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EXTENSIONS OF FUZZY IDEALS IN COUPLED Γ-SEMIRINGS

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Abstract

Let M be a Γ-semiring. In this paper, we introduce the notion of extensions of fuzzy ideals, fuzzy weakly completely prime ideals and fuzzy 3-weakly completely prime ideals of $M \times M$. In particular, we study some of the relationships between fuzzy weakly completely prime ideals, fuzzy 3-weakly prime ideals in terms of the extension of fuzzy ideals of $M \times M$. Our work is inspired by [1].

1. Some New Notions and Notations

Remark 1.1. We assume M is a Γ-semiring as defined in [1].

Definition 1.2. Let M be a Γ-semiring. A fuzzy subset μ of $M \times M$ will be

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called a fuzzy ideal if for all $(x, m), (y, v) \in M \times M$, $\alpha \in \Gamma$, it holds

(a) $\mu(x + y, m + v) \geq \min\{\mu(x, m), \mu(y, v)\}$,

(b) $\mu(x\alpha y, m\alpha v) \geq \max\{\mu(x, m), \mu(y, v)\}$.

Definition 1.3. Let M be a Γ-semiring. A fuzzy ideal f of $M \times M$ will be called a fuzzy k-ideal of $M \times M$ if $f(x, m) \geq \min\{f(x + y, m + v), f(y, v)\}$ for all $(x, m), (y, v) \in M \times M$.

Definition 1.4. Let M be a Γ-semiring, μ be a fuzzy subset of $M \times M$, and $s \in M$. The fuzzy subset $\langle s, \mu\rangle : M \times M \mapsto [0, 1]$, defined by $\langle s, \mu\rangle(x, m) = \mu(s\alpha x, s\alpha m)$ for all $(x, m) \in M \times M$, $\alpha \in \Gamma$, will be called an extension of μ by s.

Definition 1.5. Let M be a Γ-semiring. If μ is a fuzzy subset of $M \times M$, we define $\operatorname{supp} \mu = \{(s, s) \in M \times M \mid \mu(s, s) > 0\}$.

Definition 1.6. Let M be a Γ-semiring, A be a subset of $M \times M$, and $(x, m) \in M \times M$. We define $\langle x, A\rangle = \{(s, s) \in M \times M \mid (x\alpha s, m\alpha s) \in A\}$ for all $\alpha \in \Gamma$.

Definition 1.7. Let M be a Γ-semiring. We say μ is a fuzzy weakly completely prime ideal if $\mu(x_1\alpha x_2, x_1'\alpha x_2') = \max\{\mu(x_1, x_1'), \mu(x_2, x_2')\}$ for all $(x_1, x_1'), (x_2, x_2') \in M$, and $\alpha \in \Gamma$.

Definition 1.8. Let M be a Γ-semiring. A fuzzy ideal μ of $M \times M$ will be called 3-weakly completely prime ideal if for all $(x_1, x_1'), (x_2, x_2'), (x_3, x_3') \in M \times M$, $\alpha \in \Gamma$, we have

$$\begin{aligned}\mu(x_1\alpha x_2\alpha x_3, x_1'\alpha x_2'\alpha x_3') &= \max\{\mu(x_1\alpha x_2, x_1'\alpha x_2'), \mu(x_1\alpha x_3, x_1'\alpha x_3')\}\\ &= \max\{\mu(x_2\alpha x_3, x_2'\alpha x_3'), \mu(x_2\alpha x_1, x_2'\alpha x_1')\}\\ &= \max\{\mu(x_3\alpha x_1, x_3'\alpha x_1'), \mu(x_3\alpha x_2, x_3'\alpha x_2')\}.\end{aligned}$$

2. Some Properties

Proposition 2.1. *Let M be a commutative Γ-semiring. If μ is a fuzzy ideal of $M \times M$, and $s \in M$, then the extension of μ by s is a fuzzy ideal of $M \times M$.*

Proof. Obviously, $\langle s, \mu\rangle$ is a fuzzy subset of $M \times M$. Let $(x, m), (y, v) \in M \times M$, and $\alpha, \beta \in \Gamma$. Observe we have the following

$$\begin{aligned}\langle s, \mu\rangle(x + y, m + v) &= \mu(s\alpha(x + y), s\alpha(m + v)) \\ &= \mu(s\alpha x + s\alpha y, s\alpha m + s\alpha v) \\ &\geq \min\{\mu(s\alpha x, s\alpha m), \mu(s\alpha y, s\alpha v)\} \\ &= \min\{\langle s, \mu\rangle(x, m), \langle s, \mu\rangle(y, v)\}\end{aligned}$$

which implies

$$\langle s, \mu\rangle(x + y, m + v) \geq \min\{\langle s, \mu\rangle(x, m), \langle s, \mu\rangle(y, v)\}.$$

Also

$$\begin{aligned}\langle s, \mu\rangle(x\beta y, m\beta v) &= \mu(s\alpha x\beta y, s\alpha m\beta v) \\ &\geq \mu(s\alpha x, s\alpha m) \\ &= \langle s, \mu\rangle(x, m).\end{aligned}$$

Hence

$$\langle s, \mu\rangle(x\beta y, m\beta v) \geq \langle s, \mu\rangle(y, v)$$

which implies $\langle s, \mu\rangle$ is a fuzzy ideal of $M \times M$.

