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EXPANDING COUNTERPART OF THE HARDY-ROGERS MAPPING THEOREM IN CONE HEPTAGONAL METRIC SPACE

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Abstract

Very recently, we introduced a concept of Cone Heptagonal Metric Space and obtained the Chatterjea Mapping Theorem in this setting [1]. In the present paper, we obtain the expanding counterpart of the Hardy-Rogers Mapping Theorem [2, Theorem 1(a)].

1. Introduction

Let (X, d) be a metric space. Recall Hardy and Rogers [2] that a map $T : X \mapsto X$ is called a Hardy-Rogers contraction if

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

for all $x, y \in X$, where $a, b, c, e, f \geq 0$ and satisfy $\alpha := a + b + c + e + f < 1$.

Remark 1.1. If $\alpha = 1$ and

$$d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y),$$

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then we say $T : X \mapsto X$ is Hardy-Rogers non-expansive. On the other hand if $\alpha > 1$ and

$$d(Tx, Ty) \geq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y),$$

then we say $T : X \mapsto X$ is Hardy-Rogers expansive.

In Hardy and Rogers [2, Theorem 1], the authors proved under certain conditions on (X, d) that if $T : X \mapsto X$ is a Hardy-Rogers contraction or Hardy-Rogers non-expansive, then T has a unique fixed point.

In the present paper, we consider (X, d) a cone heptagonal metric space, and show under certain conditions on (X, d), that if $T : X \mapsto X$ is Hardy-Rogers expansive, then T has a unique fixed point.

At first, we obtain a theorem related to the following, and obtain the main result as a Corollary.

Definition 1.2. Let (X, d) be a metric space, and $T, g : X \mapsto X$ be two self-maps on X. We will say T is Hardy-Rogers expansive with respect to g if

$$d(Tx, Ty) \geq ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + fd(gx, gy)$$

for all $x, y \in X$, where $a, b, c, e, f \geq 0$ and satisfy $\alpha := a + b + c + e + f > 1$.

Remark 1.3. Note that if g is the identity in the above, then we simply refer to T as Hardy-Rogers expansive.

This paper is organized as follows. Section 2 gives some preliminary ideas that would be useful in the sequel. By way of Example 2.9, we showed in [1] that the notion of cone heptagonal metric space is a proper extension of cone hexagonal metric space. Example 2.10 and Example 2.11 also show that the notion of cone heptagonal metric space is a proper extension of cone hexagonal metric space. The expanding counterpart of the Hardy-Rogers mapping theorem, Hardy and Rogers [2, Theorem 1(a)] is obtained as Corollary 3.2. Finally we illustrate Theorem 3.1 with Example 3.3.

2. Preliminaries

Notation 2.1. E will denote a real Banach space.

Definition 2.2. $P \subset E$ will be called a cone iff

(a) P is closed, nonempty, and $P \neq \{0\}$,

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ implies $ax + by \in P$,

(c) $x \in P$ and $-x \in P$ implies $x = 0$.

Notation 2.3. $\leq$ will denote a partial ordering with respect to P and will be defined by $x \leq y$ iff $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P.

Definition 2.4. A cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$, the inequality $0 \leq x \leq y$ implies that $\|x\| \leq k\|y\|$. The least positive number k satisfying $\|x\| \leq k\|y\|$ is called the normal constant of P.

Remark 2.5. In this paper, we always assume that E is a real Banach space and P is a solid cone in E with $\operatorname{int}(P) \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.6. Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Remark 2.7. If we replace (c) of the previous definition with the following, which we call the heptagonal property, $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, t) + d(t, y)$ for all $x, y, z, w, u, v, t \in X$ and for all distinct points $z, w, u, v, t \in X - \{x, y\}$, then we say d is a cone heptagonal metric on X, and we call (X, d) a cone heptagonal metric space.

