho(x, y) = 0$, then from (a) and (b) of the previous definition, $x = y$. But if $x = y$, $\rho(x, y) \neq 0$.

Example 1.10 (Altun and Acar [3]). Let $X = \mathbb{R}^+$ and define $\rho : X \times X \mapsto \mathbb{R}^+$ by $\rho(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, \rho)$ is a partial metric space.

Example 1.11 (Altun and Acar [3]). For any real numbers $a \leq b$, let *I* denote the set of all intervals $[a, b]$. Define $\rho : I \times I \mapsto \mathbb{R}^+$ *by* $\rho([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$, then (I, ρ) is a partial metric space.

Remark 1.12. For other examples of partial metric spaces, see Escardo [6] and Matthews [5].

Remark 1.13 (Altun and Acar [3])**.** Each partial metric ρ on X generates a T_0 topology τ_ρ on X which has as a base the family of open ρ-balls $\{B_\rho(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\rho(x, \varepsilon) = \{y \in X : \rho(x, y) < \rho(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1.14 (Altun and Acar [3])**.** If ρ is a partial metric on X, then the functions $\rho^s, \rho^w : X \times X \mapsto \mathbb{R}^+$ defined by

$$\rho^s(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y)$$

and

$$\rho^w(x, y) = \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\}$$

are ordinary equivalent metrics on X.

Definition 1.15 (Altun and Acar [3])**.** A sequence $\{x_n\}$ in a partial metric space (X, ρ) converges, with respect to τ_{ρ^s}, to a point $x \in X$ iff

$$\lim_{n, m \to \infty} \rho(x_n, x_m) = \lim_{n \to \infty} \rho(x, x_n) = \rho(x, x).$$

Definition 1.16 (Altun and Acar [3])**.** A sequence $\{x_n\}$ in a partial metric space (X, ρ) is called Cauchy if $\lim_{n, m \to \infty} \rho(x_n, x_m) < \infty$. If $\lim_{n, m \to \infty} \rho(x_n, x_m) = 0$, then $\{x_n\}$ is called a 0-Cauchy sequence in (X, ρ).

Definition 1.17 (Altun and Acar [3])**.** A partial metric space (X, ρ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_ρ, to a point $x \in X$ such that $\rho(x, x) = \lim_{n, m \to \infty} \rho(x_n, x_m)$.

Definition 1.18 (Altun and Acar [3])**.** A partial metric space (X, ρ) is said to be

0-complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_ρ, to a point $x \in X$ such that $\rho(x, x) = 0$.

Remark 1.19 (Romaguera [7]). (X, ρ) is 0-complete iff every 0-Cauchy sequence converges with respect to τ_{ρ^s}.

Remark 1.20 (Altun and Acar [3]). Every 0-Cauchy sequence in (X, ρ) is a Cauchy sequence in (X, ρ). It follows that if (X, ρ) is complete then it is 0-complete. Moreover, a 0-complete partial metric space may not be complete, for example, see Romaguera [7].

Remark 1.21 (Altun and Acar [3]). A sequence $\{x_n\}$ is Cauchy in a partial metric space (X, ρ) iff it is Cauchy in the metric space (X, ρ^s). Moreover, (X, ρ) is complete iff (X, ρ^s) is complete. See Matthews [5].

2. Main Results

Definition 2.1. Let (X, ρ) be a partial metric space. A map $T : X \mapsto X$ will be called a $(\delta, 1-\delta)$-weak contraction if there exists $\delta \in (0, 1)$ such that for all $x, y \in X$ the following inequality holds:

$$\rho(Tx, Ty) \leq \delta\rho(x, y) + (1-\delta)\rho^w(y, Tx).$$

Now we show that any Kannan mapping is a $(\delta, 1-\delta)$-weak contraction in the sense of Definition 2.1.

Proposition 2.2. *Let (X, ρ) be a partial metric space, and $T : X \mapsto X$ be a Kannan mapping, that is, there exists $a \in [0, \frac{1}{2})$ such that*

$$\rho(Tx, Ty) \leq a[\rho(x, Tx) + \rho(y, Ty)]$$

for all $x, y \in X$. Then T is a $(\delta, 1-\delta)$-weak contraction.

