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EXPLICITLY PUT MACH'S PRINCIPLE IN GENERAL RELATIVITY

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Abstract

General relativity (GR) relies on torsion-free Riemannian geometry, which minimizes the rotational aspect of Mach's principle. The Einstein-Cartan theory (ECT) aimed to overcome this difficulty. Unfortunately, using the spin field as the source of the torsion field, ECT did not fulfill its goal. We propose using the vorticity field to replace the spin field as the source of the torsion field and a wave equation to link the two. The improved ECT has certain astronomical implications that may shed light on the dark matter.

1. Introduction

Historically, two thought experiments have played important roles in shaping human understanding of space-time. The first was performed by Galileo Galilei [1, 2]. In this experiment, two objects of unequal mass must fall at the same speed in a uniform gravity field. This eventually led

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to the equivalence principle [1, 2]. The second was performed by Isaac Newton [3, 4]. In this experiment, a bucket of water will have a flat surface if it stands still, but the surface will curve when it rotates around its axis. From this experiment, Newton conceived of both motion and space-time as absolute [3, 4]. Later, when Ernst Mach considered the same experiment, he drew a different conclusion, which eventually became Mach's principle. In Mach's principle, any motion, including rotation, is relative; if the bucket of water is the only object in the universe, then its surface must be flat, and even if it stands still, when all the surrounding objects, including the distant stars, rotate around it, the water surface will curve [5].

In Mach's thinking, he implied a global rotation in which all surrounding objects were rotating at the same angular speed. This is apparently incorrect because otherwise, the distant stars will rotate faster than the speed of light. Any rotation should be a local motion. The largest rotating objects that have been observed were the spiral galaxies and galaxy clusters. Even for these objects, the angular speed varies from place to place. If we replace the global rotation with a locally rotating matter field, Mach's principle would be both logically sound and practical.

Both the equivalence principle and Mach's principle were guideposts to Einstein's thoughts on GR [6, 10]. His goal was to include both principles in the mathematical formulation of GR [6]. Unfortunately, the only geometry he could find was Riemannian geometry. In this geometry, Riemann specifically picked the torsion-free Levi-Civita connection whose connection coefficients (Christoffel symbols) are symmetric so that the torsion tensor is always zero [7]. The consequence of this choice is that space-time can bend but not rotate. Therefore, the rotational aspect of Mach's principle was not fully incorporated into GR.

According to GR, in the vicinity of any point in space-time, an inertial reference frame can always be found by a general coordinate transformation. In non-technical terms, this implies that inside a free-falling spacecraft in a gravity field, the lab frame is a good inertial frame if the spacecraft is sufficiently small [8]. However, this cannot be true if the space rotates. For example, if the spacecraft is at a vertex of the rotating space, then regardless of how small the spacecraft is, the centrifugal force inside the spacecraft will always be felt. According to Mach's principle, even if you stand still in space, your arms will be lifted when all the surrounding objects rotate around you [5]. In this situation, the equivalence principle cannot help because it only addresses the translational motion, whereas Mach's principle addresses the rotational motion. Translation and rotation belong to two different transformation groups [9].

One may argue that a rotating space is merely a pure imagination, and there is no physical meaning to it. However, we can find a live example of a rotating space in our solar system. According to GR, the space-time around the earth is curved owing to the mass of the earth; in this locally curved space-time, the moon follows the geodesic orbit around the earth. For the same reason, the earth orbits the sun and drags its locally curved space to rotate around the sun. This rotating curved space drags the moon to move with it. In this example, we not only see a rotating space but also its impact on the objects trapped in it.

Because Mach's principle is only vaguely defined, it has many interpretations. Two important aspects are frequently considered. The first is the dependence of the inertia of the body on the proximity of the surrounding masses. In this aspect, Mach's principle is vindicated by GR because the curvature of space-time is determined by all matter fields in the universe [10, 11]. The second aspect is the distortion of space-time by the dynamical movement of the matter field, which is mostly manifested by rotation. This aspect has been called the frame-dragging effect or Lense-Thirring effect in GR. It was first studied by Einstein in 1913 on the fact that the acceleration of a rotating mass shell induces an acceleration of a body it encloses and the creation of a Coriolis field by the rotating mass shell [10, 12, 13]. The distortion of space-time by a spinning mass was further studied by Josef Lense and Hans Thirring in 1917. They derived a metric with a rotation term that is proportional to the angular momentum of the rotating body [12, 13]. The Lense-Thirring effect is extremely small and drops much faster than gravity; it becomes important only in the vicinity of a massive rotating object such as the Kerr black hole [14].

