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FURTHER STUDIES ON THE IMPROVED REPLACEMENTS OF FEYNMAN'S FREE PARTICLE PATH INTEGRALS

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Abstract

An improved formula for the transition probability amplitude for a free particle initially in an eigenstate of position is further refined. Consideration is given to the transition probability amplitude conservation aspect of the problem.

1. Introduction

Recently [1], we had given an improved formula for the transition probability amplitude of a free particle initially in an eigenstate of position $|x_0\rangle$ at time t_0 to an eigenstate $|x\rangle$ at time t (see the very last mathematical expression in reference [1]) which is

$$\langle x | e^{-iH(t-t_0)} | x_0 \rangle = |K|^2 \left(\frac{2\pi(t-t_0)}{im(x-x_0)^2} \right)^{\frac{1}{2}} \exp\left(\frac{im}{2(t-t_0)} (x-x_0)^2 \right). \qquad (1)$$

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This is supposed to replace the usual expression for the same quantity which appears in textbooks [2] (see Eq. (1) of ref. [1])

$$\langle x\,|e^{-iH(t-t_0)}|\,x_0\rangle = \left(\frac{m}{2\pi i(t-t_0)}\right)^{\frac{1}{2}} \exp\!\left(\frac{im}{2(t-t_0)}(x-x_0)^2\right). \qquad (2)$$

However, there are several things which need to be clarified before the improved version can be used by physicists. The most important of these is the unknown 'constant' $|K|^2$ appearing in Eq. (1) above which has to be specified. In fact, we show below in Section 3 that this is not an absolute constant but a function of time without, however, invalidating any of the steps used in ref. [1] for arriving at this equation. Secondly, a physical basis for conservation of matter which was a reason for not accepting Eq. (2) will be worked out in Section 2 even though this equation seems to have 'official acceptance'. This conservation of matter or in other words the conservation of probability is the same as that worked out in terms of the particle wave function [3] (see p. 62 and 63 of this reference) which in our case as will be shown below is a specialized one dimensional application of this formulation. Another problem which needs to be settled first before we can proceed to the determination of $|K|^2$ of Eq. (1) is to remove the singularity at $x = x_0$ on the right hand side of this equation. This, we feel, is an error in Eq. (1) and has crept in because we have used the stationary phase approximation for evaluating an integral in order to derive the equation. Heuristic arguments will be presented in Section 3 to replace Eq. (1) with a better alternative.

2. Probability Conservation (or its failure) in Cases of Eqs. (1) and (2) and a need for better Alternative to Eq. (1)

It is a well known fact [4] (see pages 51 and 52 of this reference except that for our case $\hbar = 1$ in the units we use) that $e^{-iH(t-t_0)}|\, x_0\rangle = |\, \psi(t)\rangle$ is the state of the particle at time t when initially at time $t = t_0$ it was in an eigenstate of position $|\, x_0\rangle$. Also it is known from ref. [4] (see p. 70) that the wave function $\psi(x, t)$ is nothing but $\langle x \,\|\, \psi(t)\rangle$. Thus we interpret the left hand side of our Eqs. (1) and (2) as nothing but the wave function $\psi(x, t)$ of a particle evolving from an eigenstate of position at $t = t_0$. As is true for all wave functions it satisfies the Schrödinger equation of wave mechanics in one dimension as the coordinates y and z do not appear in this function $\psi(x, t)$ and hence we write

$$-\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\frac{\partial\psi}{\partial t}. \tag{3}$$

Multiplying this with ψ^* and subtracting the complex conjugate of Eq. (3) multiplied with ψ yields

$$-\frac{1}{2im}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) = \frac{\partial}{\partial t}(\psi^*\psi), \tag{4}$$

where $\psi = \langle x\,|e^{-iH(t-t_0)}|\, x_0\rangle$ and this can be applied to the expressions given by Eqs. (1) and (2), respectively. One must, however, note that in order to integrate the left and right hand sides of Eq. (4) with respect to x from $-\infty$ to ∞, $\psi(x, t)$ needs to be a well-behaved function of x free of any singularities. The right hand side of Eq. (1) shows that at $x = x_0$ the function is singular and hence it has to be replaced by a better function so that we can use it in Eq. (4) to obtain an integral form of this same equation. As we have already remarked in Section 1 that this singularity may have arisen because of the approximate method used in obtaining Eq. (1) in our earlier paper [1], we can in principle modify the function at and near $x = x_0$ and this will be attempted in Section 3.

