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EXACT GENERAL SOLUTIONS FOR MHD AXIAL COUETTE FLOWS OF SECOND GRADE FLUIDS THROUGH A POROUS MEDIUM IN A CIRCULAR CYLINDER

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Abstract

Axial Couette flows of the incompressible second grade fluids through

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an infinite circular cylinder are analytically investigated when magnetic and porous effects are taken into consideration. A general expression is established for dimensionless velocity of fluid using the finite Hankel transform. It can generate exact solutions for any motion of this kind of the respective fluids and the problem in discussion is completely solved. For illustration some special cases are considered and the results validation is graphically investigated. Finally, graphical representations showed that the fluid flows faster and the steady state is later reached in the absence of magnetic field or porous medium.

1. Introduction

The exact solutions for initial-boundary value problems can describe flows of some fluids or deformations of solids in different circumstance or may be used to test numerical methods for more complex motion problems. Unfortunately, there are few exact solutions in the existing literature for magnetohydrodynamic (MHD) flows of non-Newtonian fluids in cylindrical domains. For second grade fluids, for instance, we mention the results of Jha and Apere [1] and Jamil and Zafarullah [2]. Constitutive equation of these fluids is given by the relation [3]

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2, \tag{1}$$

where $\boldsymbol{T}$ is the stress tensor, $-p\boldsymbol{I}$ is the constitutively part of the stress due to the constraint of incompressibility, μ denotes the fluid viscosity, α_1 and α_2 are material constants and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors.

In this short note, we provide a general expression of the dimensionless velocity field for MHD axial Couette flows of the electrical conducting incompressible second grade fluids (ECISGFs) through a porous medium in an infinite circular cylinder. This expression allows us to determine exact solutions for any flow of this type of the respective fluids and the problem in discussion is completely solved. For illustration, as well as for the validation of this expression, some special cases are considered and two comparative graphs are included. Finally, the effects magnetic field and porous medium on the fluid behavior are graphically depicted and discussed. It was found that the fluid flows slower and the steady state is earlier obtained in the presence of a magnetic field or porous medium.

2. The Problem Presentation

Let us consider the MHD axial Couette flow of ECISGFs through a porous medium in an infinite horizontal circular cylinder that, after the moment $t = 0^+$, moves along its axis with the velocity $Vf(t)$. Here, V is constant and the function $f(\cdot)$ is piecewise continuous and $f(0) = 0$. For such a motion we are looking for a velocity field of the form [4]

$$\boldsymbol{w} = \boldsymbol{w}(r, t) = (0, 0, w(r, t)), \tag{2}$$

in a suitable system of cylindrical coordinate r, θ and z. Substituting $\boldsymbol{w}(r, t)$ from Eq. (2) in (1) one finds that the non-trivial shear stress $\tau(r, t)$ satisfies the following relation

$$\tau(r, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\frac{\partial w(r, t)}{\partial r}; \qquad 0 < r < a, \qquad t > 0, \tag{3}$$

where a is the radius of cylinder. The continuity equation is identically verified.

In the absence of a pressure gradient in the flow direction but in the presence of a magnetic field of strength B perpendicular to the z-axis of the coordinate system, the balance of linear momentum reduces to the relevant partial differential equation [2]

$$\rho \frac{\partial w(r, t)}{\partial t} = \frac{\partial \tau(r, t)}{\partial r} + \frac{1}{r}\tau(r, t) - \sigma B^2 w(r, t) - \mu \frac{\varphi}{k}\left(1 + \frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right) w(r, t);$$

$$0 < r < a, \qquad t > 0, \tag{4}$$

in which ρ is the fluid density, σ is its electrical conductivity, φ $(0 < \varphi < 1)$ is the porosity and k (> 0) denotes the permeability of porous medium. The corresponding initial and boundary conditions are given by the relations

$$w(r, 0) = 0, \qquad 0 \le r \le a; \qquad w(a, t) = Vf(t), \qquad t > 0. \tag{5}$$