Proof. Since $a < \frac{1}{2}$ and $\frac{1}{4} < \frac{1}{2}$, we can take $a < \frac{1}{4}$. Now observe we have the following:

$$\rho(Tx, Ty) \leq a[\rho(x, Tx) + \rho(y, Ty)]$$

$$\leq a[\rho(x, y) + \rho(y, Tx) - \rho(y, y) + \rho(y, Tx) + \rho(Tx, Ty) - \rho(Tx, Ty)].$$

From the above we deduce the following:

$$(1-a)\rho(Tx, Ty) \leq a\rho(x, y) + 2a\rho(y, Tx) - a[\rho(y, y) + \rho(Tx, Tx)]$$

$$\leq a\rho(x, y) + 2a\rho(y, Tx) - 2a\min\{\rho(y, y), \rho(Tx, Tx)\}$$

$$= a\rho(x, y) + 2a[\rho(y, Tx) - \min\{\rho(y, y), \rho(Tx, Tx)\}]$$

$$= a\rho(x, y) + 2a\rho^{w}(y, Tx)$$

$$\leq a\rho(x, y) + (1-2a)\rho^{w}(y, Tx).$$

From the above, we have

$$\rho(Tx, Ty) \leq \frac{a}{1-a}\rho(x, y) + \frac{(1-2a)}{1-a}\rho^{w}(y, Tx).$$

Thus with $\delta := \frac{a}{1-a}$ and $1 - \delta := \frac{(1-2a)}{1-a}$, T is a $(\delta, 1-\delta)$-weak contraction. o

Now we introduce the nonlinear type $(\delta, 1-\delta)$-weak contraction in metric space as follows:

Definition 2.3. Let (X, d) be a metric space, and T be a self-map of X. T is called a $(\varphi, 1-\delta)$-weak contraction, if there exists a comparison function φ and some $\delta \in (0, 1)$ such that for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq \varphi(d(x, y)) + (1-\delta)d(y, Tx).$$

From the above, we have the following in the setting of partial metric spaces:

Definition 2.4. Let (X, ρ) be a partial metric space, and T be a self-map of X. T is called a $(\varphi, 1-\delta)$-weak contraction, if there exists a comparison function φ and some $\delta \in (0, 1)$ such that for all $x, y \in X$, the following inequality holds:

$$\rho(Tx, Ty) \leq \varphi(\rho(x, y)) + (1-\delta)\rho^{w}(y, Tx).$$

Now we have the following existence and uniqueness fixed point result for a map satisfying the definition immediately above as follows:

Theorem 2.5. *Let (X, ρ) be a 0-complete partial metric space and $T : X \mapsto X$ be a $(\varphi, 1-\delta)$-weak contraction with a (c)-comparison function, then T has a fixed point. Moreover, the fixed point is unique iff the (c)-comparison function is given by $\varphi(t) = \delta t$, where $\delta \in (0, 1)$.*

Proof. Let $x_0 \in X$ and $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a $(\varphi, 1-\delta)$-weak contraction, then we have the following:

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(Tx_{n-1}, Tx_n) \\ &\leq \varphi(\rho(x_{n-1}, x_n)) + (1-\delta)\rho^{w}(x_n, Tx_{n-1}) \\ &= \varphi(\rho(x_{n-1}, x_n)). \end{aligned}$$

By induction, we obtain $\rho(x_n, x_{n+1}) \leq \varphi^{n}(\rho(x_0, x_1))$ for all $n \in \mathbb{N}$. By triangle inequality, for $m > n$, we have

$$\begin{aligned} \rho(x_n, x_m) &\leq \sum_{k=n}^{m-1} \rho(x_k, x_{k+1}) - \sum_{k=n}^{m-2} \rho(x_{k+1}, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \rho(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \rho(x_k, x_{k+1}) \end{aligned}$$

$$\leq \sum_{k=n}^{\infty} \varphi^k(\rho(x_0, x_1)).$$

Since φ is a (c)-comparison function, then $\sum_{k=n}^{\infty} \varphi^k(\rho(x_0, x_1))$ is convergent, and so $\{x_n\}$ is a 0-Cauchy sequence in *X*. Since *X* is 0-complete, then $\{x_n\}$ converges, with respect to τ_ρ, to a point $z \in X$ such that $\lim_{n\to\infty} \rho(x_n, z) = \rho(z, z) = 0$. Now we claim that $\rho(z, Tz) = 0$. Suppose not, suppose $\rho(z, Tz) > 0$. Since φ is a (c)-comparison function, then $\varphi(t) < t$ for $t > 0$. Also since $\lim_{n\to\infty}$ $\rho(x_n, z) = \rho(z, z) = 0$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\rho(x_n, z) < \frac{\rho(z, Tz)}{2}$. Now observe we have the following:

