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# FEYNMAN'S DYNAMICAL ROUTE TO SPECIAL RELATIVITY VIA WORK-TO-ENERGY CONVERSION AND NEWTON'S SECOND LAW

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## Abstract

Adapting a derivation due to Feynman, expressions for the relativistic energy and momentum of a ponderable object are obtained from work-energy equivalence and Newton's Second Law of mechanics. Transformation laws of relativistic energy and momentum are found and the time dilation relation and the Lorentz transformation of time intervals are obtained from the invariant relation connecting the Newtonian mass of an object to its relativistic energy and momentum. Lorentz transformations of space-time events are then derived and discussed. Finally, space-time geometrical and kinematical velocity transformation formulas are compared. Although identical in Galilean relativity, they are found to differ at order $(v/c)^2$ in special relativity.

Keywords and phrases: mass-energy equivalence, relativistic mass variation, Lorentz transformations, relative velocity transformations.

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## 1. Introduction

The origins of special relativity theory (SRT) are to be found in considerations related to classical electromagnetic theory. A system of transformation equations where time intervals are predicted to have different values for observers in different inertial frames, was proposed by Voigt [1] in 1887 in order to render covariant the wave equation. These transformations were adapted by Lorentz [2] in order to obtain the eponymous space-time transformation equations that are the basis of SRT. As demonstrated by Lorentz [2], as well as earlier by Larmor [3] and slightly later by Poincaré [4], the Lorentz Transformations (LT) render covariant Maxwell's equations, and so predict that electromagnetic waves have the same speed in vacuum, $c$, in all inertial frames. In Einstein's seminal paper on SRT of 1905 [5], this prediction was elevated (though tacitly[1]) to the status of a postulate and used to derive the LT from first principles.

It was soon realised that Einstein's 'second postulate' concerning the constancy of the speed of light was not necessary in order to derive the LT. Following the pioneering papers of Ignatowski [6], Frank and Rohe [7] and Pars [8], the number of published axiomatic derivations of the LT, not invoking light-speed-constancy, is now vast. Work prior to 1968 is cited in ref. [9], more recent work in ref. [10] by the present author. The majority of these derivations, following that of Einstein in 1905, were 'kinematical' in the sense that no dynamical laws of physics, or the properties of particular physical systems, were considered. In contrast, in the original applications of the LT by Lorentz [2] and Poincaré [4] to the

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[1]The actual statement of Einstein's 'principle of the constancy of the speed of light' is [5]: 'Any ray of light moves in the "stationary" system of coordinates with a determined velocity $c$ whether the ray is emitted by a stationary or by a moving body'. This is only a statement of source-speed-independence of the speed of light in a particular frame. In the subsequent derivation of the LT in ref. [5], the equality of the speed of light in the 'stationary' system $K$ and the 'moving' system $k$ is asserted without further comment.

electromagnetic structure of the electron, kinematics and dynamics were inextricably intertwined. It was stressed by Pais [11] that one of the most important conceptual advances contributed by Einstein to the development of SRT was the demonstration of the 'kinematical' and hence universal nature of the LT.

Some postulates that have been employed by different authors to derive the space-time LT are:

Einstein [5]:

- The laws of physics are the same in all inertial frames
- The speed of light is the same in all inertial frames
- Space-time homogeneity
- Spatial isotropy
- The Reciprocity Principle

Pars [8]:

- Linear transformation equations
- Galilean relativity at small velocities
- Group property of transformations

Field (1) [10]:

- The LT is a single-valued function of its arguments
- Reciprocal space-time measurements of similar rulers and clocks at rest in different inertial frames yield identical results
- Spatial isotropy

Field (2) [12]:

- The LT is a single-valued function of its arguments

• Equations describing laws of physics are invariant with respect to exchange of unidimensional space and time coordinates, or, more generally, to exchange of unidimensional spatial and temporal components of four-vectors

In almost all cases the derivations are purely mathematical manipulations of equations linear in space-time coordinates, with no discussion of the operational meaning of the symbols employed. Einstein did not actually state the frame independence of the speed of light - only that it is independent of the speed of its source. However, this second postulate, as well as the last three listed above, were tacitly invoked in his derivation of the LT [5]. The 'Reciprocity Principle' (to be further discussed below) is the assertion that if the relative velocity of two objects A and B in the rest frame of A is $\vec{v}$ then that of A in the rest frame of B is $-\vec{v}$. The derivation of Pars is the first of many appearing in the literature (for example [14, 15, 16]) invoking the group property. This property is critically discussed in the Appendix of the present paper. The motivation of the two derivations of the LT by the present author cited above was to minimise the number of necessary postulates, and to make them as weak as possible. Only the second postulate of the first derivation concerns the operational meaning (rulers and clocks are invoked) of the space-time coordinate symbols.

Advantages of axiomatic derivations of the LT are the simplicity and almost 'obvious' nature of the postulates used, and mathematical elegance. However, a disadvantage, presciently pointed out in 1921 by Pauli [13] is that such derivations shed little light on the physical meaning of the equations:

*From the group theoretical assumption, it is only possible to determine the general form of the transformation formulae not of their physical content.*

Indeed the still controversial and anti-common-sense nature of some predictions of SRT are due, not to the equations of the theory, but rather how the equations are physically interpreted.

In contrast, the derivation of the LT to be presented in the present paper is based not on abstract considerations of space-time geometry or relativistic invariance but instead on the relativistic generalisation of a particular physical law - Newton's Second Law of Motion of classical mechanics. The space-time LT is then derived from the equations of relativistic kinematics (i.e., the transformation laws of energy and momentum) not, as conventionally done, in the inverse order. The 'relativistic generalisation' is obtained by introducing dimensionally-correct ansätze for energy and momentum valid for any physically-allowed velocity of a moving object in any inertial frame. The advantage of this approach is that the operational meaning of all coordinate symbols in the equations is clear from the outset.

In the work of Einstein, and in particular in axiomatic derivations of later authors, as well as in text books, the LT are presented as transformations between an event: $(x, y, z, t)$ observed in one frame S and the same event: $(x', y', z', t')$ as observed in another frame S'. In order to specify such an event, it is evident that spatial coordinate systems must be defined in the different frames. Also the existence of the epochs $t, t'$ of the occurrence of the event presuppose the existence of some clocks, at rest in the frames S, S', respectively, to measure them. The problem of the synchronisation of these clocks, in relative motion, must then be addressed. As pointed out in previous work by the present author [17, 18, 19, 20, 21, 22, 23], and as will be seen below, certain incorrect predictions of SRT that have survived, in the literature, text books and the popular understanding of SRT from 1905 until the present day, are due not to any errors in the LT per se, but rather incorrect handling of coordinate systems and clock synchronisation constants when they are used to transform space-time events rather than space-time *intervals*.

In the axiomatic derivation of SRT in the present paper, firstly the formulas for relativistic energy, $E$, and momentum, $\mathbf{p}$ are obtained, as well as transformation laws relating different kinematical configurations $(E, \mathbf{p})$ of the same physical object in the same inertial frame. In a second step the time dilation (TD) relation and the time LT for space and time intervals are derived. Thirdly, on introducing spatial coordinate systems, the LT for space-time events are derived. Lastly, relativistic transformations of *relative velocities*, as previously discussed in the literature, are derived and discussed. Unlike in previous work, the axioms employed make no reference to the special relativity principle, light speed, group properties, or any considerations of spatial geometry or of symmetry. They are:

(I) Work-to-energy conversion: $dE = dW = \mathbf{F} \cdot d\mathbf{s}$.

(II) Newton's Second Law of mechanics: $d\mathbf{p} / dt = \mathbf{F}$.

(III) Ansätze, proportional to its Newtonian mass, $m$, for the velocity dependence of the relativistic energy, $E$, and momentum, $\mathbf{p}$, of a physical object.