Remark 2.8. A metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2.9. Let $X = \{r, s, t, u, v, w, k\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ be a cone in E. Define $d : X \times X \mapsto E$ by

$d(x, x) = 0$ for all $x \in X$,

$d(r, s) = d(s, r) = (6, 12)$,

$d(r, t) = d(r, u) = d(r, v) = d(r, w) = d(s, t) = d(s, u) = d(s, v) = d(s, w) = d(t, u) =$

$d(t, v) = d(t, w) = d(u, v) = d(u, w) = d(v, w) = d(t, r) = d(u, r) = d(v, r) = d(w, r) =$

$d(t, s) = d(u, s) = d(v, s) = d(w, s) = d(u, t) = d(v, t) = d(w, t) = d(v, u) = d(w, u) =$

$d(w, v) = (1, 2)$,

$d(k, r) = d(k, s) = d(k, t) = d(k, u) = d(k, v) = d(k, w) = d(r, k) = d(s, k) = d(t, k) =$

$d(u, k) = d(v, k) = d(w, k) = (5, 10)$.

Then it is easy to see that (X, d) is a cone heptagonal metric space, but it is not a cone hexagonal metric space, since it lacks the hexagonal property of Auwalu and Hincal [3] since $(6, 12) = d(r, s) > d(r, t) + d(t, u) + d(u, v) + d(v, w) + d(w, s) = (1, 2) + (1, 2) + (1, 2) + (1, 2) + (1, 2) = (5, 10)$ as $(6, 12) - (5, 10) = (1, 2) \in P$.

Example 2.10. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$. Define $d : X \times X \mapsto E$ as follows

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ (6a, 6) & \text{if } x \text{ and } y \text{ are both in } \{1, 2\},\ x \neq y, \\ (a, 1) & \text{if } x \text{ and } y \text{ are not both at time in } \{1, 2\},\ x \neq y, \end{cases}$$

where $a > 0$ is a constant. Then (X, d) is a cone heptagonal metric space, but not a cone hexagonal metric space. Observe that, $(6a, 6) = d(1, 2) > d(1, 3) + d(3, 4) + d(4, 5) + d(5, 6) + d(6, 1) = (5a, 5)$, thus the hexagonal property of Auwalu and Hincal [3] does not hold.

Example 2.11. Let $X = \mathbb{N}$, $E = \mathbb{C}^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and

$P = \{x \in E : x(t) \geq 0\}$ for $t \in [0, 1]$. Then this cone is not normal. Define $d : X \times X \mapsto E$ as follows

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 6e^t & \text{if } x \text{ and } y \text{ are both in } \{1, 2\}, x \neq y, \\ e^t & \text{if } x \text{ and } y \text{ are not both at time in } \{1, 2\}, x \neq y. \end{cases}$$

Then (X, d) is a cone heptagonal metric space, but not a cone hexagonal metric space, since it lacks the hexagonal property of Auwalu and Hincal [3].

Definition 2.12. Let (X, d) be a cone heptagonal metric space, and $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We sometimes write $\lim_{n\to\infty} x_n = x$.

Definition 2.13. Let (X, d) be a cone heptagonal metric space, and $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is said to be a Cauchy sequence in X.

Definition 2.14. Let (X, d) be a cone heptagonal metric space. If every Cauchy sequence in X converges to a point in X, then X is called a complete cone heptagonal metric space.

Definition 2.15. Let f and g be two self-maps of a nonempty set X. If $fx = gx = y$ for some $x \in X$, then x is called a coincidence point of f and g and y is called the point of coincidence of f and g.

Definition 2.16. Two self-maps f and g of a nonempty set X are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies that $fgx = gfx$.

In the sequel, we will need the following from M. Abbas and G. Jungck [4].

Proposition 2.17. *If f and g are weakly compatible self-maps of a nonempty set X such that they have a unique point of coincidence, that is, $fx = gx = y$, then y is the unique common fixed point of f and g.*

In the sequel, we will need the following from Malhotra et al. [5].

Remark 2.18. Let P be a cone in a real Banach space E and let $a, b, c \in E$, then

(a) If $a \leq b$ and $b \ll c$, then $a \ll c$,

(b) If $a \ll b$ and $b \ll c$, then $a \ll c$,

(c) If $0 \leq u \ll c$, for each $c \in \operatorname{int}(P)$, then $u = 0$,

(d) If $c \in \operatorname{int}(P)$ and $a_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll c$,

(e) If $0 \leq a_n \leq b_n$, for each n and $a_n \to a$, $b_n \to b$, then $a \leq b$,

(f) If $a \leq \lambda a$, where $0 < \lambda < 1$, then $a = 0$.