The minuscule of the frame-dragging effect is because GR is based on the torsion-free Riemannian geometry. Therefore, the second aspect of Mach's principle is only partially vindicated.

To overcome the above problem, Elie Cartan in 1922 proposed the Riemann-Cartan geometry, which used the affine connection instead of the Levi-Civita connection. In the new geometry, the connection coefficients are not symmetric. The asymmetric parts are factored into an antisymmetric torsion tensor [15]. In 1928, Einstein was very interested in the new geometry and collaborated to reformulate GR in the Riemann-Cartan geometry. This new theory was coined the Einstein-Cartan theory [16, 17, 18]. In the new theory, space-time can not only curve but also rotate. The bending character of space-time is described by the Ricci tensor, which is a contracted form of the Riemann curvature tensor. The rotational character was captured using the torsion tensor [16, 17].

In Einstein-Cartan theory, Cartan chose the spin field as the source of the torsion field [15]. This choice had two consequences. First, the spin field is weak under most circumstances. For example, in classical fluid mechanics, the spin field that describes the smallest whirls is almost negligible unless large whirls break into smaller whirls in turbulence [19, 20]. This turbulence is negligible in the galactic disk of a spiral galaxy. Therefore, in a spiral galaxy, there is a strong rotation in the disk, but the spin field in the disk is very weak. The second consequence is that the field equation that links the torsion tensor and the spin tensor is an algebraic equation. This implies that the torsion field cannot propagate out of the matter field [17]. This is clearly not what Mach's principle states. In the bucket of water thought experiment, the rotating neighbors could be far away from the bucket. Owing to these limitations, the Einstein-Cartan theory failed to materialize. The rotational aspect of Mach's principle is not fully factored into GR.

In fluid mechanics, vorticity is a much better measurement of fluid rotation; it accurately measures both large whirls and small whirls [19, 20]. If we use the vorticity field to replace the spin field in Einstein-Cartan theory by applying the curl operator to the stress-energy-momentum tensor, then we will overcome the two shortcomings in Einstein-Cartan theory because vorticity is not weak in a rotating field, and the torsion-vorticity equation becomes a second-order differential field equation. Such an improvement would bring the full glory of Mach's principle to GR.

2. Equations

In GR, there are two field equations:

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} P_{ab}, \tag{1}$$

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} = 0. \tag{2}$$

Equation (1) is the well-known Einstein field equation in which R_{ab} is the Ricci tensor, R is the Ricci scalar, g_{ab} is the metric tensor, and P_{ab} is the stress-energy-momentum tensor [8].

Equation (2) is the torsion equation in which T^c_{ab} is the torsion tensor and Γ^c_{ab} is the Christoffel symbol or the connection coefficient. From

Equation (2), the Christoffel symbol must be symmetric. This is the definition of the Levi-Civita connection, which is a requirement of Riemannian geometry [7].

When people talk about GR, they normally refer only to Equation (1). However, without Equation (2), the Ricci tensor on the left-hand side of Equation (1) is not symmetric because it also depends on the first-order derivative of the antisymmetric torsion tensor. Therefore, Equation (1) cannot be solved without Equation (2).

The left-hand side of Equation (1) only measures how space-time is curved. For example, the Ricci tensor only measures how volume changes along a bundle of geodesics [7]. This only reflects the equivalence principle but not Mach's principle on rotation. People may argue that the curvature of space-time is affected by all matter fields in the universe, so Equation (1) is the manifestation of Mach's principle. However, we cannot find any rotational signature of Mach's principle from this equation.

In Einstein-Cartan theory, Equation (2) is replaced by Equation (3):

$$T_{ab}^{c} + g_{a}^{c}T_{bd}^{d} - g_{b}^{c}T_{ad}^{d} = \frac{8\pi G}{c^{4}}\sigma_{ab}^{c}. \tag{3}$$

In Equation (3), σ_{ab}^{c} is the spin tensor. From this equation, we can calculate the torsion tensor T_{ab}^{c} which is nonzero [15]. This also fixes the antisymmetric part of the Ricci tensor in ECT.

As Equation (3) is an algebraic equation instead of a differential equation, the torsion field cannot propagate. They can only remain inside the matter field [17]. Therefore, Equation (3) is not a good reflection of Mach's principle on rotation. The spin field is generally too weak to represent the rotational motion of the matter field.