Proposition 2.2. *Let M be a Γ-semiring. If μ is a fuzzy k-ideal of $M \times M$, and $s \in M$, then the extension of μ by s is a fuzzy k-ideal of $M \times M$.*

Proof. By the previous Proposition, $\langle s, \mu\rangle$ is a fuzzy ideal of $M \times M$. Since μ

is a fuzzy k - ideal of $M \times M$, then

$$\mu(s\alpha x,\ s\alpha m) \geq \min\{\mu(s\alpha x + s\alpha y,\ s\alpha m + s\alpha v),\ \mu(s\alpha y,\ s\alpha v)\}$$

for all $(x, m), (y, v) \in M \times M$, $\alpha \in \Gamma$. Thus,

$$\langle s, \mu\rangle(x, m) \geq \min\{\langle s, \mu\rangle(x + y.\ m + v),\ \langle s, \mu\rangle(y, v)\}$$

for all $(x, m)(y, v) \in M \times M$, $\alpha \in \Gamma$. So $\langle s, \mu\rangle$ is a fuzzy k - ideal of $M \times M$.

Proposition 2.3. *Let M be a Γ-semiring, $s \in M$, and μ be a fuzzy ideal of $M \times M$. Then*

(a) $\mu \subseteq \langle s, \mu\rangle$,

(b) $\left\langle (s\alpha)^{n-1} s,\ \mu\right\rangle \subseteq \left\langle (s\alpha)^{n} s,\ \mu\right\rangle$ *for every natural number* n, $\alpha \in \Gamma$,

(c) *If* $\mu(s, s) > 0$, *then* $supp\langle \mathrm{s}, \mu\rangle = M \times M$.

Proof. Let M be a Γ-semiring, $s \in M$, μ be a fuzzy ideal of $M \times M$, and $\alpha \in \Gamma$.

For (a) Since μ is a fuzzy ideal of $M \times M$, by Proposition 2.1, $\langle \mathrm{s}, \mu\rangle$ is an ideal of $M \times M$. Now

$$\langle s, \mu\rangle(x, m) = \mu(s\alpha x,\ s\alpha m) \geq \mu(x, m)$$

for all $(x, m) \in M \times M$ and α in Γ, which implies $\mu \subseteq \langle s, \mu\rangle$.

For (b) For all natural numbers n and for all $(x, m) \in M \times M$, $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}\left\langle (s\alpha)^{n} s,\ \mu\right\rangle(x, m) &= \mu((s\alpha)^{n} s\beta x,\ (s\alpha)^{n} s\beta m)) \\ &= \mu(s\alpha(s\alpha)^{n-1} s\beta x,\ s\alpha(s\alpha)^{n-1} s\beta m) \\ &\geq \mu((s\alpha)^{n-1} s\beta x,\ (s\alpha)^{n-1} s\beta m)\end{aligned}$$

$$= \left\langle (s\alpha)^{n-1} s,\ \mu \right\rangle (x,\ m)$$

which implies $\left\langle (s\alpha)^{n} s,\ \mu \right\rangle \subseteq \left\langle (s\alpha)^{n-1} s,\ \mu \right\rangle$.

For (c) Let $\mu(s,\ s) > 0,\ \ (x,\ m) \in M \times M$. Now

$$\langle s,\ \mu \rangle (x,\ m) = \mu(s\alpha x,\ s\alpha m) \geq \mu(s,\ s) > 0$$

which implies

$$\langle s,\ \mu \rangle (x,\ m) \geq 0$$

which implies

$$x \in \operatorname{supp}\langle s,\ \mu \rangle$$

for all $(x,\ m) \in M \times M$. Thus

$$M \subseteq \operatorname{supp}\langle s,\ \mu \rangle.$$

By Definition 1.5,

$$\operatorname{supp}\langle s,\ \mu \rangle \subseteq M$$

which implies

$$M = \operatorname{supp}\langle s,\ \mu \rangle.$$

Proposition 2.4. *Let M be a Γ-semiring, A be a subset of $M \times M$, and $(x,\ m) \in M \times M$. Then*

$$\langle s,\ \lambda_A \rangle = \lambda_{\langle s,\ A \rangle}$$

for all $(s,\ s) \in M \times M$, where λ_A denotes the characteristic function of $A \subseteq M \times M$.