Here $dE$ is the change of energy of a physical object due to the work $dW$ done by its displacement $ds$ under the action of an impressed force $\mathbf{F}$. The derivation proceeds by first obtaining expressions for $E$ and $\mathbf{p}$ as a function of velocity and then considering the transformation $(E, \mathbf{p}) \to (E', \mathbf{p}')$ of kinematical configurations of the object in a single reference frame, due to the action of the force. The transformation law so obtained allows the definition of two kinematical invariants, from the first of which the TD relation and the LT for time intervals may be obtained. Because these transformation laws contain only space or time *intervals*, these equations unlike the LT for *events* do not depend on the definition of coordinate systems or of clock synchronisation constants. On introducing coordinate systems and synchronised clocks, the LT for space-time events are then obtained and compared with the conventional event LT of SRT. Finally, predictions of space-time geometry (i.e., for observations of spatial positions at different epochs of the same object in different inertial frames in the same space-time experiment) are compared and contrasted with those of relativistic kinematics. In particular, transformation formulas for relative velocities are derived and compared with the velocity transformation formulas of conventional SRT as derived by Einstein [5].

The organisation of the paper is as follows: In the following section, explicit formulas for relativistic energy and momentum are derived by adapting a calculation of Feynman [25] invoking Newton's Second Law[2]. An initially unknown universal constant occurring, for dimensional reasons, in the initial ansätze is identified with the square of the speed of light in free space. In Section 3, the transformation laws of relativistic energy and momentum are derived from an invariant relation obtained in the previous section. In Section 4, the space-time LT for intervals or events are derived and interpreted to demonstrate the spurious nature of the 'length contraction' and 'relativity of simultaneity' effects of conventional SRT. In Section 5, space-time geometrical and kinematical velocity transformations are compared and contrasted. Group properties of different velocity transformations and their physical interpretation are discussed in an appendix.

## 2. Relativistic Energy and Momentum

Relativistic energy and momentum as attributes of a physical object

[2]A similar calculation was published much earlier by Lewis [24]. This fact came to the notice of the author only after the completion of the first version of the present paper.

of Newtonian mass $m$, and velocity $v$ in some inertial frame, are defined here according to the ansätze:

$$E(v) \equiv \kappa m \gamma(v), \tag{2.1}$$

$$\mathbf{p}(v) \equiv \frac{E(v)}{\kappa}\mathbf{v} = m\gamma(v)\mathbf{v}. \tag{2.2}$$

In these equations $\kappa$ is a universal constant of dimensions $L^2T^{-2}$, the physical significance of which will be elucidated in the following. The initially unknown, dimensionless, function $\gamma(v)$, $(v \equiv |\mathbf{v}|)$ may be assumed to satisfy the condition $\gamma(0) = 1$. In this way, the classical definition of momentum is recovered by (2.2) in the low velocity limit.

Following a proof due to Feynman [25], the function $\gamma(v)$ and hence the velocity dependence of $E$ and $\mathbf{p}$ will now be determined from the solution of a first order differential equation obtained by combining the similar equations describing the conversion of work into energy:

$$dE = dW = \mathbf{F} \cdot d\mathbf{s} \tag{2.3}$$

and Newton's Second Law of mechanics, expressed in terms of momentum:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}. \tag{2.4}$$

The quantity $d\mathbf{s}$ is the infinitesimal displacement of the object during the time interval $dt$ under the action of an arbitrary force $\mathbf{F}$. Since $\mathbf{v} \equiv d\mathbf{s}/dt$ (2.1)-(2.4) may be combined to give:

$$dE = \kappa m d\gamma = \mathbf{F} \cdot d\mathbf{s} = \mathbf{v} \cdot d\mathbf{p} = m\mathbf{v} \cdot d(\gamma\mathbf{v}). \tag{2.5}$$

Canceling throughout the common factor $m$ and multiplying through by $\gamma$, (2.5) gives:

$$\kappa\gamma d\gamma = \frac{\kappa}{2} d(\gamma^2) = \gamma\mathbf{v} \cdot d(\gamma\mathbf{v}) = \frac{1}{2} d(\gamma^2 v^2) \tag{2.6}$$

so that $\gamma(v)$ is the solution of the differential equation:

$$\kappa d(\gamma^2) = d(\gamma^2 v^2). \tag{2.7}$$

Direct integration of (2.7) with the boundary condition $\gamma(0) = 1$ gives:

$$\kappa[\gamma(v)^2 - 1] = \gamma(v)^2 v^2. \tag{2.8}$$

Solving this equation for $\gamma(v)$, it is found that:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{\kappa}}}. \tag{2.9}$$

The above calculation is identical to that given by Feynman [25] except that Feynman assumed initially that $\kappa = c^2$, where $c$ is the speed of light in vacuum. In the present work, this identification [10] will be seen to follow by assuming that light actually consists of particles - photons.

Since the relation (2.9) is physical only if $v < \sqrt{\kappa} = v_{MAX}$, the physical significance of the constant $\kappa$ is clear: $\kappa = v_{MAX}^2$, *where* $v_{MAX}$ *is the maximum possible velocity of a ponderable object in an inertial frame allowed by the application of forces that obey Newton's Second Law.*

To identify the limiting velocity $v_{MAX}$ with the speed of light in free space, (2.2) is written as:

$$p = \sqrt{\kappa} m\gamma(v)\beta(v), \tag{2.10}$$

where $\beta(v) \equiv v / \sqrt{\kappa}$. Then use of the identity: $\gamma^2 - \gamma^2\beta^2 \equiv 1$ together with (2.1) and (2.10) gives

$$E^2 - \kappa p^2 = \kappa^2 m^2(\gamma^2 - \gamma^2\beta^2) = \kappa^2 m^2. \tag{2.11}$$

For any object which is light[3] in the sense that $E, \sqrt{\kappa}p >> \kappa m$, (2.1), (2.2) and (2.11) give the relation

$$v = \frac{\kappa p}{E} \cong \frac{\kappa p}{\sqrt{\kappa}p} = \sqrt{\kappa} = v_{MAX}. \tag{2.12}$$

If light is therefore identified as the manifestation of the propagation of massless (or almost massless) particles in free space, it follows that $c \cong v_{MAX} = \sqrt{\kappa}$. The second of Einstein's postulates of special relativity, when (see below) it is correctly physically interpreted, is therefore derived in the present approach. An experimentally verified corollory is that all particles satisfying the conditions $E, \sqrt{\kappa}p >> \kappa m$ will move isotropically, in the inertial frame in which they are produced, at the same speed as light in free space. This derivation by the present author of the relation: $\sqrt{\kappa} = v_{MAX} = c$ for massless particles was first given in Ref. [10].