In fluid mechanics, vorticity is the correct term for describing fluid rotation. This describes whirls of any size. It is defined as $V = \nabla \times u$, where u is the velocity field [19, 20]. In GR, we can redefine the covariant vorticity tensor as follows:

$$V_{ab}^{c} = \nabla_a \times P_b^c = D_a P_b^c - D_b P_a^c. \tag{4}$$

In Equation (4), $P_b^c = g^{cd} P_{db}$ is the stress-energy-momentum tensor with one contravariant index and one covariant index and D_a is the covariant derivative [7]. The vorticity tensor is antisymmetric in the two lower indices; therefore, it has the same signature as that of the torsion tensor. This is the first-order derivative of the stress-energy-momentum tensor.

There are two methods to determine the torsion equation that link the torsion tensor to the vorticity tensor. The first method is to find the correct Lagrangian and then apply the least action principle to derive the field equation. The second method uses Einstein's principle of general covariance, which states that all laws of physics are to be expressed in covariant form using tensors. Any physical equation must have the same form under a general coordinate transformation.

The second method is more heuristic and requires more physical insight. The first method is regarded as mathematically rigorous, but its first step in finding the right Lagrangian is heuristic and requires physical insight. In GR textbooks, both methods are used to derive the Einstein field equation (Equation (1)) [8].

We used the second method to determine the torsion equation. Starting from Equation (1) on the left-hand side, the Einstein tensor is in the first-order derivative of the Christoffel symbol; it matches the stress-energy-momentum tensor on the right-hand side. If we were to replace the stress-energy-momentum tensor with the vorticity tensor on the right-hand side, the left-hand side must be replaced with something in the second-order derivative of the Christoffel symbol because the vorticity tensor is in the first-order derivative of the stress-energy-momentum tensor. The left-hand side must be antisymmetric to match the antisymmetric property of the vorticity tensor on the right-hand side. The only antisymmetric tensor that originates from the Christoffel symbol is the torsion tensor. The only second-order derivative of the torsion tensor that conforms to Einstein's principle of general covariance is the left-hand side of Equation (5).

$$\nabla^2 T^c_{ab} = \frac{8\pi G}{c^4} V^c_{ab}. \tag{5}$$

In Equation (5), $\nabla^2 = D^{\mu} D_{\mu}$ is the 4-Laplacian operator in the covariance form, D_{μ} is the covariant derivative, T^c_{ab} is the torsion tensor, and V^c_{ab} is the vorticity tensor given by Equation (4). It can be easily verified that Equation (5) maintains the same form under the general coordinate transform.

By no means are the above arguments the derivation of Equation (5), but they are a good reading of Einstein's principle of general covariance and Equation (1). Equation (5) is a wave equation; therefore, the torsion of space-time can propagate out of the matter field. This is exactly what Mach's principle requires.

Equations (1) and (5) form the complete set of field equations of GR in the Riemann-Cartan geometry. This pair of equations describes how space-time is curved and rotates under the influence of the matter field; thus, they are the correct reflection of both the equivalence principle and Mach's principle, as Einstein had envisioned before publication of his GR in 1915 [21, 22, 23].

It may be argued that the derivation of Equation (5) lacks mathematical rigor. This is somewhat true. However, no torsion equation can be rigorously derived from any existing theory through deduction. For example, Equation (2) was hand-picked by Riemann without a reason [7].

Although Cartan derived Equation (3) by using the first method, the way he chose the "right" Lagrangian was heuristic. The only guidelines for choosing the correct Lagrangian are symmetry, simplicity, and good guess [24]. This is true for the derivation of many well-known physics equations: (1) Einstein's field equation in GR by Hilbert [8], (2) Dirac's field equation in relativistic quantum mechanics [25], and (3) Yang-Mills theory in quantum field theory [26].

If we were to use the first method to derive the torsion equation by choosing the right Lagrangian, we would simply shift the burden of justifying Equation (5) to the justification of the specific Lagrangian we choose. The best way to justify Equation (5) is to use it in astronomical applications and to see if the results agree with observations.

If we examine Equations (3), (1) and (5) in terms of connection coefficients, they are in the zeroth-order, first-order and second-order derivatives of the Christoffel symbol, respectively. Equation (1) is symmetric (if we decouple curvature and torsion). Equations (3) and (5) are antisymmetric. They all conform to Einstein's principle of general covariance.

Although it is difficult to mathematically prove that they are the only forms when imposing the above symmetries and derivative orders, their choices are very limited. These equations describe the dynamical properties (curvature and torsion) of space-time.

Mach's principle has been in use for a long time. Its relativistic relational concept can be traced back to the Leibnitz era. Unlike the equivalence principle, there is no mathematical equation to describe it. Cartan and Einstein attempted to develop an equation to describe it, but their efforts fell short. We hope that Equation (5) can fill in the missing pieces.