For the moment assuming that $\psi(x, t)$ is a well behaved function of x (which the right hand side of Eq. (2) really is), we obtain an integral form of Eq. (4) which expresses conservation of probability in one dimension

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\psi dx = -\frac{1}{2im}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]_{x=-\infty}^{x=\infty}. \tag{5}$$

This is the one dimensional analogue of the Eq. (2-111) of reference [3]. One can apply $\psi = \langle x | e^{-iH(t-t_0)} | x_0 \rangle$ of Eq. (2) to Eq. (5) to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\psi dx = \left.\frac{m(x - x_0)}{2\pi(t - t_0)^2}\right|_{x=-\infty}^{x=\infty} = \infty. \tag{6}$$

Thus conservation of probability is violated in this form of transition probability amplitude although it is given by Feynman and accepted in all modern textbooks. It was the reason for our original motivation to find an alternative to this formulation of path integral in non relativistic quantum mechanics. One must seek a form of $\psi = \langle x | e^{-iH(t-t_0)} | x_0 \rangle$ which satisfies $\frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\ \psi dx = 0$ and having this in mind we seek to modify Eq. (1) suitably in Section 3.

3. A suitable Modification of Eq(1) to get rid of the Singularity at $x = x_0$ and hence to Evaluate $|K|^2$

First we note from Eq. (1)

$$\psi\psi^* = |K|^4\ \frac{2\pi}{m}\frac{t - t_0}{(x - x_0)^2}. \tag{7}$$

One of the ways to get rid of the singularity is to replace this function in the interval $-\sqrt{t-t_0}+x_0 < x < \sqrt{t-t_0}+x_0$ with $f(t, x)$ which satisfies

$$\int_{-\sqrt{t-t_0}+x_0}^{\sqrt{t-t_0}+x_0} f(t, x)\,dx = 1 - |K|^4 \frac{4\pi}{m}\sqrt{t-t_0} \tag{8}$$

and

$$f\left(t, -\sqrt{t-t_0}+x_0\right) = f\left(t, \sqrt{t-t_0}+x_0\right) = |K|^4 \frac{2\pi}{m}. \tag{9}$$

This form of the function $\psi^*\psi$ has no singularities and can be applied to Eq. (5) to give $\frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\psi dx = 0$ that is the probability density integral is conserved. Normalization of the function ψ can be obtained by noting from Eq. (7)

$$\int_{-\infty}^{-\sqrt{t-t_0}+x_0} |K|^4 \frac{2\pi}{m}\frac{t-t_0}{(x-x_0)^2}\,dx$$

$$= \int_{\sqrt{t-t_0}+x_0}^{\infty} |K|^4 \frac{2\pi}{m}\frac{t-t_0}{(x-x_0)^2}\,dx = 2|K|^4 \frac{\pi}{m}\sqrt{t-t_0}. \tag{10}$$

Thus we have $\int_{-\infty}^{-\sqrt{t-t_0}+x}\psi^*\ \psi dx + \int_{-\sqrt{t-t_0}+x}^{\sqrt{t-t_0}+x}\psi^*\ \psi dx + \int_{\sqrt{t-t_0}+x}^{\infty}\psi^*\ \psi dx = 1.$ A simple choice $f(t, x) = |K|^4 \frac{2\pi}{m}$ helps us to evaluate the left hand side of Eq. (8) and give $|K|^4 = \frac{m}{8\pi\sqrt{t-t_0}}$ from which we get an improved formula to replace Eq. (1). In the interval $-\sqrt{t-t_0}+x_0 < x < \sqrt{t-t_0}+x_0$, we have

$$\langle x|e^{-iH(t-t_0)}|x_0\rangle = \left(\frac{1}{4i\sqrt{t-t_0}}\right)^{\frac{1}{2}} \exp\left(\frac{im}{2(t-t_0)}(x-x_0)^2\right) \tag{11a}$$

and for other values of x

$$\langle x \,| e^{-iH(t-t_0)} |\, x_0 \rangle = \left(\frac{\sqrt{t - t_0}}{4i(x - x_0)^2} \right)^{\frac{1}{2}} \exp\left(\frac{im}{2(t - t_0)} (x - x_0)^2 \right), \quad \text{(11b)}$$

A fact that needs to be taken notice of is that in the interval $-\sqrt{t - t_0} + x_0 < x < \sqrt{t - t_0} + x_0$, the above expression is similar in some ways to Feynman's original expression that is Eq. (2). However, the entire analysis in the present note has been made to show the unphysical nature of Eq. (2) and that Eqs. (11a) and (11b) are in line with modern physics.

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