## 3. Transformation Laws of Relativistic Energy and Momentum

It follows from Equation (2.11) that different kinematical configurations, $(E, \mathbf{p}), (E', \mathbf{p}')$ of an object of Newtonian mass $m$ are related by the equation:

$$E^2 - \kappa p^2 = (E')^2 - \kappa(p)^2 = \kappa^2 m^2. \tag{3.1}$$

It is germane to consider generation of the configuration $(E', \mathbf{p}')$ from $(E, \mathbf{p})$ by a two parameter boost operation. Since the momentum vectors $\mathbf{p}$ and $\mathbf{p}'$ define a plane in the inertial frame of observation, it is

[3]No pun intended

convenient, as shown in Figure 1, to decompose these vectors into components parallel to (longitudinal, L, component) and transverse to (T component) the vector $\mathbf{p}' - \mathbf{p}$ :

$$\mathbf{p} = \mathbf{p}_L + \mathbf{p}_T, \quad \mathbf{p}' = \mathbf{p}'_L + \mathbf{p}'_T, \tag{3.2}$$

$$p_L = p\cos\theta,\ p'_L = p'\cos\theta',\ p_T = p\sin\theta = p'_T = p'\sin\theta'. \tag{3.3}$$

It therefore follows from (3.1)-(3.3) that there are two independent frame-invariant relations [26] involving, respectively, the longitudinal and transverse components of relativistic momentum:

$$E^2 - \kappa p_L^2 = (E')^2 - \kappa(p'_L)^2, \tag{3.4}$$

$$p_T^2 = (p'_T)^2. \tag{3.5}$$

Linear transformation equations between $(E, \mathbf{p}_L)$ and $(E', \mathbf{p}'_L)$ may, in general, be written as:

$$E' = c_1 E + c_2\sqrt{\kappa}\, p_L, \tag{3.6}$$

$$p'_L = c_3 p_L + \frac{c_4 E}{\sqrt{\kappa}}, \tag{3.7}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are dimensionless coefficients. Setting $E = m\kappa$ and $p_L = p_T = 0$ in (3.6) and (3.7) gives, on comparing with the relations $E' = \kappa m\gamma'$, $p'_L = p' = \sqrt{\kappa}m\gamma'\beta'$ : $c_1 = \gamma'$ and $c_4 = \gamma'\beta'$. This suggests writing the transformation equations as:

$$E' = \gamma''(E + a\sqrt{\kappa}\, p_L), \tag{3.8}$$

$$p'_L = \gamma''(b p_L + \frac{\beta'' E}{\sqrt{\kappa}}), \tag{3.9}$$

where the two parameters specifying the boost are $\theta$ and $\beta''$, and the dimensionless coefficients $a$ and $b$ are determined by consistency of (3.8) and (3.9) with (3.4). Substituting (3.8) and (3.9) in (3.4), it is found that $\gamma''$ and $\beta''$ satisfy the identity: $(\gamma'')^2 - (\gamma''\beta'')^2 \equiv 1$, and that $a = \beta''$ and $b = 1$, so that the complete transformation equations of relativistic energy and momentum are:

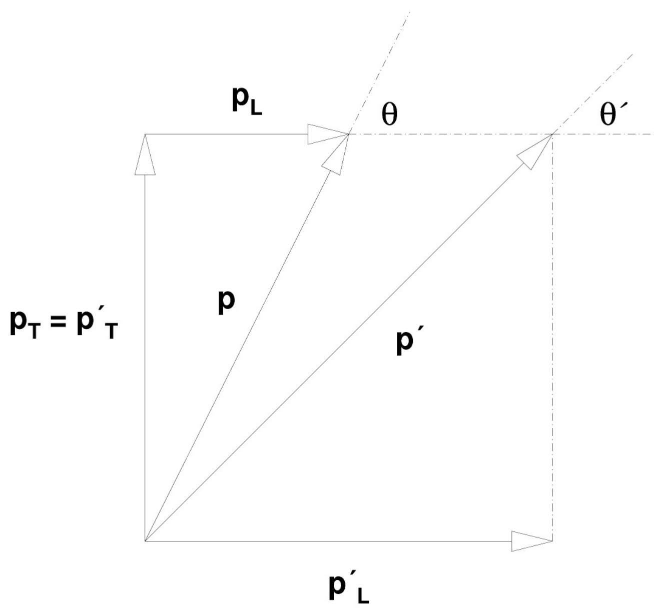


**Figure 1.** Geometrical definitions for the transformation of relativistic energy-momentum configurations: $(E, \mathbf{p}) \to (E', \mathbf{p}')$. Momenta and angles are shown in the plane spanned by the relativistic momentum vectors $\mathbf{p}$ and $\mathbf{p}'$.

$$E' = \gamma''(E + \beta''\sqrt{\kappa}p_L), \tag{3.10}$$

$$p'_L = \gamma''\left(p_L + \frac{\beta'' E}{\sqrt{\kappa}}\right), \tag{3.11}$$

$$p'_T = p_T. \tag{3.12}$$

On setting $\sqrt{\kappa} = c$, the well-known transformation equations of relativistic energy and momentum are recovered.

Noting that $\beta = \sqrt{\kappa}p / E$, $\beta' = \sqrt{\kappa}p' / E'$ and $\beta'' = \sqrt{\kappa}p'' / E''$, multiplying (3.11) by $\sqrt{\kappa}$ and dividing it by (3.10) gives:

$$\beta' \cos\theta' = \frac{\beta\cos\theta + \beta''}{1 + \beta''\beta\cos\theta}, \tag{3.13}$$

while dividing (3.12) by (3.10) gives

$$\beta' \sin\theta' = \frac{\beta\sin\theta}{\gamma''(1 + \beta''\beta\cos\theta)}. \tag{3.14}$$

Solving (3.13) for $\beta''$ gives the boost parameter $\beta''$ in terms of the parameters $(\beta, \theta)$ and $(\beta', \theta')$ specifying the configurations: $(E, \mathbf{p})$ and $(E', \mathbf{p}')$:

$$\beta'' = \frac{\beta'\cos\theta' - \beta\cos\theta}{1 - \beta\beta'\cos\theta\cos\theta'}. \tag{3.15}$$

The ratio of (3.14) to (3.13) gives the angle $\theta'$ in terms of the boost parameter $\beta''$ and the parameters $(\beta, \theta)$ of the original configuration $(E, \mathbf{p})$:

$$\tan\theta' = \frac{\beta\sin\theta}{\gamma''(\beta\cos\theta + \beta'')}. \tag{3.16}$$

It is important to contrast the physical meaning of the relations (3.10)-(3.16), just derived, from the interpretation given to them in conventional special relativity theory. The configurations $(E, \mathbf{p})$ and $(E', \mathbf{p}')$ are different ones of the same object as viewed in the same reference frame. The physical meaning of the boost $(\theta, \beta'')$ is similar to that of an active rotation of the position vector of an object in a fixed reference frame. An invariant of the latter transformation is the length of the position vector; the invariant of the transformation between $(E, \mathbf{p})$ and $(E', \mathbf{p}')$ is that defined by Equation (3.1) above. In contrast (3.10) to (3.12) have previously been interpreted as giving the energy and momentum $(E', \mathbf{p}')$ of an object as observed in a frame with boost $(\theta, -\beta'')$ relative to the frame with configuration $(E, \mathbf{p})$. In a similar manner, (3.15) has been erroneously interpreted as giving the relative velocity, in the rest frame of the object with the configuration $(E, \mathbf{p})$, of the object rest frame where the configuration is $(E', \mathbf{p}')$. This problem of interpretation will be rediscussed below after consideration of the space-time transformation equations.

## 4. Derivation and Interpretation of the Space-Time Lorentz Transformation Equations

The second member of Equation (2.11) may be written as:

$$[\kappa m\gamma(v)]^2 - \kappa[m\gamma(v)v]^2 = (\kappa m)^2. \tag{4.1}$$

Dividing through by a factor $\kappa m^2$ and using the definition of the velocity $v : v \equiv ds / dt,$ where $ds$ is a spatial displacement of the object, gives:

$$\kappa\gamma(v)^2 - \left[\gamma(v)\frac{ds}{dt}\right]^2 = \kappa \tag{4.2}$$

or, on re-arrangement:

$$\kappa(dt)^2 - (ds)^2 = \kappa(d\tau)^2 \tag{4.3}$$

where:

$$d\tau \equiv \frac{dt}{\gamma(v)}. \tag{4.4}$$

Inspection of (4.3) shows that $dt = d\tau$ when $ds = 0,$ so that $d\tau$ is a time interval in the rest frame of the object. Transposition of (4.4) leads to the time dilation (TD) relation:

$$dt = \gamma(v)d\tau \tag{4.5}$$

giving the correspondence between a proper time interval $d\tau$ (a time interval registered by a clock at rest relative to the object) and the time interval $dt$ recorded by a clock, relative to which, the object moves at speed $v$. It is important to note that the TD relation has no spatial dependence. In consequence, on integrating (4.5) to give the relation between clock settings: $t = \gamma(v)\tau$, an arbitrary pair of clocks 1, 2 with a common velocity (and so at rest, at any position, in the same frame) that are synchronised so that, at a given instant, $\tau_1 = \tau_2$ are also observed to be so by a stationary observer:

$$t_1 = \gamma(v)\tau_1 = \gamma(v)\tau_2 = t_2$$

- there is no 'relativity of simultaneity' effect.