The particle still follows the geodesic equation in curved and rotating space-time,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \tag{6}$$

However, in this equation, the Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ is no longer symmetrical. The antisymmetric part is described by the torsion tensor $T^\mu_{\alpha\beta}$, which can be solved using Equation (5). The symmetric part corresponds to the usual gravity. The particle's motion is influenced by both gravity and torsion.

3. Astronomical Implications

GR has been one of the most successful theories of the last century. However, it also leads to two dark mysteries (dark matter and dark energy) and two singularities (black hole singularity and big bang singularity) in cosmology. As American cosmologist Janna Levin put it, "Nature abhors an infinity. The limits of general relativity are often signaled by infinities: infinite curvature as in the center of a black hole, the infinite energy of the singular big bang. We might be inclined to add an infinite universe to the list of intolerable infinities" [27]. As I tried to prove in my previous publications that microscopic infinity is most likely due to some poor postulations of existing theories [28, 29], the same can be true for macroscopic infinity.

GR is incomplete. As discussed earlier, this was realized by both Cartan and Einstein in the 1920s, and they worked very hard to complete it by deriving a torsion equation of space-time. In fact, the aforementioned four cosmological problems can be traced back to the failure to derive the correct torsion equation, and Mach's principle is not fully incorporated into GR. This is a big statement that requires extensive mathematical proof and astronomical evidence. For example, to demystify dark matter, we must solve at least three problems without dark matter: (1) the flat rotation curve of the spiral galaxy, (2) galaxy formation, and

(3) the flat universe, which is revealed by the CMB data [27]. For the third problem, eliminating dark matter is insufficient, and dark energy must also be removed. To do so, we must also explain the accelerating expansion of the universe [30, 31] without dark energy. It is impossible to cover all of these topics in a single paper. In the rest of this section, we solve three outstanding problems and defer the rest of the problems in subsequent studies.

3.1. Flat rotation curve of spiral galaxies

In astronomy, when the rotation curve of a spiral galaxy is observed, the curve is very flat at a large distance from the galactic center. This contradicts the theoretical prediction when considering all known matters in the galaxy. At such a high speed, the gravity force of all existing matter cannot hold the stars, so they will fly away from the galaxy [32, 33]. There are two camps on this problem. In the first camp, people proposed the dark matter solution: there must be some unknown matter in the halo of the galaxy to hold the stars together [33]. In the second camp, researchers proposed Modified Newtonian Dynamics (MOND) [34, 35]. In MOND, there exists an empirical critical acceleration a_0, and the Newtonian force is $F_N = m\mu\left(\frac{a}{a_0}\right)a.$ Function $\mu(x)$ has the following properties:

$$\mu(x) \to 1, \;\; x >> 1,$$

$$\mu(x) \to x, \;\; x << 1.$$

According to MOND, at small distances when gravity is strong, gravity follows the Newtonian inverse square law; at large distances when gravity is weak, it follows the inverse law. Each solution has both advantages and disadvantages [36].

According to the new theory proposed in this study, stars are influenced by both gravity and torsion. In Equation (6), gravity originates from the symmetric part of the Christoffel symbol, and torsion originates from the antisymmetric part of the symbol. Without torsion, gravity alone is not strong enough to maintain the high velocity of the stars. The torsion of space-time is caused by the rotation disk of the spiral galaxy. After considering torsion, the dark matter may no longer be needed.

Let us consider a simple galactic disk with cylindrical symmetry, where every star in the disk rotates in a circular orbit. We describe the disk in polar coordinates (r, θ, z). The thickness of the disk is $-\frac{L}{2} < z < \frac{L}{2}$.

For a static field, Equation (5) becomes the familiar Poisson equation with three spatial components and is very similar to the following vector potential equation in electromagnetism:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}. \tag{7}$$

Therefore, we can borrow many tricks from electromagnetism to solve Equation (5). For example, when viewed from a distance, we can ignore the details of the current field and assume that all currents flow through a thin wire.

It is unsurprising that there is a resemblance between torsion and magnetism. Torsion to mass is similar to a magnetic field to charge. The rotating charge produces a magnetic field, which causes the charge to rotate. Similarly, the rotating mass produces torsion in space-time, and torsion influences the rotation of the mass.

Now, we solve the simple galactic disk problem. In a cylinder between $[r, r + dr]$, the vorticity is:

$$dV = \frac{\oint (\rho u^2 L 2\pi r dr) dl}{\pi r_0^2} = \frac{4\pi\rho u^2 L r^2 dr}{r_0^2}. \tag{8}$$

In Equation (8), ρ is the mass density, u is the velocity (letter V is saved for vorticity), and r_0 is a small radius near the galactic center. Here, we take one trick from electromagnetism to pretend that all vorticities flow through a thin wire with radius r_0.