3. Main Results

Theorem 3.1. *Let (X, d) be a complete cone heptagonal metric space and let $T, g : X \mapsto X$ satisfy*

$$d(Tx, Ty) \geq ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + fd(gx, gy)$$

for all $x, y \in X$, where $a, b, c, e, f \geq 0$ and satisfy $a + b + 2e + f > 1$, $f + c + e > 1$, $f > 1$, $b + c < 1$, and $a + e < 1$. If $g(X) \subseteq T(X)$ and either $T(X)$ or $g(X)$ is complete, then T and g have a unique point of coincidence in X. If T and g are weakly compatible, then they have a unique common fixed point in X.

Proof. Let $x_0 \in X$, since $g(X) \subseteq T(X)$, we can choose $x_1 \in X$ such that $gx_0 = Tx_1$. Continuing this process we can construct a sequence $\{x_n\}$ in X such that $Tx_n = gx_{n-1}$, for all $n \geq 1$. If $gx_{n-1} = gx_n$ for some $n \geq 1$, then $Tx_n = gx_n$ and x_n is a coincidence point of T and g. Hence assume that $x_n \neq x_{n-1}$ for all $n \geq 1$. Now observe that

$$d(gx_{n-1}, gx_n) = d(Tx_n, Tx_{n+1})$$

$$\geq ad(gx_n, Tx_n) + bd(gx_{n+1}, Tx_{n+1}) + cd(gx_n, Tx_{n+1}) + ed(gx_{n+1}, Tx_n)$$

$$
\begin{aligned}
&+fd(gx_n, gx_{n+1}) \\
&\geq ad(gx_n, gx_{n-1}) + bd(gx_{n+1}, gx_n) + cd(gx_n, gx_n) + ed(gx_{n+1}, gx_{n-1}) \\
&\quad +fd(gx_n, gx_{n+1}) \\
&\geq ad(gx_n, gx_{n-1}) + (b+f)d(gx_{n+1}, gx_n) + ed(gx_{n+1}, gx_{n-1}).
\end{aligned}
$$

From the above, one has,

$$(b+f)d(gx_{n+1}, gx_n) \leq (1-a)d(gx_{n-1}, gx_n) - ed(gx_{n+1}, gx_{n-1}).$$

Now using the triangle inequality in the expression immediately above, one deduces that

$$(b+f+e)d(gx_{n+1}, gx_n) \leq (1-a-e)d(gx_{n-1}, gx_n).$$

Thus, it follows that $d(gx_{n+1}, gx_n) \leq \gamma d(gx_{n-1}, gx_n)$, where, $\gamma := \frac{1-a-e}{b+f+e} \in (0, 1)$, and by induction, we have $d(gx_{n+1}, gx_n) \leq \gamma^n d(gx_0, gx_1)$. Now observe that

$$
\begin{aligned}
&d(gx_{n-1}, gx_{n+1}) = d(Tx_n, Tx_{n+2}) \\
&\geq ad(gx_n, Tx_n) + bd(gx_{n+2}, Tx_{n+2}) + cd(gx_n, Tx_{n+2}) + ed(gx_{n+2}, Tx_n) \\
&\quad +fd(gx_n, gx_{n+2}) \\
&\geq ad(gx_n, gx_{n-1}) + bd(gx_{n+2}, gx_{n+1}) + cd(gx_n, gx_{n+1}) + ed(gx_{n+2}, gx_{n-1}) \\
&\quad +fd(gx_n, gx_{n+2}).
\end{aligned}
$$

From the above, we deduce that

$$
\begin{aligned}
&fd(gx_n, gx_{n+2}) \\
&\leq d(gx_{n-1}, gx_{n+1}) - ad(gx_n, gx_{n-1}) - bd(gx_{n+2}, gx_{n+1}) \\
&\quad -cd(gx_n, gx_{n+1}) - ed(gx_{n+2}, gx_{n-1}) \\
&\leq d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+2}) + d(gx_{n+2}, gx_{n+1}) - ad(gx_n, gx_{n-1}) \\
&\quad -bd(gx_{n+2}, gx_{n+1}) - cd(gx_n, gx_{n+1}) - ed(gx_{n+2}, gx_{n-1}).
\end{aligned}
$$