Space and time intervals along the worldline of a moving clock that obeys the TD relation (4.5) satisfy the relation:

$$ds = vdt = \sqrt{\kappa}\beta dt. \tag{4.6}$$

Combining this equation with (4.4) yields the Lorentz transformation equation for time intervals:

$$d\tau = \frac{dt}{\gamma(v)} = \frac{\gamma(v)dt}{\gamma(v)^2} = \gamma(v)(dt - \beta^2 dt) = \gamma(v)\left(dt - \frac{\beta ds}{\sqrt{\kappa}}\right). \tag{4.7}$$

Denoting, as is conventional, by S the inertial frame in which the clock has speed $v$, and S' (with interval of proper time $dt' = d\tau$ and displacement $ds'$) that in which it is at rest, then the Lorentz transformation for spatial intervals corresponding to (4.7) is:

$$ds' = \gamma(v)(ds - vdt) = 0 \tag{4.8}$$

since the displacement, $ds'$, of any object at rest in S' vanishes. Note that the factor $\gamma(v)$ on the right side of (4.8) can be replaced by any finite constant or function of $v$ without affecting any physical consequence of the transformations (4.7) and (4.8). In fact, these equations are physically equivalent, respectively, to:

$$d\tau = \frac{dt}{\gamma(v)}, \tag{4.9}$$

$$ds' = 0, \qquad ds = vdt. \tag{4.10}$$

The differential worldline equations in (4.10) are simply the description of uniform motion in the frame S, and so are the same in special and Galilean relativity. Then the only modification of space-time geometry in passing from Galilean to special relativity is the replacement of an interval of Newtonian universal time: $dT = dt = d\tau$ by the transposed TD relation (4.9).

In order to relate the transformation equations for time and space *intervals* (4.7) and (4.8) to those relating observation of a space time event in the frame S to the same event as observed in the frame S', spatial coordinate systems must be introduced in these frames to specify the positions of events. Choosing Cartesian coordinate systems with arbitrary origins but parallel axes, enables space-time coordinates of an event $(x, y, z; t)$ in S and $(x', y', z'; t')$ in S' to be specified. With the direction of the $x$ and $x'$ axes parallel to $\mathbf{v}$, the worldline of an object at rest in S' is written in the frame S as:

$$x(t) = x(t_0) + v(t - t_0), \tag{4.11}$$

$$y(t) = y(t_0), \tag{4.12}$$

$$z(t) = z(t_0) \tag{4.13}$$

and in the frame S' as:

$$x'(t) = x'(t_0), \tag{4.14}$$

$$y'(t) = y'(t_0), \tag{4.15}$$

$$z'(t) = z'(t_0). \tag{4.16}$$

In virtue of the linearity of the interval transformation Equations, (4.7) and (4.8) may be written in terms of the coordinates introduced in (4.11)-(4.16) as:

$$\Delta\tau \equiv \tau(t) - \tau(t_0) = t'(t) - t'(t_0) = \gamma(v)\left[t - t_0 - \frac{\beta}{\sqrt{\kappa}}[x(t) - x(t_0)]\right], \tag{4.17}$$

$$\Delta s' \equiv x'(t) - x'(t_0) = \gamma(v)[x(t) - x(t_0) - v(t - t_0)] = 0 \tag{4.18}$$

and (4.9) and (4.10) as:

$$\Delta\tau \equiv \tau(t) - \tau(t_0) = t'(t) - t'(t_0) = \frac{t - t_0}{\gamma(v)}, \qquad \Delta t \equiv t - t_0, \tag{4.19}$$

$$\Delta s' \equiv x'(t) - x'(t_0) = 0, \qquad \Delta s \equiv x(t) - x(t_0) = v(t - t_0). \tag{4.20}$$

The conventional space-time Lorentz transformation equations for space-time events, as derived by Einstein [5], to be found in text books on special relativity, correspond to a particular choice of coordinate origins and clock synchronisation constants in (4.17) and (4.18):

$$x'(t_0) = x(t_0) = 0, \qquad t_0 = \tau(t_0) = 0$$

to yield

$$t'(t) = \tau(t) = \gamma(v)\left[t - \frac{\beta}{\sqrt{\kappa}}[x(t)]\right], \tag{4.21}$$

$$x'(t) = \gamma(v)[x(t) - vt] = 0 \tag{4.22}$$

which, on ignoring the temporal dependence of the coordinates and the proper time as well as the '= 0' in the last member of (4.22) gives, on setting $\sqrt{\kappa} = c$, the conventional Lorentz transformations:

$$t' = \tau = \gamma(v)\left[t - \frac{\beta}{c}x\right], \tag{4.23}$$

$$x' = \gamma(v)[x - vt]. \tag{4.24}$$

The spurious 'length contraction' and 'relativity of simultaneity' effects arise essentially from the missing conjunction $'= 0'$ on the right side of Equation (4.24). Since $x'$ specifies the position of an object in a frame in which it is rest, it must be independent of time. Since the right side of (4.24) vanishes for some value of $t$ (when $t = x/v$), whatever the value of $x$, it must therefore, because the left side of (4.24) is independent of $t$, vanish for all values of $t$. *That is* (4.24) *is only valid, as an event transformation function, when both* $x' = 0$ *and* $x = vt$.

The transformations (4.21) and (4.22) show how an event $(x' = 0, t' = 0)$ in the frame S' is observed as $(x = 0, t = 0)$ in the frame S. This means that clocks at rest placed at the origins of S and S' are synchronised so that $t = t' = 0$, when the origins are aligned in the $x$, $x'$ direction. With the same choice of coordinate origins in the frames S and S' and the same synchronisation condition, but placing an object (on the worldline of which the transformed event is to be situated) instead at $x' = x'(0) \neq 0$ corresponds to the choice of initialisation constants in the general formulas (4.17) and (4.18)[4]:

$$x'(t_0) = x(t_0) \neq 0, \qquad t_0 = \tau(t_0) = 0$$

to give the transformation equations:

[4]Note that the freedom to choose, independently, coordinate origins in the frames S and S' without changing any physical predictions (translational invariance) always allows the choice $x'(0) = x(0)$ to specify the initial conditions of the problem.

$$t'(t) - \tau(t) = \gamma(v)\left[t - \frac{\beta}{\sqrt{\kappa}}[x(t) - x(0)]\right], \tag{4.25}$$

$$x'(t) - x(0) = \gamma(v)[x(t) - x(0) - vt] = 0 \tag{4.26}$$

or, equivalently:

$$t'(t) = \tau(t) = \frac{t}{\gamma(v)}, \tag{4.27}$$

$$x'(t) - x(0) = 0, \qquad x(t) - x(0) = vt. \tag{4.28}$$

Comparing (4.26) with the conventional space Lorentz transformation (4.24), it can be seen that the latter corresponds to the choice $x(0) = 0$. Since $x'(t) = x(0)$ for all values of $t$, it can again be seen that (4.24) can only be a valid event transformation if $x' = 0$. The distance, in S', between an object at the origin of S' and one at $x' = x(0)$ is $x(0)$. As the objects are at rest in S' this separation, $\Delta x'$, is time-independent. Since the world line of an object at the origin of S' is, according to (4.22), $x(t) = vt$, it follows from (4.28) and (4.22) that, for all values of $t$:

$$\Delta x \equiv x[t, x' = x(0)] - x[t, x' = 0] = x(0) = \Delta x'. \tag{4.29}$$

There is therefore no 'length contraction' effect. See refs. [17, 18, 19, 20, 21, 22, 23] for an explanation of how the spurious and correlated 'length contraction' and 'relativity of simultaneity' effects arise from manipulation of the conventional Lorentz transformation equations (4.23) and (4.24), under the invalid assumption that $x' \neq 0$ in (4.24), so that these equations are erroneously assumed to be valid for all values $(x, t)$ and $(x', t')$.