Introducing the following non-dimensional functions and variables

$$w^* = \frac{1}{V} w, \quad \tau^* = \frac{a}{\mu V} \tau, \quad r^* = \frac{1}{a} r, \quad t^* = \frac{\nu}{a^2} t, \quad f^*(t^*) = f\left(\frac{a^2}{\nu} t^*\right) \tag{6}$$

and dropping out the star notation one finds the dimensionless forms

$$\tau(r, t) = \left(1 + \alpha \frac{\partial}{\partial t}\right) \frac{\partial w(r, t)}{\partial r}; \quad 0 < r < 1, \quad t > 0, \tag{7}$$

$$\frac{\partial w(r, t)}{\partial t} = \frac{\partial \tau(r, t)}{\partial r} + \frac{1}{r} \tau(r, t) - Mw(r, t) - K\left(1 + \alpha \frac{\partial}{\partial t}\right) w(r, t);$$

$$0 < r < 1, \quad t > 0, \tag{8}$$

of the governing equations. In above relations $\nu = \mu/\rho$ is the cinematic viscosity of the fluid, the constant $\alpha = \alpha_1 / (\rho a^2)$ and the magnetic and porous parameters M and K, respectively, are defined by the next relations

$$M = (\sigma B^2/\rho)(a^2/\nu) = (a^2/\mu)\sigma B^2, \quad K = (\varphi/k) a^2. \tag{9}$$

Eliminating $\tau(r, t)$ between Eqs. (7) and (8) one obtains the governing equation

$$\frac{\partial w(r, t)}{\partial t} = \left(1 + \alpha \frac{\partial}{\partial t}\right) \left[\frac{\partial^2 w(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial w(r, t)}{\partial r}\right]$$

$$- Mw(r, t) - K\left(1 + \alpha \frac{\partial}{\partial t}\right) w(r, t), \tag{10}$$

for the dimensionless velocity field $w(r, t)$. The corresponding initial and boundary conditions are given by the relations

$$w(r, 0) = 0, \quad 0 \le r \le 1; \quad w(1, t) = f(t), \quad t > 0. \tag{11}$$

3. Solution of the Problem

Applying the finite Hankel transform to Eq. (10) and bearing in mind conditions (11) and the relation (61) from Section 14 of the reference [5] one finds the ordinary differential equation

$$\frac{\partial w_H(r_n, t)}{\partial t} + \frac{r_n^2 + K_{eff}}{1 + \alpha(r_n^2 + K)} w_H(r_n, t) = \frac{r_n J_1(r_n)}{1 + \alpha(r_n^2 + K)} [f(t) + \alpha f'(t)];$$

$$t > 0, \tag{12}$$

for finite Hankel transform $w_H(r_n, t)$ of $w(r, t)$. Here, r_n are positive roots of $J_0(r) = 0$.

Solving this equation with the initial condition $w_H(r_n, 0) = 0$ and applying inverse finite Hankel transform to the obtained result one obtains the dimensionless fluid velocity

$$w(r, t) = f(t) - 2(1 + \alpha K) f(t) \sum_{n=1}^{\infty} \frac{J_0(r r_n)}{r_n J_1(r_n)[1 + \alpha(r_n^2 + K)]} + 2(1 - \alpha M)$$

$$\times \sum_{n=1}^{\infty} \frac{r_n J_0(r r_n)}{J_1(r_n)[1 + \alpha(r_n^2 + K)]^2} \int_0^t f(t - s) \exp\left[- \frac{r_n^2 + K_{eff}}{1 + \alpha(r_n^2 + K)} s \right] ds. \tag{13}$$

The corresponding shear stress can be immediately obtained substituting $w(r, t)$ from Eq. (13) in (7). Since $f(\cdot)$ is an arbitrary function, the problem in discussion is completely solved. For illustration, as well as to verify the correctness of obtained results and to bring to light some characteristics of the fluid behavior, some special cases are considered.