Let us assume the following simple empirical laws within the disk based on observations [32, 37]:

$$u = u_0\left(\frac{r}{r_0}\right)^{\alpha}, \tag{9}$$

$$\rho = \rho_0 e^{-\beta\frac{r}{r_0}}. \tag{10}$$

In Equations (9) and (10), $\alpha > 0$, $\beta > 0$. The vorticity within radius r is:

$$V(r) = \int_{r_0}^{r} dV = \int_{r_0}^{r} \frac{4\pi\rho u^2 Lr^2 dr}{r_0^2} = \int_{r_0}^{r} 4\pi\rho_0 u_0^2 Le^{-\beta\frac{r}{r_0}}\left(\frac{r}{r_0}\right)^{2+2\alpha} dr$$

$$= 4\pi\rho_0 u_0^2 Lr_0 \beta^{-2\alpha-3}\left[\Gamma(2\alpha+3,\ \beta) - \Gamma\left(2\alpha+3,\ \beta\frac{r}{r_0}\right)\right]. \tag{11}$$

In Equation (11), $\Gamma(s,\ x)$ is the upper incomplete gamma function. The vorticity per unit thickness is:

$$I(r) = \frac{V}{L} = 4\pi\rho_0 u_0^2 r_0 \beta^{-2\alpha-3}\left[\Gamma(2\alpha+3,\ \beta) - \Gamma\left(2\alpha+3,\ \beta\frac{r}{r_0}\right)\right]. \tag{12}$$

Let R be the radius of the galactic disk. According to Equation (5), at $r > R$, the torsion from the disk is:

$$T(r) = -\frac{8\pi G}{c^4}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{I(R)dz}{\sqrt{r^2+z^2}} = -\frac{16\pi GI}{c^4}\int_0^{\frac{L}{2r}} \frac{dx}{\sqrt{1+x^2}}$$

$$= -\frac{16\pi GI}{c^4}\ln\left(\sqrt{1+\left(\frac{L}{2r}\right)^2} + \frac{L}{2r}\right) \cong -\frac{8\pi GI}{c^4}\frac{L}{r}. \tag{13}$$

Here, we take another trick from electromagnetism to pretend that the torsion produced by the disk comes from the vorticity in a thin wire through the galactic center. Specifically, we make the following mathematical analogies: T (torsion) to $\boldsymbol{A}$ (magnetic vector potential) and V (vorticity) to I (electrical current). This approach stems from the similarity between Equation (7) and the static version of Equation (5).

Combining Equations (12) and (13):

$$T(r) = -\frac{8\pi GL}{c^4 r} 4\pi\rho_0 u_0^2 r_0 \beta^{-2\alpha-3}\left[\Gamma(2\alpha+3,\ \beta) - \Gamma\left(2\alpha+3,\ \beta\frac{R}{r_0}\right)\right]$$

$$\cong -\frac{8\pi GL}{c^4 r} 4\pi\rho_0 u_0^2 r_0 \beta^{-2\alpha-3}\Gamma(2\alpha+3,\ \beta). \tag{14}$$

The mass of the galactic disk is:

$$M = \int_{r_0}^{R} dM = \int_{r_0}^{R} 2\pi r\rho L dr = \int_{r_0}^{R} 2\pi\rho_0 r L e^{-\beta\frac{r}{r_0}} dr$$

$$= \int_{1}^{\frac{R}{r_0}} 2\pi\rho_0 r_0^2 L x e^{-\beta x} dx = 2\pi\rho_0 r_0^2 L\beta^{-2}\left[\Gamma(2,\ \beta) - \Gamma\left(2,\ \beta\frac{R}{r_0}\right)\right]$$

$$\cong 2\pi\rho_0 r_0^2 L\beta^{-2}\Gamma(2,\ \beta). \tag{15}$$

In the galactic disk, the speed of the stars is nonrelativistic. Thus, Equation (6) can be written as:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\{\alpha\beta\}}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} - T^i_{xy}\frac{dx}{dt}\frac{dy}{dt} = 0. \tag{16}$$

The first term is acceleration. The second term originates from the symmetric part of the Christoffel symbol and is gravity:

$$\Gamma^i_{\{\alpha\beta\}}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \cong c^2\Gamma^i_{00} = \partial^i\phi = \partial^i\left(-\frac{GM}{r}\right). \tag{17}$$