## 5. Relativistic Space-Time Geometry is Distinct from Relativistic Kinematics

Consider observation of a point on the worldline of a moving object in two inertial frames. Contrary to the usual convention as employed in the previous section, and in order to facilitate direct comparison with the kinematical transformations considered in Section 3, the inertial frame S is assumed to move uniformly with speed $v''$ along the $x'$-axis of the frame S'. As shown in Figure 2, the Cartesian coordinate axes $x$, $x'$ and $y$, $y'$ are assumed to be parallel. In the following, without any loss of generality, only motion in the $x$ - $y$ plane is considered. At time $t = t' = 0$, the origins O and O' of the coordinate systems are coincident. Starting at this instant, an object moves in a straight line at speed v from O to P in S [Figure 2a)] and at speed v' from O' to P' in S'. [Figure 2b)].

The postulates to be used in the following space-time calculation in order to relate velocities and angles observed in one frame to those observed in another are:

(i) The definition of uniform velocity:

$$\text{velocity} \equiv \mathbf{v} = \frac{\text{displacement of object}}{\text{elapsed time}} \equiv \frac{\mathbf{s}}{t}. \tag{5.1}$$

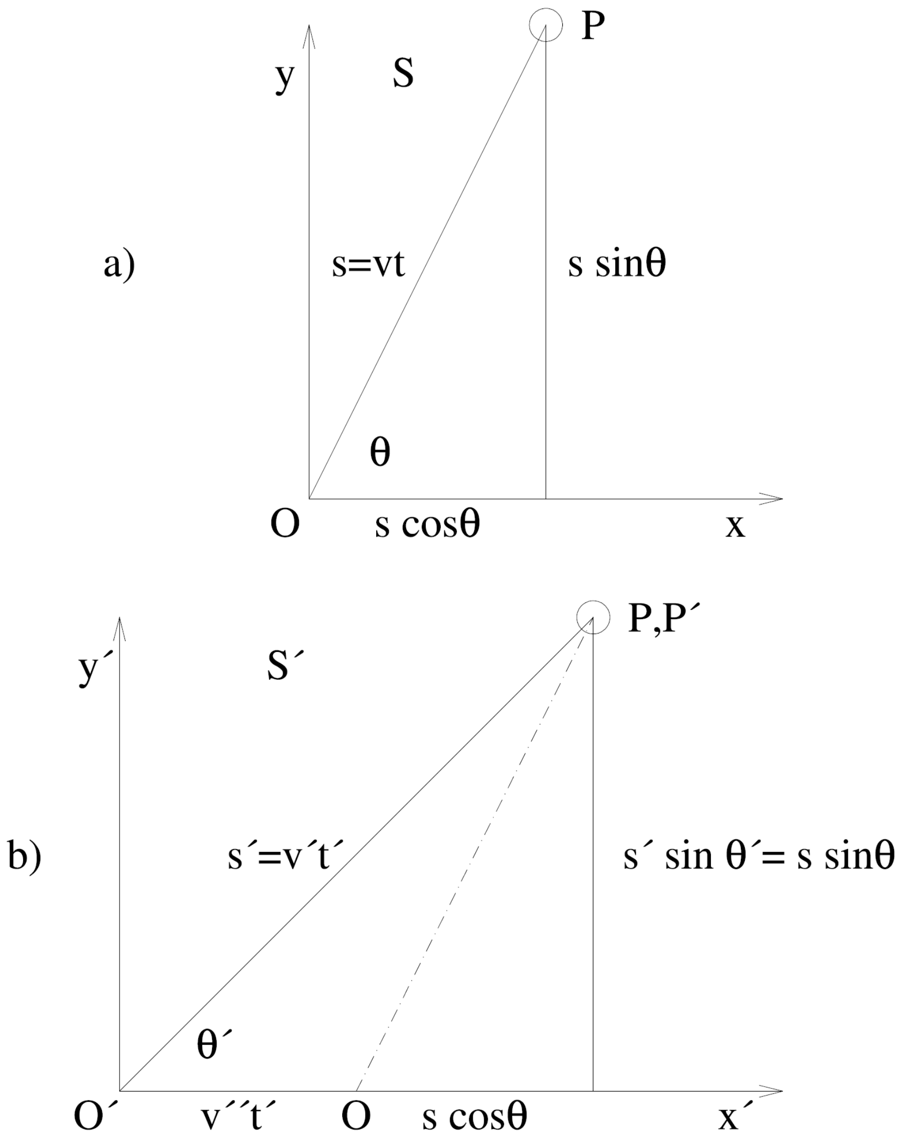


**Figure 2.** Space-time geometry of the world-lines of a uniformly-moving object in different inertial frames: (a) frame S, world-line segment OP, speed $v$; (b) frame S', world-line segment O'P', speed $v'$. The frame S moves with speed $v''$ in S' along the common $x$, $x'$ axis. See text for discussion.

(ii) The TD relation (4.5) for elapsed time intervals in the frames S and S' defined above:

$$t' = \gamma'' t \equiv \gamma(v'')t. \tag{5.2}$$

Considering the $x$, $x'$-components of the displacement, inspection of Figure 2b gives:

$$s' \cos\theta' = v't' \cos\theta' = v''t' + s\cos\theta = v''t' + vt\cos\theta. \tag{5.3}$$

Note the absence, as discussed in the previous section, of any 'length contraction' effect in Equation (5.3). Using (5.2) and re-arranging gives the transformation formula for $x$, $x'$-components of the velocities $\mathbf{v}$ and $\mathbf{v}'$:

$$v_x = v\cos\theta = \gamma''(v'\cos\theta' - v'') = \gamma''(v'_{x'} - v''). \tag{5.4}$$

For the $y$ - $y'$-components of the displacement:

$$s' \sin\theta' = v't' \sin\theta' = s\sin\theta = vt\sin\theta \tag{5.5}$$

so that application of (5.2) gives:

$$v_y = v\sin\theta = \gamma'' v' \sin\theta' = \gamma'' v'_{y'}. \tag{5.6}$$

Denoting the relative velocity of the moving object and O as $\mathbf{u}$ in S and $\mathbf{u}'$ in S', so that

$$\mathbf{u} = \mathbf{v} = \hat{i}v_x + \hat{j}v_y, \tag{5.7}$$

$$\mathbf{u}' = \hat{i}(v'_{x'} - v'') + \hat{j}v'_{y'}, \tag{5.8}$$

where $\hat{i}$, $\hat{j}$ are unit vectors parallel to the $(x, x')$, $(y, y')$ axes, (5.4) and (5.6) may be combined to give a simple vector equation for the transformation of relative velocities:

$$\mathbf{u} = \gamma''\mathbf{u}'. \tag{5.9}$$

Relations derived by combining Equations (5.4) and (5.6) are now compared with similar ones for transformations of velocity and angles as derived in Section 2 above using relativistic kinematics. The former set of relations are denoted by 'RSTG', for 'relativistic space-time geometry' the latter by 'RKIN' for 'relativistic kinematics'. Transformed quantities in RSTG relations are distinguished by a tilde accent, those in RKIN relations with a circumflex accent. For ease of comparison, all velocities are scaled by the factor $1/\sqrt{\kappa}$.