Proof. Let $(x,\ m) \in M \times M,\ \ \alpha \in \Gamma$. Observe

$\langle s,\ \lambda_A\rangle(x,\ m) = \lambda_A(s\alpha x,\ s\alpha m) = 1$ or $0.$

(a) If $\langle s\ \alpha_A\rangle\langle x,\ m\rangle = 1,$ then $\alpha_A(s\alpha x,\ s\alpha m) = 1$ for all $(s,\ s)(x,\ m) \in M \times M,$ $\alpha \in \Gamma,$ which implies

$(s\alpha x,\ s\alpha m) \in A$

for all $(s,\ s)(x,\ m) \in M \times M,$ $\alpha \in \Gamma,$ which implies

$(x,\ m) \in \langle s,\ A\rangle$

which implies

$\alpha_{\langle s,\ A\rangle}(x,\ m) = 1.$

(b) If $\langle s,\ \lambda_A\rangle(x,\ m) = 0,$ then $\lambda_A(s\alpha x,\ s\alpha m) = 0$ for all $(s,\ s) \in M \times M,$ and $\alpha \in \Gamma,$ which implies

$(s\alpha x,\ s\alpha m) \notin A$

which implies

$(x,\ m) \notin \langle s,\ A\rangle$

which implies

$\lambda_{\langle s,\ A\rangle}(x,\ m) = 0$

which implies

$\langle s,\ \lambda_A\rangle = \lambda_{\langle s,\ A\rangle}.$

Theorem 2.5. *Let* M *be a* Γ*-semiring, and* μ *be a fuzzy weakly completely prime ideal of* $M \times M$*. If* $(x,\ m) \in M \times M$ *is such that* $\mu(x,\ m) =$ $inf_{(y,\ v) \in M \times M}\mu(y,\ v),$ *then* $\langle (x,\ m),\ \mu\rangle = \mu.$

Proof. Let $(y,\ v) \in M \times M.$ Clearly, $\inf_{(y,\ v) \in M \times M} \mu(y,\ v)$ is in $[0,\ 1].$ Thus

$\langle (x, m), \mu\rangle\langle y, v\rangle = \mu(x\alpha y, m\alpha v)$. Since μ is a fuzzy weakly completely prime ideal of $M \times M$, then

$$\mu(x\alpha y, m\alpha v) = \max\{\mu(x, m), \mu(y, v)\}.$$

Thus either

$$\mu(x\alpha y, m\alpha v) = \mu(x, m)$$

or

$$\mu(x\alpha y, m\alpha v) = \mu(y, v).$$

Suppose $\mu(x\alpha y, m\alpha v) \neq \mu(y, v)$. Then $\mu(x\alpha y, m\alpha v) = \mu(x, m)$. Since μ is a fuzzy ideal of $M \times M$, by Proposition 2.3, $\mu \subseteq \langle (x, m), \mu\rangle$ which implies

$$\mu(y, v) \leq \langle (x, m), \mu\rangle(y, v)$$

which implies

$$\mu(y, v) \leq \mu(x\alpha y, m\alpha v)$$

which implies

$$\mu(y, v) \leq \mu(x, m).$$

Also since $\mu(x, m) = \inf_{(y, v) \in M \times M} \mu(y, v)$, this implies

$$\mu(x, m) \leq \mu(y, v).$$

From $\mu(x, m) = \mu(y, v)$ and $\mu(x\alpha y, m\alpha v) = \mu(y, v)$, we get a contradiction. It follows that $\mu(x\alpha y, m\alpha v) = \mu(y, v)$ which implies

$$\langle x, \mu\rangle(y, v) = \mu(y, v)$$

for all $(y, v) \in M \times M$. Hence $\langle (x, m), \mu\rangle = \mu$.

Theorem 2.6. *Let M be a commutative Γ-semiring, μ be a fuzzy subset of*

$M \times M$ such that $\langle (s, s'), \mu \rangle = \mu$ for every $(s, s') \in M \times M$, then μ is constant.

Proof. Let M be a commutative Γ-semiring, μ be a fuzzy subset of $M \times M$ such that $\langle (s, s'), \mu \rangle = \mu$ for every $(s, s') \in M \times M$ and $(x, m)(y, v) \in M \times M$. Then $\langle (x, m), \mu \rangle = \mu$ and $\langle (y, v), \mu \rangle = \mu$. Thus

$$\begin{aligned}\mu(y, v) &= \langle (x, m), \mu \rangle (y, v)\\ &= \mu(x\alpha y, m\alpha v)\\ &= \mu(y\alpha x, v\alpha m)\\ &= \langle (y, v), \mu \rangle (x, m)\\ &= \mu(x, m)\end{aligned}$$

which implies μ is constant.