From the above, we deduce that

$$
\begin{aligned}
&(f-1)d(gx_n,\ gx_{n+2})\\
&\leq (1-a)d(gx_n,\ gx_{n-1})+(1-b)d(gx_{n+2},\ gx_{n+1})\\
&-cd(gx_n,\ gx_{n+1})-ed(gx_{n+2},\ gx_{n-1})\\
&\leq (1-a)d(gx_n,\ gx_{n-1})+(1-b)d(gx_{n+2},\ gx_{n+1})\\
&-c[d(gx_n,\ gx_{n+2})+d(gx_{n+2},\ gx_{n+1})]\\
&-e[d(gx_n,\ gx_{n-1})+d(gx_n,\ gx_{n+2})].
\end{aligned}
$$

From the above, one deduces that

$$
\begin{aligned}
&(f+c+e-1)d(gx_n,\ gx_{n+2})\\
&\leq (1-a-e)d(gx_n,\ gx_{n-1})+(1-b-c)d(gx_{n+2},\ gx_{n+1}).
\end{aligned}
$$

It follows that $d(gx_n,\ gx_{n+2})\leq \alpha_1 d(gx_n,\ gx_{n-1})+\alpha_2 d(gx_{n+2},\ gx_{n+1})$, where $\alpha_1 := \dfrac{1-a-e}{f+c+e-1}>0$ and $\alpha_2 := \dfrac{1-b-c}{f+c+e-1}>0$. Now observe that

$$
\begin{aligned}
&d(gx_{n-1},\ gx_{n+2})=d(Tx_n,\ Tx_{n+3})\\
&\geq ad(gx_n,\ Tx_n)+bd(gx_{n+3},\ Tx_{n+3})+cd(gx_n,\ Tx_{n+3})+ed(gx_{n+3},\ Tx_n)\\
&+fd(gx_n,\ gx_{n+3})\\
&\geq ad(gx_n,\ gx_{n-1})+bd(gx_{n+3},\ gx_{n+2})+cd(gx_n,\ gx_{n+2})+ed(gx_{n+3},\ gx_{n-1})\\
&+fd(gx_n,\ gx_{n+3}).
\end{aligned}
$$

From the above, we deduce that

$$
\begin{aligned}
&fd(gx_n,\ gx_{n+3})\\
&\leq d(gx_{n-1},\ gx_{n+2})-ad(gx_n,\ gx_{n-1})-bd(gx_{n+3},\ gx_{n+2})\\
&-cd(gx_n,\ gx_{n+2})-ed(gx_{n+3},\ gx_{n-1})\\
&\leq d(gx_{n-1},\ gx_n)+d(gx_n,\ gx_{n+3})+d(gx_{n+3},\ gx_{n+2})-ad(gx_n,\ gx_{n-1})
\end{aligned}
$$

$$-bd(gx_{n+3}, gx_{n+2}) - cd(gx_n, gx_{n+2}) - ed(gx_{n+3}, gx_{n-1}).$$

From the above, we deduce that

$$\begin{aligned}
&(f-1)d(gx_n, gx_{n+3}) \\
&\le (1-a)d(gx_n, gx_{n-1}) + (1-b)d(gx_{n+3}, gx_{n+2}) \\
&\quad -cd(gx_n, gx_{n+2}) - ed(gx_{n+3}, gx_{n-1}) \\
&\le (1-a)d(gx_n, gx_{n-1}) + (1-b)d(gx_{n+3}, gx_{n+2}) \\
&\quad -c[d(gx_n, gx_{n+3}) + d(gx_{n+3}, gx_{n+2})] \\
&\quad -e[d(gx_n, gx_{n-1}) + d(gx_n, gx_{n+3})].
\end{aligned}$$

From the above, one deduces that

$$\begin{aligned}
&(f+c+e-1)d(gx_n, gx_{n+3}) \\
&\le (1-a-e)d(gx_n, gx_{n-1}) + (1-b-c)d(gx_{n+3}, gx_{n+2}).
\end{aligned}$$