- velocity transformation formulas

$$\tilde{\beta}'_{x'} = \frac{\beta_x}{\gamma''} + \beta'', \qquad \tilde{\beta}_x = \gamma''(\beta'_{x'} - \beta''), \quad \text{(RSTG)} \tag{5.10}$$

$$\hat{\beta}'_{x'} = \frac{\beta_x + \beta''}{1 + \beta''\beta_x}, \qquad \hat{\beta}_x = \frac{\beta'_{x'} - \beta''}{1 - \beta''\beta'_{x'}}, \quad \text{(RKIN)} \tag{5.11}$$

$$\tilde{\beta}'_{y'} = \frac{\beta_y}{\gamma''}, \qquad \tilde{\beta}_y = \gamma''\beta'_{y'}, \quad \text{(RSTG)} \tag{5.12}$$

$$\hat{\beta}'_{y'} = \frac{\beta_y}{\gamma''(1 + \beta''\beta_x)}, \quad \hat{\beta}_y = \frac{\beta'_{y'}}{\gamma''(1 - \beta''\beta'_{x'})}. \quad \text{(RKIN)} \tag{5.13}$$

- angle transformation formulas

$$\tan\tilde{\theta}' = \frac{\beta\sin\theta}{\beta\cos\theta + \gamma''\beta''}, \qquad \tan\tilde{\theta} = \frac{\beta'\sin\theta'}{\beta'\cos\theta' - \beta''}, \quad \text{(RSTG)} \tag{5.14}$$

$$\tan\hat{\theta}' = \frac{\beta\sin\theta}{\gamma''(\beta\cos\theta + \beta'')}, \qquad \tan\hat{\theta} = \frac{\beta'\sin\theta'}{\gamma''(\beta'\cos\theta' - \beta'')}. \quad \text{(RKIN)} \tag{5.15}$$

In these formulas, the unaccented velocity symbols correspond to initial conditions in the space-time experiments described, i.e., to velocities of objects created or placed in certain reference frames (the frames S or S' in

Equations (5.10)-(5.15)). All such velocities therefore respect the condition: $-1 \le \beta \le 1$ imposed by Equation (2.9) above. The same condition is seen to be respected by the transformed velocities $\hat{\beta}$ in the RKIN formulas. Since, however, the velocities $\tilde{\beta}$ in the RSTG formulas are, by definition *relative velocities of different objects in the same reference frame* they are not subject to the constraint imposed by Equation (2.9). It fact it is found that:

$$-\sqrt{2} \le \tilde{\beta}'_{x'} \le \sqrt{2}, \qquad -\infty \le \tilde{\beta}_x \le \infty, \tag{5.16}$$

$$0 \le \tilde{\beta}'_{y'} \le 1, \qquad 0 \le \tilde{\beta}_y \le \infty. \tag{5.17}$$

The limits on $\tilde{\beta}'_{x'}$ are found by differentiating the first formula in (5.10) w.r.t. $\beta''$, to find the maximum value of the function of $\beta''$ on the right side of this equation. It is found to occur when $\beta_x = 1$ and $\beta'' = 1/\sqrt{2}$. Note that with the coordinate axes shown in Figure 2, $\beta_y$ and $\beta'_{y'}$ are positive quantities.

Neglecting terms of $O(\beta^2)$ and higher in (5.10)-(5.15) (that is in Galilean relativity) gives identical space-time-geometrical and kinematical transformation formulas. Unlike the RKIN formulas the RSTG formulas and their inverses are not form-invariant. The RSTG formula (5.10) does not respect the 'Reciprocity Principle' [9] that holds in Galilean relativity. Setting $\beta'_{x'} = 0$ in (5.11) gives $\hat{\beta}_x = -\beta''$, i.e., the origin of S' moves with velocity $-\beta''$ in the $x$-direction when (see Figure 2) the origin of S moves with velocity $\beta''$ in the $x'$-direction. Setting $\beta'_{x'} = 0$ in the RSTG formula, (5.10) gives instead $\tilde{\beta}_x = -\gamma''\beta''$. This violates the Reciprocity Principle that states that: 'If the velocity of B as observed by A is **u** then the velocity of A as observed by B is **−u**'.

As demonstrated in the appendix, the RSTG formulas, unlike the RKIN ones, do not constitute a group.

The RSTG transformation formulas of Equation (5.10) for the case $\theta = \theta' = 0$ have previously been derived in the context of the analysis of a circular Sagnac interferometer by Post [27] and Klauber [28]. Setting $\theta = \theta' = 0, \beta'_{x'} = 1$ in the RKIN formula (5.11) gives $\beta_x = 1$, so predicting in accordance with the tacit hypothesis made in Einstein's original 1905 paper on special relativity [5] that: 'the speed of light is the same in all inertial frames'. The incompatiblity of this hypothesis with the interference effect observed by Sagnac [29], that is correctly predicted by the RSTG formulas in (5.10) (or by their $O(\beta)$ approximations) was already pointed out in 1937 by Dufour and Prunier [30], as well as, more recently, by Selleri [31], Klauber [28] and the present author [32].

According to the argument given at the end of Section 3 of the present paper, all particles with rest energy much smaller than their actual energy have a fixed speed *c* in any inertial frame - apparently therefore confirming the correctness of Einstein's second postulate of special relativity. How now is the apparent paradox that the latter is inconsistent with the results of Sagnac's experiment to be resolved? The answer lies in careful consideration of the exact operational definitions of the various symbols appearing in the transformation Equations (5.10)-(5.15). Since the unaccented velocities are all those of ponderable or massless objects either created or placed in one of the inertial frames S or S', they are subject to the restriction of Equation (2.9): $-1 \le \beta \le 1$. However, in the RSTG equations the velocities $\tilde{\beta}$ and $\tilde{\beta}'$ are, in all cases, *relative velocities of two distinct objects in the same inertial frame* as compared to the unaccented velocities which are instead 'frame velocities', i.e., *the velocity of a single object in an inertial frame as fixed by the initial conditions of the space-time experiment under consideration*. For example, in the second equation in (5.10), $\beta'_{x'} - \beta''$ is the $x'$ component of the relative velocity of the origin O of the frame S and the object considered in Figure 2, in the frame S'. The same relative velocity, as observed in the frame S is $\tilde{\beta}_x$. The RSTG equations do relate observations of relative velocities of the same two objects in different frames, in the same experiment. In contrast, as is clear from the considerations of Section 3 above, the RKIN equations instead relate different kinematical configurations $(E, \mathbf{p})$ and $(E', \mathbf{p}')$, in the same frame, of a single object, via the boost parameters $(\theta, \beta'')$. The application of Newton's second law in the derivation of Equation (3.1) shows that the two configurations are both specified in the same reference frame: the frame in which this law is applied in Equation (2.5). The situation is analogous to the active rotation of a spatial vector $\mathbf{r}$ into another vector $\mathbf{r}'$ of the same length, by rotation through an angle $\theta$. The invariant of the transformation is:

$$\mathbf{r} \cdot \mathbf{r} = \mathbf{r}' \cdot \mathbf{r}'$$

to be compared with the invariant relation (3.1) for relativistic energy and momentum.