3.1. The cylinder oscillates along its symmetry axis

Replacing $f(t)$ by $H(t)\cos(\omega t)$ or $H(t)\sin(\omega t)$ (where $H(t)$ is the unit step function) in Eq. (13) one obtains dimensionless starting velocities $w_c(r, t)$ and $w_s(r, t)$ corresponding to the MHD Couette flows of ECISGFs through a porous medium in an infinite circular cylinder that oscillates along its symmetry axis with the velocity $V\cos(\omega t)$ or $V\sin(\omega t)$, respectively. It is well known the fact that these motions become steady in time and, as expected, the respective starting velocities can be presented as sums of their steady state (permanent or long time) and transient components, namely

$$w_c(r, t) = w_{cp}(r, t) + w_{ct}(r, t), \quad w_s(r, t) = w_{sp}(r, t) + w_{st}(r, t);$$

$$0 < r < 1, \quad t > 0. \tag{14}$$

In above relations

$$w_{cp}(r, t) = \cos(\omega t) - 2(1 + \alpha K)\cos(\omega t)\sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)[1 + \alpha(r_n^2 + K)]}$$

$$+ 2(1 - \alpha M)\cos(\omega t)\sum_{n=1}^{\infty} \frac{r_n(r_n^2 + K_{eff})J_0(rr_n)}{a_n J_1(r_n)[1 + \alpha(r_n^2 + K)]}$$

$$+ 2\omega(1 - \alpha M)\sin(\omega t)\sum_{n=1}^{\infty} \frac{r_n J_0(rr_n)}{a_n J_1(r_n)}, \tag{15}$$

$$w_{ct}(r, t) = -2(1 - \alpha M)\sum_{n=1}^{\infty} \frac{r_n(r_n^2 + K_{eff})J_0(rr_n)}{a_n J_1(r_n)[1 + \alpha(r_n^2 + K)]}$$

$$\times \exp\left[-\frac{r_n^2 + K_{eff}}{1 + \alpha(r_n^2 + K)}t\right], \tag{16}$$

$$w_{sp}(r,\ t) = \sin(\omega t) - 2(1+\alpha K)\sin(\omega t)\sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)[1+\alpha(r_n^2+K)]}$$

$$+ 2(1-\alpha M)\sin(\omega t)\sum_{n=1}^{\infty} \frac{r_n(r_n^2+K_{eff})J_0(rr_n)}{a_n J_1(r_n)[1+\alpha(r_n^2+K)]}$$

$$- 2\omega(1-\alpha M)\cos(\omega t)\sum_{n=1}^{\infty} \frac{r_n J_0(rr_n)}{a_n J_1(r_n)}, \tag{17}$$

$$w_{st}(r,\ t) = 2\omega(1-\alpha M)\sum_{n=1}^{\infty} \frac{r_n J_0(rr_n)}{a_n J_1(r_n)} \exp\left[-\frac{r_n^2+K_{eff}}{1+\alpha(r_n^2+K)}\, t\right], \tag{18}$$

in which the constants $a_n = (r_n^2 + K_{eff})^2 + \omega^2[1+\alpha(r_n^2+K)]^2$ with $n = 1,\ 2,\ 3,\ \ldots$.

Direct computations, using the governing equation for steady motions, show that the steady state velocities $w_{cp}(r,\ t)$ and $w_{sp}(r,\ t)$ can be presented in simpler forms, namely

$$w_{cp}(r,\ t) = \mathrm{Re}\left\{\frac{I_0\left(r\sqrt{\delta}\right)}{I_0(\delta)}\, e^{i\omega t}\right\},\quad w_{sp}(r,\ t) = \mathrm{Im}\left\{\frac{I_0\left(r\sqrt{\delta}\right)}{I_0(\delta)}\, e^{i\omega t}\right\};$$

$$0 < r < 1,\quad t > 0, \tag{19}$$

where $\delta = [K_{eff} + i\omega(1+\alpha K)]/(1+i\omega\alpha)$, $K_{eff} = M + K$ is effective permeability and $I_0(\cdot)$ is the modified Bessel function of zero order. Figures 1 clearly shows the equivalence of the expressions of $w_{cp}(r,\ t)$ and $w_{sp}(r,\ t)$ given by Eqs. (15), $(19)_1$ and (17), $(19)_2$, respectively.