Theorem 2.7. *Let M be a Γ-semiring, μ be a fuzzy ideal of $M \times M$, $Im\mu = \{1, \alpha\}$. Suppose $\langle (y, v), \mu \rangle = \mu$, for all those $(y, v) \in M \times M$, for which $\mu(y, v) = \alpha$. Then μ is a fuzzy weakly completely prime ideal of $M \times M$.*

Proof. Let $(x_1, x_1'), (x_2, x_2') \in M \times M$. Since μ is a fuzzy ideal of $M \times M$, $\mu(x_1\beta x_2, x_1'\beta x_2') \geq \mu(x_1, x_1')$ and $\mu(x_1\beta x_2, x_1'\beta x_2') \geq \mu(x_2, x_2')$.

Case 1. $\mu(x_1\beta x_2, x_1'\beta x_2') = \mu(x_1, x_1')$

Then $\mu(x_1, x_1') \geq \mu(x_2, x_2')$ which implies

$\max \mu(x_1, x_1'), \mu(x_2, x_2') = \mu(x_1, x_1') = \mu(x_1\beta x_2, x_1'\beta x_2')$.

Case 2. $\mu(x_1\beta x_2, x_1'\beta x_2') \neq \mu(x_2, x_2')$

Then $\mu(x_1, x_1')$ is not maximal of $\mu(M \times M)$, otherwise $\mu(x_1, x_1') = 1 = \mu(x_1\beta x_2, x_1'\beta x_2')$ which is a contradiction. Thus, $\mu(x_1, x_1') = \alpha$, and by hypothesis $\langle \mu(x_1, x_1'), \mu \rangle = \mu$ which implies

$$\langle\mu(x_1, x_1'), \mu\rangle\mu(x_2, x_2') = \mu(x_2, x_2')$$

which implies

$$\mu(x_1\beta x_2, x_1'\beta x_2') = \mu(x_2, x_2')$$

which implies

$$\mu(x_2, x_2') = \mu(x_1\beta x_2, x_1'\beta x_2') \geq \mu(x_1, x_1')$$

which implies

$$\mu(x_1\beta x_2, x_1'\beta x_2') = \max \mu(x_1, x_1'), \mu(x_2, x_2').$$

It follows that μ is a fuzzy weakly completely prime ideal of $M \times M$.

Theorem 2.8. *Let* M *be a commutative* Γ*-semiring and* μ *be a fuzzy ideal of* $M \times M$. *Then* μ *is a fuzzy* 3-*weakly prime ideal of* $M \times M$ *iff any extension of* μ *by* (x, m), *where* $(x, m) \in M \times M$, *is a fuzzy weakly completely prime ideal of* $M \times M$.

Proof. Suppose μ is a fuzzy 3-weakly completely prime ideal of Γ-semiring M and $(x, m) \in M$, $\alpha \in \Gamma$. Observe

$$\begin{aligned}\langle(x, m), \mu\rangle &= \mu(x\alpha x_1\alpha x_2, m\alpha x_1'\alpha x_2') \\ &= \max\{\mu(x\alpha x_1, m\alpha x_1'), \mu(x\alpha x_2, m\alpha x_2')\} \\ &= \max\{\langle(x, m), \mu\rangle(x_1, x_1'), \langle(x, m), \mu\rangle(x_2, x_2')\}\end{aligned}$$

for all $(x_1, x_1'), (x_2, x_2') \in M \times M$, $\alpha \in \Gamma$. Thus $\langle(x, m), \mu\rangle$ is a fuzzy weakly completely prime fuzzy ideal of $M \times M$, for every $(x, m) \in M \times M$. Conversely, suppose $\langle(x, m), \mu\rangle$ is a fuzzy weakly completely prime ideal of $M \times M$, for every $(x, m) \in M \times M$. Observe we have the following

$$\mu(x_1\alpha x_2\alpha x_3, x_1'\alpha x_2'\alpha x_3') = \langle(x, m), \mu\rangle(x_2\alpha x_3, x_2'\alpha x_3')$$

$$= \max\{\langle(x_1, x_1'), \mu\rangle(x_2, x_2'), \langle(x_1, x_1'), \mu\rangle(x_3, x_3')\}$$

$$= \max\{\mu(x_1\alpha x_2, x_1'\alpha x_2'), \mu(x_1\alpha x_3, x_1'\alpha x_3')\}$$

and since μ is commutative, then by Defintion 1.8, μ is a fuzzy 3-weakly completely prime ideal of $M \times M$.

3. Open Problem

To our knowledge the following is unsolved:

Conjecture 3.1. Let M be a commutative Γ-semiring, and μ be a fuzzy ideal of $M \times M$. If μ is a weakly completely prime ideal of $M \times M$, then μ is a fuzzy 3-weakly completely prime ideal of $M \times M$.

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