The experimental correctness of the RSTG equations and the inapplicability of the RKIN ones, for the case of photon propagation in the vicinity of the Earth is vouchsafed not only by Sagnac interferometers but also by GPS operation [33] and time transfer by microwave signals via a satellite in low-Earth orbit [34, 35, 36]. It is a prediction of the Schwarzschild metric equation of General Relativity [37, 38] that the ECI (Earth Centered Inertial) frame[5] is a preferred one for the propagation of light at speeds close to $c$ in the vicinity of the Earth. The velocity of light signals in the frame of a receiver fixed on the surface of the Earth, and

[5]This is a frame co-moving with the centroid of the Earth with axes pointing in fixed directions relative to the Celestial Sphere.

therefore rotating in the ECI frame, is given by the RSTG not the RKIN velocity transformations. Indeed, in GPS operation the Galilean (order $v/c$) approximation to the RSTG formulas is used [33]. The possibility to further test the RSTG predictions in long-baseline neutrino beams has been discussed in a recently published paper [39].

To give a concrete example of the very different predictions for space-time effects of the RSTG and RKIN velocity transformations consider the case of two protons colliding, head on, in the LHC collider at CERN. Suppose that they are each, initially, at a distance of 5m from their collision point. Each one is assumed to have an energy, as during the experimental program of 2012, of 4 TeV, corresponding to a velocity in the laboratory such that $v/c = 1 - 5.5\times 10^{-8}$. The *relative* velocity of the two protons in the laboratory frame S is then $2c(1 - 5.5\times 10^{-8})$. According to Equation (5.10), the velocity of the left-moving proton in the rest frame, S', of the right-moving one is:

$$\gamma(v)2c(1 - 5.5\times 10^{-8}) \cong 8528\,c$$

and since length intervals are invariant, the initial spatial separation of the protons, in both S and S', is 10m. In the laboratory frame, the time interval $\Delta t = 5m\,/\,[c(1 - 5.5\times 10^{-8})] \cong 16\,\mathrm{ns}$ elapses until the protons collide. The corresponding time interval in the frame S', according to the TD relation is $\Delta t' = \Delta t\,/\,\gamma(v) = 3.75\,\mathrm{ps}$, in agreement with the relative velocity and spatial separation in this frame just quoted. Using instead the RKIN formula (5.11) to calculate the relative velocity of the two protons in the frame S' gives $c(1 - 1.35\times 10^{-15})$. According to the TD relation this corresponds to an initial separation $\cong c\Delta t' = 1.13\,\mathrm{mm}$ of the colliding protons in the frame S', instead of 10 m. Finally it may be noted that, if a conceptual 'ruler' of length $L'$ in the frame S', is used to measure the initial separation of the protons in this frame, then according to the length contraction effect of conventional SRT, the initial separation of the protons in the frame S which is, by hypothesis, 10 m, must be equal to $L'/\gamma(v)$ so that $L' = 4264 \times 10\,\mathrm{m} = 42.6\,\mathrm{km}$, to be compared with $L' = 1.13\,\mathrm{mm}$ as required by TD and the RKIN velocity transformation formula! So consistent predictions for observations in the frames S and S' are obtained by application of the RSTG velocity transformations, whereas contradictory predictions follow from application of the RKIN transformation formulas, time dilation and the conventional length contraction effect.

**Appendix**

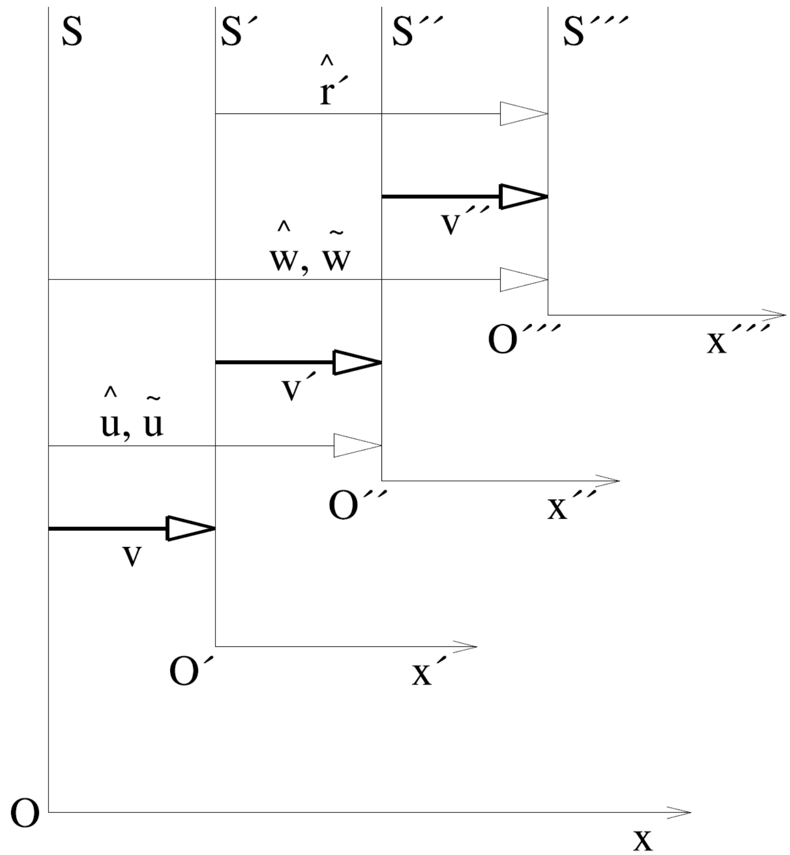


**Figure 3.** Inertial frames $\mathrm{S}', \mathrm{S}'', \mathrm{S}'''$ move parallel to the $x$ axis of the frame S. The velocity of S′ relative to S in S, $v$, of $\mathrm{S}''$ relative to S′ in $\mathrm{S}', v'$, and of $\mathrm{S}'''$ relative to $\mathrm{S}''$ in $\mathrm{S}'', v''$ (indicated by bold arrows) are fixed initial conditions. Application of the RSTG transformation formulas gives velocities $\widetilde{u}, \widetilde{w}$ in S, while application of the RKIN formulas gives velocities $\hat{u}, \hat{w}$ in S and $\hat{r}'$ in S′. See text for discussion.

In order to discuss the group properties and physical interpretation of the RSTG and RKIN velocity transformation formulas in Equation (5.10) and (5.11) it is convenient to set $\theta = \theta' = 0$ (velocities parallel to the boost direction) and consider a nested series of inertial frames: $S, S', S'', S'''$ as shown in Figure 3. The relative velocities $v$, $v'$ and, $v''$ of $SS'', S'S''$ and $S''S'''$, respectively, are fixed initial conditions and the velocities $\tilde{u}$ and $\hat{u}$ of $S''$ relative to S in the frame S or $\tilde{w}$ and $\hat{w}$ of $S'''$ relative to S, are calculated according to the first formula in (5.10) (for $\tilde{u}$ and $\tilde{w}$) or the first formula in (5.11) (for $\hat{u}$ and $\hat{w}$). This gives, in an evident notation:

$$\hat{\beta}_u = \frac{\beta_v + \beta_{v'}}{1 + \beta_v \beta_{v'}}, \quad \hat{\beta}_w = \frac{\hat{\beta}_u + \beta_{v''}}{1 + \hat{\beta}_u \beta_{v''}}. \quad \text{(RKIN)} \tag{A.1}$$

Substituting $\hat{\beta}_u$ from the first of these equations in the second one it is found that

$$\hat{\beta}_w = \frac{\beta_v + \hat{\beta}_{r'}}{1 + \beta_v \hat{\beta}_{r'}}, \tag{A.2}$$

where

$$\hat{\beta}_{r'} = \frac{\beta_{v'} + \beta_{v''}}{1 + \beta_{v'} \beta_{v''}}. \tag{A.3}$$

Comparing (A.1)-(A.3) shows that successive applications of the transformation relations (substituting $\hat{\beta}_u$ in the formula for $\hat{\beta}_w$ in (A.1)) gives the form-invariant equation for $\hat{\beta}_w$, again in terms of velocities in S $(\beta_v)$ and S′ $(\hat{\beta}_{r'})$. The RKIN transformations are therefore form-invariant. The transformations in (A.1) can be formally written in a more symmetrical manner as [10]:

$$\beta_A - \beta_B + \beta_C - \beta_A \beta_B \beta_C = 0 \tag{A.4}$$

equivalent to the three transformations:

$$\beta_A = \frac{\beta_B - \beta_C}{1 - \beta_B\beta_C}, \quad \beta_B = \frac{\beta_C + \beta_A}{1 + \beta_C\beta_A}, \quad \beta_C = \frac{\beta_B - \beta_A}{1 - \beta_B\beta_A}. \qquad \text{(A-5)}$$

Setting $\beta_C = 0$ gives the identity operator $\beta_A = \beta_B$ while substituting the equation for $\beta_B$ in that for $\beta_A$ recovers $\beta_A$ independently of the value of $\beta_C$. The second transformation in (A.5) is therefore the inverse of the first. The transformations in (A.4) or (A.5) are therefore form invariant and contain both identity and inverse operations, so that they constitute a group.