It follows that $d(gx_n, gx_{n+3}) \le \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+3}, gx_{n+2})$, where $\alpha_1 := \frac{1-a-e}{f+c+e-1} > 0$ and $\alpha_2 := \frac{1-b-c}{f+c+e-1} > 0$. Similarly, we have,

$$d(gx_n, gx_{n+4}) \le \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+4}, gx_{n+3}),$$

$$d(gx_n, gx_{n+5}) \le \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+5}, gx_{n+4}),$$

where $\alpha_1 := \frac{1-a-e}{f+c+e-1} > 0$ and $\alpha_2 := \frac{1-b-c}{f+c+e-1} > 0$. For the sequence $\{gx_n\}$, we consider $d(gx_n, gx_{n+p})$ in two cases, p is even and p is odd. When p is even, let $p = 2 + 2m$, where $m \ge 2$, and when p is odd let $p = 5 + 2m$, where $m \ge 1$. In the case $p = 5 + 2m$, we have

$$\begin{aligned}
&d(gx_n, gx_{n+5m+2}) \\
&\le 2d(gx_n, gx_{n+1}) + 2d(gx_n, gx_{n+3}) \\
&\quad +d(gx_n, gx_{n+5}) + \cdots + d(gx_{n+5m+1}, gx_{n+5m+2}) \\
&\le 2d(gx_n, gx_{n+1}) + 2\alpha_1 d(gx_n, gx_{n-1}) + 2\alpha_2 d(gx_{n+3}, gx_{n+2})
\end{aligned}$$

$$+\alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+5}, gx_{n+4}) + \cdots + d(gx_{n+5m+1}, gx_{n+5m+2})$$

$$\leq 2\gamma^n d(gx_0, gx_1) + 2\alpha_1\gamma^{n-1} d(gx_0, gx_1) + 2\alpha_2\gamma^{n+2} d(gx_0, gx_1)$$

$$+ \alpha_1\gamma^{n-1} d(gx_0, gx_1) + \alpha_2\gamma^{n+4} d(gx_0, gx_1) + \cdots + \gamma^{n+5m+1} d(gx_0, gx_1)$$

$$\leq 3\alpha_1\gamma^{n-1} d(gx_0, gx_1) + 2\alpha_2\gamma^{n+2} d(gx_0, gx_1) + \alpha_2\gamma^{n+4} d(gx_0, gx_1)$$

$$+ \frac{2\gamma^n}{1-\gamma} d(gx_0, gx_1).$$

In the case $p = 2 + 2m$, we have,

$$d(gx_n, gx_{n+2m+2})$$

$$\leq 2d(gx_n, gx_{n+2}) + d(gx_n, gx_{n+4}) + \cdots$$

$$+d(gx_{n+2m}, gx_{n+2m+1}) + d(gx_{n+2m+1}, gx_{n+2m+2})$$

$$\leq 2\alpha_1 d(gx_n, gx_{n-1}) + 2\alpha_2 d(gx_{n+1}, gx_{n+2}) + \alpha_1 d(gx_n, gx_{n-1})$$

$$+\alpha_2 d(gx_{n+3}, gx_{n+4}) + \cdots + d(gx_{n+2m}, gx_{n+2m+1})$$

$$+d(gx_{n+2m+1}, gx_{n+2m+2})$$

$$\leq 2\alpha_1\gamma^{n-1} d(gx_0, gx_1) + 2\alpha_2\gamma^{n+1} d(gx_0, gx_1) + \alpha_1\gamma^{n-1} d(gx_0, gx_1)$$

$$+ \alpha_2\gamma^{n+3} d(gx_0, gx_1) + \cdots + \gamma^{n+2m} d(gx_0, gx_1) + \gamma^{n+2m+1} d(gx_0, gx_1)$$

$$\leq 2\alpha_1\gamma^{n-1} d(gx_0, gx_1) + 2\alpha_2\gamma^{n+1} d(gx_0, gx_1) + \alpha_1\gamma^{n-1} d(gx_0, gx_1)$$

$$+ \alpha_2\gamma^{n+3} d(gx_0, gx_1) + \cdots + \frac{\gamma^n}{1-\gamma} d(gx_0, gx_1).$$