The third term originates from the antisymmetric torsion. In polar coordinates, Equation (16) is reduced to:

$$-\frac{u^2}{r} + \frac{GM}{r^2} - T(r)u^2 = 0. \tag{18}$$

Using Equation (14), Equation (18) becomes

$$-\frac{u^2}{r} + \frac{GM}{r^2} + \frac{16\pi GMu_0^2}{c^4 r_0 \beta^{2\alpha+1}} \frac{\Gamma(2\alpha + 3, \beta)}{\Gamma(2, \beta)} \frac{u^2}{r} = 0. \tag{19}$$

For $r >> R$, torsion dominates gravity, so we can drop the second term in Equation (19):

$$1 \cong \frac{16\pi GMu_0^2}{c^4 r_0 \beta^{2\alpha+1}} \frac{\Gamma(2\alpha + 3, \beta)}{\Gamma(2, \beta)}. \tag{20}$$

At $r = R$, gravity still dominates torsion, so

$$\frac{u^2}{R} = \frac{GM}{R^2}. \tag{21}$$

Combining Equations (9), (20) and (21)

$$\left(\frac{u}{c}\right)^4 = \frac{\beta^{2\alpha+1}}{16\pi} \frac{\Gamma(2, \beta)}{\Gamma(2\alpha + 3, \beta)} \left(\frac{r_0}{R}\right)^{1-2\alpha}. \tag{22}$$

In Equation (22), let us fix r_0 to be the Schwarzschild radius of the galactic disk $\frac{2GM}{c^2}$ such that

$$\left(\frac{u}{c}\right)^4 = \frac{\beta^{2\alpha+1}}{16\pi} \frac{\Gamma(2, \beta)}{\Gamma(2\alpha + 3, \beta)} \left(\frac{2GM}{Rc^2}\right)^{1-2\alpha}. \tag{23}$$

In Equation (23), we require

$$0 < \alpha < \frac{1}{2}, \quad \beta > 0. \tag{24}$$

Therefore, the terminal velocity depended on the mass, size, and shape of the mass density curve and velocity curve of the disk. Equation (23) is fully testable. A comparison with observational data will be compiled in a subsequent paper.

In this simple exercise, we show that within the galactic disk, gravity dominates torsion; thus, the centripetal force falls according to an inverse square law. Outside the galactic disk, torsion dominates gravity, so that the centripetal force falls as an inverse law. Now, we know that the inverse law originates from torsion, and Newtonian gravity always follows the inverse square law.

By no means is this simplified galactic disk a real description of spiral galaxies. For example, spiral galaxies have spiral arms that are denser than the regions in between. They normally have a central bulge that is more luminous than the disk, and the barred spiral galaxies have a rotating bar in the center. Nevertheless, this example demonstrates that the torsion field has a significant impact on the motion of stars. This perfectly explains the flat rotation curve of the spiral galaxies.

The flat rotation curve of the spiral galaxies is not due to MOND or dark matter. This is simply due to torsion in space-time. This conclusion can be easily verified through observations.

Let us consider the rotation curves of both the spiral galaxies and isolated non-rotating elliptical galaxies. The stars in an elliptical galaxy also rotate around the galactic center. However, because the orientation of each star's orbit is random, stars do not form a rotation disk. However, we can still observe the orbit of each star to calculate the rotation curve of the galaxy. If the dark matter is the reason for the flat rotation curve, then it must be true for both elliptical and spiral galaxies. However, if the flat rotation curve is caused by the torsion of space-time, then we should expect a declining rotation curve outside the elliptical galaxy because the lack of a rotation disk will not cause a significant torsion of space-time.

For this test, we selected elliptical galaxies without rotation. Otherwise, the spin of the elliptical galaxy can cause torsion around it. This defeats the purpose of the test. Gravitational lensing can also be used in these elliptical galaxies to determine whether dark matter is absent. In fact, there have been some observations that already point to this conclusion [38, 39, 40].

It is worth mentioning that in addition to the two popular solutions to the problem of the flat rotation curve of spiral galaxies in the literature, Cooperstock and Tieu proposed a new relativistic solution in 2007 [37]. In [37], the galactic disk was modeled as a pressure-free fluid and a metric owing to this rotational fluid in Equation (25):

$$ds^2 = -e^{v-w}(udz^2 + dr^2) - r^2 e^{-w} d\phi^2 + e^w (cdt - Nd\phi)^2. \tag{25}$$

The most important part of the metric is the rotation-induced field N. They related it to the local angular velocity, as shown in Equation (26):

$$\omega = \frac{Nce^w}{r^2 e^{-w} - N^2 e^w} \approx \frac{Nc}{r^2}. \tag{26}$$

They further derived a differential field equation for this field and related it to the mass density in Equation (27):

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0, \quad \frac{N_r^2 + N_z^2}{r^2} = \frac{8\pi G\rho}{c^2}. \tag{27}$$

By fitting the rotation curve, they determined the field N. Then, using Equation (27), they calculated the mass density. By comparing the mass density with the observations for several galaxies, they concluded that the flat rotation curve is a relativistic phenomenon of the fluid [37].