Applying the initial conditions specified above to the first RSTG transformation in (5.10) gives:

$$\tilde{\beta}_u = \frac{\beta_{v'}}{\gamma(v)} + \beta_v, \quad \tilde{\beta}_w = \frac{\beta_{v''}}{\gamma''} + \tilde{\beta}_u. \quad \text{(RSTG)} \qquad \text{(A.6)}$$

The operational definition of $\tilde{\beta}_u$ is the velocity of the frame $S''$ observed in S. The quantity $\beta_{v''} / \gamma''$ is the velocity of the frame $S'''$ relative to $S''$ as observed in the frame S. The velocity, $\beta_{v''}$ of $S'''$ in $S''$, as fixed by the initial conditions, is scaled by the TD factor $\gamma''$ between the frames $S''$ and S: $\Delta t = \gamma''\Delta t''$.[6] The TD factor $\gamma''$ is given, in terms of the fixed velocities $v$ and $v'$ by the transformation equation:

$$V_0'' \equiv \gamma'' = \gamma(v)\,[\gamma(v') + \beta_v\beta_{v'}\gamma(v')]. \qquad \text{(A.7)}$$

As can be seen in Figure 3, this is the Lorentz transformation of the temporal component of the dimensionless 4-vector velocity of the frame $S''$: $(V_0'; V') \equiv (\gamma(v'); \beta_{v'}\gamma(v'))$, between the frames S' and S. The transformation formula (A.7) follows from that of relativistic energy, (3.10), on dividing through by the factor $\kappa m$. Combining (A.6) and (A.7)

[6]Note the different meaning of the symbol $\gamma''$ here as compared to that in Equation (5.2).

the observed velocity, $\tilde{\beta}_w$ of the frame $S'''$, in the frame S, in terms of the initial velocities $v$, $v'$ and $v''$ is:

$$\tilde{\beta}_w = \frac{\beta_{v''}}{\gamma''} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v$$

$$= \frac{\beta_{v''}}{\gamma(v)\gamma(v')[1+\beta_v\beta_{v'}]} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v. \quad \text{(A.8)}$$

Comparison of this formula with that for $\tilde{\beta}_u$ in (A.6) shows that RSTG velocity transformations are not form-invariant. Furthermore, applying first the transformation giving $\tilde{\beta}_u$ in (A.6), and then the same transformation with the replacement: $v \to -v$, gives

$$\frac{\beta_{v'}}{\gamma(v)^2} + \beta_v\left(\frac{1}{\gamma(v)} - 1\right) \neq \beta_{v'} \quad \text{(A.9)}$$

which, unlike the same sequence of operations for the RKIN transformation giving $\hat{\beta}_u$ in (A.1), does not yield the inverse transformation. Since the left side of (A.9) does reduce to $\beta_{v'}$ in the Galilean limit $\gamma(v) \to 1$, the breakdown of the group property evident in (A.8) and (A.9) is a purely relativistic, $O(\beta^2)$, effect.

Defining:

$$\beta'' \equiv \frac{\sqrt{(\gamma'')^2 - 1}}{\gamma''} \quad \text{(A.10)}$$

and using (A.7) to eliminate $\gamma''$ on the right side of this equation it is found that

$$\beta'' = \frac{\beta_v + \beta_{v'}}{1 + \beta_v\beta_{v'}} = \hat{\beta}_u \quad \text{(A.11)}$$

which enables (A.8) to be written as:

$$\widetilde{\beta}_w = \frac{\beta_{v''}}{\gamma(\hat{u})} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v. \tag{A.12}$$

It is then seen that the RKIN velocity transformation formula in (A.1) enables calculation of the TD factor $\gamma''$ that appears in the RSTG transformation which correctly describes velocities observed in different frames in the same space-time experiment. This is a correct space-time physics (not kinematical) application of the RKIN velocity transformation.

Dividing the relativistic momentum transformation formula through by the factor $m\sqrt{\kappa}$ yields the Lorentz transformation of the spatial component of the 4-vector velocity $(V_0';\ V')$:

$$V'' \equiv \gamma''\beta'' = \gamma(v)\,[\gamma(v')\beta_{v'} + \beta_v\gamma(v')]. \tag{A.13}$$

The ratio of Equation (A.13) to (A.7) recovers Equation (A.11) above. In fact Equation (A.7), (A.11) and (A.13) are algebraically equivalent: if any one of the equations is postulated then the remaining two may be derived by purely algebraical manipulation.

It is instructive to compare the Lorentz transformation equations for the 4-vector velocity $(V_0';\ V')$, (A.7) and (A.13):

$$V_0'' = \gamma(v)\,[V_0' + \beta_v V'], \tag{A.14}$$

$$V'' = \gamma(v)\,[V' + \beta_v V_0'] \tag{A.15}$$

with the inverse of the conventional space-time LT in (4.23) and (4.24):

$$x_0 = \gamma(v)\,[x_0' + \beta_v x'], \tag{A.16}$$

$$x = \gamma(v)\,[x' + \beta_v x_0'], \tag{A.17}$$

where $x_0 \equiv \sqrt{\kappa}t$ and $x'_0 \equiv \sqrt{\kappa}t'$. Although the right sides of (A.14) and (A.16) or (A.15) and (A.17) are formally identical, the physical meanings of the equations are completely different. Quantities in three inertial frames, S, S′ and S″ are connected by Equation (A.14) and (A.15), quantities in only two, S and S′ by (A.16) and (A.17). Since $x'$ is, by definition, time-independent, then, as explained in Section 4 above, it necessarily vanishes. In contrast in (A.14) and (A.15) the vanishing of $V'$ requires $v' = 0$ so that the frames S′ and S″ are identical. The space-time transformation Equations (A.16) and (A.17), for $x' = 0$, are equivalent to the equations:

$$x_0 = \gamma(v)x'_0, \qquad \text{(TD)} \tag{A.18}$$

$$x' = 0, \qquad \text{(Worldline in S}'\text{)} \tag{A.19}$$

$$x = \gamma(v)\beta_v x'_0 = \beta_v x_0 \qquad \text{(Worldline in S)}, \tag{A.20}$$

whereas setting $V' = 0$ in (A.16) and (A.17) corresponds to:

$$V''_0 \to V'_0 = \gamma(v), \quad V'' \to V' = \gamma(v)\beta_v.$$

The choice $x' = 0$ on (A.16) and (A.17) corresponds only to a particular choice of co-ordinate system and leaves the physical content of the equations unchanged. On the other hand, setting $V' = 0$ in (A.14) and (A.15) changes the physical meaning of the equations - it actually destroys the transformation information relating quantities in the frames S, S' and S" that they contain. It can be seen here that equations with an identical mathematical structure can have a widely disparate physical significance.

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