Since $\alpha_1, \alpha_2 > 0$ and $\gamma \in (0, 1)$, if we take limits as $n \to \infty$ in the even and odd cases above, we deduce from Remark 2.18, that for every $c \in E$ with $0 \ll c$, there exists a natural number n_0 such that $d(gx_n, gx_{n+p}) \ll c$ for all $n > n_0$. Hence $\{gx_n\}$ is a Cauchy sequence. Suppose $g(X)$ is a complete subspace of X, then there exists $y \in g(X) \subseteq T(X)$ such that $\lim_{n\to\infty} gx_n = y$ and $\lim_{n\to\infty} Tx_n = y$ and if

$T(X)$ is complete, this holds also with $y \in T(X)$. Let $u \in X$ be such that $Tu = y$. Now observe that,

$$d(gx_{n-1}, Tu) = d(Tx_n, Tu)$$

$$\geq ad(gx_n, Tx_n) + bd(gu_n, Tu) + cd(gx_n, Tu) + ed(gu, Tx_n) + fd(gx_n, gu)$$

$$\geq fd(gx_n, gu).$$

Thus, $d(gx_n, gu) \leq \frac{1}{f} d(gx_{n-1}, Tu)$. Now observe that,

$$\begin{aligned} d(y, gu) &\leq d(y, gx_{n-4}) + d(gx_{n-4}, gx_{n-3}) + d(gx_{n-3}, gx_{n-2}) \\ &\quad + d(gx_{n-2}, gx_{n-1}) + d(gx_{n-1}, gx_n) + d(gx_n, gu) \\ &\leq d(y, gx_{n-4}) + d(gx_{n-4}, gx_{n-3}) + d(gx_{n-3}, gx_{n-2}) \\ &\quad + (gx_{n-2}, gx_{n-1}) + d(gx_{n-1}, gx_n) + \frac{1}{f} d(gx_{n-1}, Tu). \end{aligned}$$

Now for $0 \ll c$, we can choose a natural number n_0 such that

$$d(y, gx_{n-4}) \ll \frac{c}{6}, \ d(gx_{n-4}, gx_{n-3}) \ll \frac{c}{6}, \ d(gx_{n-3}, gx_{n-2}) \ll \frac{c}{6},$$

$$d(gx_{n-2}, gx_{n-1}) \ll \frac{c}{6}, \ d(gx_{n-1}, gx_n) \ll \frac{c}{6}, \text{ and, } d(gx_{n-1}, Tu) \ll \frac{cf}{6}.$$

Thus,

$$d(y, gu) \ll 6 \cdot \frac{c}{6} = c$$

for all $n > n_0$ and $gu = y$, hence $Tu = gu = y$, which means y is a coincidence point of T and g. If there exists y^* such that $gu^* = Tu^* = y^*$ for some $u^* \in X$, then by the expanding condition of the theorem, we deduce that,

$$d(y, y^*) \leq \frac{1}{f} d(y, y^*).$$

Since $f > 1$, then by Remark 2.18, $d(y, y^*) = 0$, that is, $y = y^*$. Therefore g and T have a unique point of coincidence in X. If g and T are weakly compatible, then by Proposition 2.17, they have a unique common fixed point in X

If g is the identity in the previous theorem, then we obtain the following.

Corollary 3.2. *Let* (X, d) *be a complete cone heptagonal metric space and let* $T : X \mapsto X$ *be an onto mapping satisfying*

$$d(Tx, Ty) \geq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

for all $x, y \in X$, *where* $a, b, c, e, f \geq 0$ *and satisfy* $a + b + 2e + f > 1$, $f + c + e > 1$, $f > 1$, $b + c < 1$, *and* $a + e < 1$. *Then T has a unique fixed point in* X.

Example 3.3. Let X, E, P, and $d : X \times X \mapsto E$ be defined as in Example 2.8, Ampadu [1]. As that example shows, (X, d) is a cone heptagonal metric space but not a cone hexagonal metric space. Now define mappings $T, g : X \mapsto X$ as follows: $Tx = x$ for all $x \in X$, and $g := f$, where $f : X \mapsto X$ is the mapping in Example 3.2, Ampadu [1]. It follows that all the conditions of Theorem 3.1 hold for $f \in (1, 2]$, $a = b = c = e = 0$. Moreover, $6 = w \in X$ is the unique common fixed point of T and g.

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