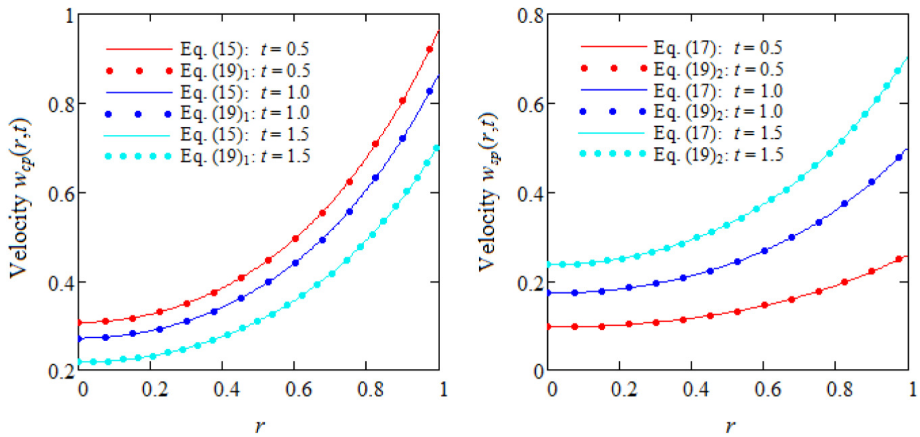

Figure 1. Equivalence of the expressions of $w_{cp}(r, t)$ and $w_{sp}(r, t)$ given by Eqs. (15), $(19)_1$ and (17), $(19)_2$, respectively, for $\alpha = 0.8$, $\beta = 0.5$, $\omega = \pi/6$, $M = 2$, $K = 4$ and three values of t.

3.2. The cylinder slides along its axis with the constant velocity V

Substituting $f(t)$ by $H(t)$ in Eq. (13) or making $\omega = 0$ in $(14)_1$ one finds the dimensionless starting velocity $w_C(r, t)$ corresponding to the motion of ECISGFs induced by cylinder that moves along its axis with the constant velocity V. This velocity can be also written as sum of its steady and transient components, i.e., $w_C(r, t) = w_{Cp}(r) + w_{Ct}(r, t)$ in which

$$w_{Cp}(r) = 1 - 2K_{eff} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)(r_n^2 + K_{eff})}, \tag{20}$$

$$w_{Ct}(r, t) = -2(1 - \alpha M) \sum_{n=1}^{\infty} \frac{r_n J_0(rr_n)}{(r_n^2 + K_{eff})[1 + \alpha(r_n^2 + K)]J_1(r_n)} \times \exp\left[-\frac{r_n^2 + K_{eff}}{1 + \alpha(r_n^2 + K)} t\right]. \tag{21}$$

An equivalent form for the steady velocity $w_{Cp}(r)$, namely $w_{Cp}(r) = I_0(r\sqrt{K_{eff}})/I_0(\sqrt{K_{eff}})$, can be obtained making $\omega = 0$ in Eq. $(19)_1$. From Eq. (20) it clearly results that the steady velocity $w_{Cp}(r)$, which is the same for Newtonian and second grade fluids, does not depend on the parameters M and K independently but only by means of the effective permeability K_{eff}. Consequently, a two parameter approach in this steady motion of the incompressible Newtonian or second grade fluids is superfluous. In all other cases the similar solutions corresponding to same motions of the incompressible Newtonian fluids can be immediately obtained making $\alpha = 0$ in the general solutions for second grade fluids.

4. Some Graphical Representations and Conclusions

In this note the motion problem of ECISGFs through a porous medium in an infinite circular cylinder that moves along its symmetry axis is completely solved when influence of magnetic field and porous medium is taken into account. General expression that has been established for the dimensionless velocity of the fluid can generate exact solutions for any motion of this kind of the respective fluids. For illustration some special cases were considered and results correctness was graphically proved. Velocity field for the incompressible Newtonian fluids performing the same motion can be immediately obtained putting $\alpha = 0$ in general solution.

Now, in order to bring to light the influence of magnetic field and porous medium on the fluid behavior, Figures 2 and 3 are prepared for a fixed value of the material constant α and increasing values of magnetic and porous parameters M or K. From these figures it clearly results that the fluid velocity $w_C(r, t)$ increases in time but is a decreasing function with respect to the two parameters. Consequently, the fluid flows faster in the absence of a magnetic field or porous medium.