$$\rho(z, Tz) \leq \rho(z, x_{n+1}) + \rho(x_{n+1}, Tz)$$

$$= \rho(z, x_{n+1}) + \rho(Tx_n, Tz)$$

$$\leq \rho(z, x_{n+1}) + \varphi(\rho(x_n, z)) + (1-\delta)\rho^w(z, x_{n+1})$$

$$< \rho(z, x_{n+1}) + \frac{\rho(z, Tz)}{2} + (1-\delta)\rho^w(z, x_{n+1}).$$

Taking limits in the above as $n \to \infty$, we deduce

$$0 < \rho(z, Tz) < \frac{\rho(z, Tz)}{2}$$

which is a contradiction. It follows that $\rho(Tz, z) = 0$, and thus $z = Tz$. For the uniqueness of the fixed point, suppose k is another fixed point of *T*, if $\rho(z, k) = 0$, then $z = k$ is clear. So we assume $\rho(z, k) > 0$. Now observe we have the following:

$$0 < \rho(z, k)$$

$$= \rho(Tz, Tk)$$

$$\leq \varphi(\rho(z, k)) + (1-\delta)\rho^w(k, Tz)$$

$$= \delta\rho(z, k) + (1-\delta)[\rho(k, Tz) - \min\{\rho(k, k), \rho(Tz, Tz)\}]$$

$$= \delta\rho(z, k) + (1-\delta)[\rho(k, z) - \min\{\rho(k, k), \rho(z, z)\}]$$

$$\leq \delta\rho(z, k) + (1-\delta)\rho(k, z)$$

$$= \rho(z, k)$$

which is a contradiction, so $z = k$ and uniqueness follows. o

Since any map satisfying Definition 2.1 also satisfies Definition 2.4, we have the following:

Corollary 2.6. *Let* (X, ρ) *be a* 0-*complete partial metric space and* $T : X \mapsto X$ *satisfy Definition* 2.1, *then T has a unique fixed point*.

Finally we illustrate the above Corollary with the following.

Example 2.7. Let $X = [\frac{1}{2}, 1]$. If $x \neq y$, put $\rho(x, y) = \max\{x, y\}$; if $x = y$, put $\rho(x, y) = 0$. It is clear that ρ is a partial metric and (X, ρ) is 0-complete. Also $\rho^w(x, y) = \rho(x, y)$. If $x \in [\frac{1}{2}, 1)$, put $Tx = 0$; if $x = 1$, put $Tx = 1$. Clearly all the conditions of the Corollary immediately above are satisfied and $x = 1$ is the unique fixed point. It remains to show the contractive condition of the Corollary holds. Observe we must check for some $\delta \in (0, 1)$ that

$$\rho(Tx, Ty) \leq \delta\rho(x, y) + (1-\delta)\rho^w(y, Tx)$$

holds for all $x, y \in X$. It might be necessary also to check the dual of the inequality immediately above, that is,

$$\rho(Tx, Ty) \leq \delta\rho(x, y) + (1-\delta)\rho^w(x, Ty).$$

Therefore it is sufficient to check that for some $\delta \in (0, 1)$ and for all $x, y \in X$, the following holds:

$$\rho(Tx, Ty) \leq \delta\rho(x, y) + (1-\delta)\min\{\rho^w(y, Tx), \rho^w(x, Ty)\}.$$

Case 1. $x = y$. In this case $\rho(Tx, Ty) = 0$, and the result is clear.

For the remaining cases, we assume $x \neq y$.

Case 2. $x, y \in [\frac{1}{2}, 1)$. In this case we also have $\rho(Tx, Ty) = 0$, and the result is clear.

Case 3. $x \in [\frac{1}{2}, 1)$ and $y = 1$ or $y \in [\frac{1}{2}, 1)$ and $x = 1$. In this case by symmetry we may exchange "x" with "y", thus only need to consider $x \in [\frac{1}{2}, 1)$ *and* $y = 1$. Now observe we have the following with $\delta = \frac{1}{2}$

$$\begin{aligned}
\rho(Tx, Ty) &= 1 \\
&= \frac{1}{2} + \frac{1}{2} \\
&= \frac{1}{2}\max\{x, 1\} + \frac{1}{2}\min\{\max\{1, 0\}, \max\{x, 1\}\} \\
&= \frac{1}{2}\rho(x, y) + \frac{1}{2}\min\{\rho(y, Tx), \rho(x, Ty)\} \\
&= \frac{1}{2}\rho(x, y) + \frac{1}{2}\min\{\rho^w(y, Tx), \rho^w(x, Ty)\}.
\end{aligned}$$

3. Concluding Remarks

This paper improves upon the existence results given by Theorem 3 and Corollary 1 contained in Altun and Acar [3].

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