However, they did not mention the nature of these fluids. If the fluid is composed of gravitationally bound matter in the disk, the rotational field N is caused by the viscosity of the fluid owing to gravitational binding. This field will then be too weak at the outer edge of the disk owing to the falling mass density. If the fluid is composed of a rotating space, it is too weak owing to the feeble frame-dragging effect [13]. Equation (27) is counterintuitive because the rotational field N should be induced by the rotation of the matter field instead of the static mass density. It would be more intuitive to calculate the N field from the rotating matter field rather than vice versa.

According to the new theory presented in this paper, the rotational fluid can be explained as the torsion field, which is induced by the vorticity of the rotating matter. It should be calculated using Equation (5).

3.2. Torsion in galaxy clusters

MOND is very successful in explaining many phenomena in spiral galaxies but is not so successful in describing galaxy clusters [36]. The most famous example of falsifying MOND is the bullet cluster [41]. In this example, two galaxy clusters pass through each other, and the dark matter distribution, which is measured by gravitational lensing, and the luminous matter distribution are not well aligned.

The offset between the two distributions in the bullet cluster can be easily explained by the torsion of space-time. According to Equation (5), the torsion equation is a wave equation, and the torsion field can propagate out of the matter field. Second, in a galaxy cluster, the torsion and the luminous matter fields are not necessarily aligned. For example, when two galaxies rotate with each other, the region with the highest torsion is between the two galaxies. It is far from the luminous matter field inside both galaxies.

Geodesic Equation (6) applies to both mass particles and photons. When we used gravitational lensing to measure the dark matter distribution, we most likely measured the discrepancy offset caused by the torsion of space-time. The torsion of space-time is caused by galaxies rotating around each other.

Owing to the lack of symmetry, we cannot analytically derive the torsion field in a galaxy cluster, as in a spiral galaxy. A galaxy cluster with more than two galaxies is the standard many-body problem. However, numerical simulations can be used to calculate both the gravity field and the torsion field in a galaxy cluster by treating each galaxy as a mass particle. In such a simulation, Equations (1) and (5) must be used. If we use only Equation (1), as we always did in the past, we will have to resort to the mysterious dark matter to explain the discrepancy between theoretical predictions and observations. A bold prediction is that when both equations are used, the dark matter field will be replaced by a torsion field.

3.3. Torsion and galaxy formation

Galaxy formation requires dark matter, and gravity from the baryonic matter alone will not be strong enough to pull enough matter together to form a spiral galaxy [42]. Observations also confirm the presence of a supermassive black hole at the center of each spiral galaxy [43, 44]. This supermassive black hole is essential for the formation of an accretion disk near the galactic center [45]. However, it is unclear how these supermassive black holes were formed. Both the supermassive black hole and dark matter must have been created before the galaxy formation, so they must have occurred in the very early universe.

Recent observations also show that most quasars lie on cosmic filaments, which range from a few hundred million light years to a few billion light years [46, 47]. A quasar is normally thought to be a galaxy at an early age [48]. The most surprising discovery from these observations is that the quasars' spin aligns with the cosmic filament where the quasars reside [46, 47]. These observations indicate that cosmic filaments are likely the largest spinning objects in the universe. However, to date, these phenomena have not been explained.

Is there an intrinsic connection between the cosmic filament, the supermassive black hole and dark matter? If we assume that a cosmic filament is a spinning tube with vorticity, then according to Equation (5), it will create a torsion field around it. If so, can this torsion field play the role of dark matter?