Moreover, as expected, the diagrams of the starting velocity $w_C(r, t)$ tend to overlap with that of its steady component $w_{Cp}(r)$ for increasing values of the time t. In this way the required time to reach the steady or permanent state is obtained. It declines for increasing values of M or K. It means that the steady state is earlier reached in the presence of the magnetic field or porous medium.

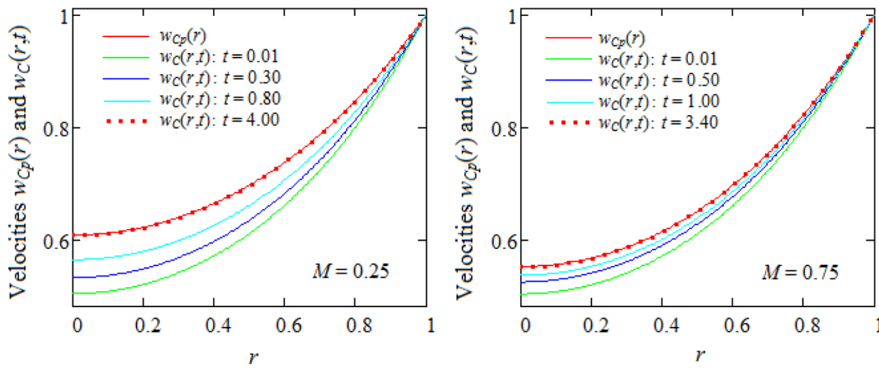

Figure 2. Convergence of starting velocity $w_C(r, t)$ to its steady component $w_{Cp}(r)$ for $\alpha = 0.3$, $K = 2$, $M = 0.25$ or $M = 0.75$ and increasing values of t.

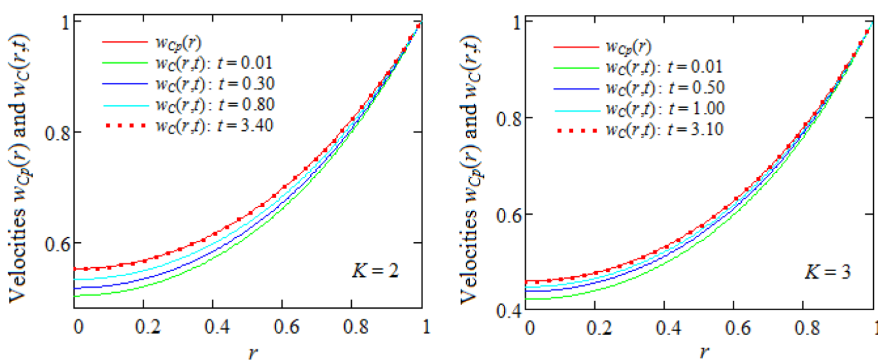

Figure 3. Convergence of starting velocity $w_C(r, t)$ to its steady component $w_{Cp}(r)$ for $\alpha = 0.3$, $M = 0.75$, $K = 2$ or $K = 3$ and increasing values of t.

Finally, we conclude with an important remark concerning the governing equation for shear stress that can be obtained for such motions of ECISGFs. More precisely, eliminating $w(r, t)$ between Eqs. (7) and (8) one finds the following governing equation

$$\frac{\partial\tau(r, t)}{\partial t} = \left(1 + \alpha\frac{\partial}{\partial t}\right)\left[\frac{\partial^2\tau(r, t)}{\partial r^2} + \frac{1}{r}\frac{\partial\tau(r, t)}{\partial r} - \frac{1}{r^2}\tau(r, t)\right]$$

$$- M\tau(r, t) - K\left(1 + \alpha\frac{\partial}{\partial t}\right)\tau(r, t), \tag{22}$$

for the non-trivial shear stress $\tau(r, t)$. This equation, which is first time mentioned here, can be useful for the study of MHD motions of the same fluids through a porous medium in an infinite circular cylinder that applies a shear stress $Sf(t)$ to the fluid. The shear stress corresponding to such motion can be determined following the same way as before while the adequate velocity field can be obtained solving the linear ordinary differential equation (8).

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