Let us assume a simple cylindrical symmetry and use the polar coordinates. The cosmic filament lies on the z-axis ranging between $\left[-\frac{L}{2}, \frac{L}{2}\right]$ and I is the vorticity per unit length. According to Equation (13), the torsion field is:

$$T(r) = -\frac{8\pi G}{c^4}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{Idz}{\sqrt{r^2 + z^2}} = -\frac{16\pi GI}{c^4}\int_{0}^{\frac{L}{2r}} \frac{dx}{\sqrt{1 + x^2}}$$

$$= -\frac{16\pi GI}{c^4}\ln\left(\sqrt{1 + \left(\frac{L}{2r}\right)^2} + \frac{L}{2r}\right). \tag{28}$$

Let us define $\eta(r)$ as the ratio of the centripetal force from torsion to the centrifugal force from the circular motion of a particle.

$$\eta(r) = \frac{-mT(r)u^2}{m\frac{u^2}{r}} = \frac{8\pi GIL}{c^4}\frac{2r}{L}\ln\left(\sqrt{1 + \left(\frac{L}{2r}\right)^2} + \frac{L}{2r}\right). \tag{29}$$

If $\eta(r) = 1$, then we will have a stable circular orbit at radius r, if $\eta(r) > 1$, then the centripetal force overwhelms the centrifugal force, and the particle is pulled radially inward; if $\eta(r) < 1$, then the centripetal force is not strong enough to hold the particle in a circular motion, and the particle will fly radially outward.

From Equation (29),

$$\begin{cases} \eta(0) = 0, \\ \eta(\infty) = \dfrac{8\pi GIL}{c^4}. \end{cases} \tag{30}$$

If $\frac{8\pi GIL}{c^4} > 1$, then there exists a critical radius r_c, so that $\eta(r_c) = 1$; when $r > r_c$, a particle will be pulled radially inward. This means that if the vorticity of the cosmic filament is strong enough, its torsion field can attract the distant matter toward it. This truly removes the need for dark matter because, unlike torsion, gravity follows the inverse square law; therefore, it requires a large amount of dark matter to pull ordinary matter together to form a galaxy.

At a large distance $r > r_c$, torsion is effective in pulling matter. At a small distance $r < r_c$, gravity is effective in pulling matter.

When matter is pulled within the vicinity of the cosmic filament, rather than falling into a supermassive black hole, it can be channeled through the spinning tube of the cosmic filament, as in a tornado. Observations show that at the ends of a cosmic filament, there are large lobes of luminous matter, so it is postulated that a cosmic filament accretes matter both radially in an accretion disk and vertically through its spin axis [46]. If so, then the supermassive black hole at the galactic center of each spiral galaxy is the cross-section of the galactic disk and the spinning tube of the cosmic filament.

In Section 3.1, when we calculate the torsion field in a spiral galaxy, we only take the vorticity from the galactic disk and ignore the vorticity from the cosmic filament. This is because they are in two different phases of the life of a galaxy. Owing to the conservation law of angular momentum, when galaxies are formed on a cosmic filament, the angular momentum is transferred from the cosmic filament to the galactic disks, as is the vorticity. In the later phase, the vorticity from the cosmic filament was negligible compared to that from the galactic disk.

After a galaxy is formed, at the inner disk owing to torsion, the motion of a particle is corrected as follows:

$$[1 - \eta(r)]\frac{u^2}{r} = \frac{GM}{r^2}. \tag{31}$$

Even if $\eta(r)$ is very small, it still boosts the particle velocity for the same amount of central mass. This will have an adverse effect on black hole formation. Of course, this is a rough estimation. In a detailed analysis, we must solve the full set of relativistic equations: Equations (1), (5), and (6). This analysis deserves a full-length article.

We can also use the spin properties of cosmic filaments to trim the catalog of cosmic filaments [49]. For example, if a long cosmic filament has two segments of opposite spin directions, then it is most likely that we artificially connect two unrelated cosmic filaments. This work is important because it not only places an upper limit on the length of the cosmic filament but also places a lower limit on the size of the universe.

4. Conclusion

By generalizing Riemannian geometry to Riemann-Cartan geometry, space-time can have both curvature and torsion; GR then becomes the Einstein-Cartan theory. By replacing the spin tensor with the vorticity tensor in Einstein-Cartan theory, the torsion equation becomes a wave equation. The improved Einstein-Cartan theory accurately reflects both the equivalence principle and Mach's principle, as Einstein originally envisioned for GR.

A particle in the Riemann-Cartan geometry is influenced by both gravity and torsion of space-time. The improved Einstein-Cartan theory can quantitatively explain the flat rotation curve of spiral galaxies. It also predicts the lack of dark matter in isolated non-rotating elliptical galaxies. This prediction can be used to falsify various solutions for the flat rotation curve problem of galaxies.

The new theory can at least qualitatively explain the discrepancy between the theoretical predictions and observations in galaxy clusters. It also explains the galaxy formation without dark matter. In conclusion, if the torsion field is included, dark matter in the galaxies and galaxy clusters is not required.

Finally, a possible explanation of the spin of cosmic filaments and the origin of a supermassive black hole at the galactic center is provided based on the new theory.

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