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ELECTROMAGNETIC FIELD IN 3-DIMENSIONAL COMPACT HYPERBOLIC MANIFOLDS

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Abstract

Compact three dimensional spaces of constant curvature without border can be considered as cavities for free electromagnetic fields. For space-times of the form: $t \otimes$ *static space*, this note shows that the electromagnetic field spectrum is the same as the Laplacian eigenvalue spectrum of a scalar field in those spaces.

Introduction

An attempt to compute numerically the first Laplacian eigenvalues of a scalar field in 3-dimensional constant curvature compact hyperbolic manifolds without border, has been presented in [1]. These geometries can be considered as cavities for free electromagnetic fields. The goal of the present note is to extend the results obtained in the scalar field case to the free electromagnetic field and to show that the eigenvalue spectra

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are the same.

The space manifolds considered here, which are called M, are isometric to the quotient H^n/Γ, where H^n is the n-dimensional constant curvature hyperbolic space (Appendix A), and Γ is a group of isometries of H^n called the covering group in the following (in this note $n = 3$). For a complete study of constant curvature spaces, see [2]. In this note, the space-times in which the fields evolve are of the form: $V = t \otimes H^3$, $V = t \otimes M$, where t is the time coordinate.

In the case of the scalar field there are, at least, two ways to compute the first Laplacian eigenvalues.

– Find a set of common eigenvectors of the Laplacian Δ in H^3 and one generator of the covering group Γ, which will be named γ^0, then expand the eigenvectors of the Laplacian in M on this set, and then impose the other periodicity constraints.

– Choose a Γ periodic test function in H^3 and use the Rayleigh theorem [3] to find bounds on the eigenvalues.

Another numerical method has been presented in [4].

The first method was chosen in [1]. It could be directly applied to the case of vector fields but this becomes complicated. The main part of this note shows that the Laplacian eigenvalue spectra of a scalar field and a free electromagnetic field are the same without calculating these eigenvalues. Once the eigenvalues are known it is easier to compute numerically the eigenvectors. The necessary elements are provided in appendices, but no attempt has been done to compute the eigenvectors explicitly.

The identity of the spectra is shown in Sections 2, 3 and 4. The technique used is to express scalar and vector fields as tensors built from spinors. Appendix A shows that, using cylindrical coordinates, the γ^0 periodicity condition are the same for scalar, vector or spinor fields.

The first section sets the notations and recalls very briefly some basic geometrical equations. Some technical calculations have been gathered in Appendix B. Appendix C, although a little bit long for our purpose, discusses the advantages of considering the second order Dirac equation. At last, Appendix D provides solutions to the free electromagnetic field equations in $V = t \otimes H^3$ which can be used to build eigenvectors in $M = H^3 / \Gamma$, and discussed the existence of "plane wave" solutions.

1. Notations

The space-time coordinates $\{x^\alpha\}$ of a point x are labelled with Greek letters: $\alpha, \beta, \gamma \ldots, 0 \le \alpha, \beta, \gamma, \ldots < n$. The time coordinate is: x^0. The vectors of the local natural frame are written $\overrightarrow{e_\alpha}, \overrightarrow{e_\beta}, \ldots$. When tensors are expressed with respect to local orthonormal frames they are labelled with Latin letters: $a, b, c \ldots$. The orthonormal local frame basis vectors are called $\overrightarrow{h_a}$, and we set $\overrightarrow{h_a} = h_a^\alpha \overrightarrow{e_\alpha}$. The metric tensor is $g_{\alpha\beta}$, and $g^{\alpha\beta}$ is its inverse. The determinant of the metric tensor is called g. The signature of the metric is: $(+ - - -)$. In the case of local orthonormal frames, the metric tensor is written: η_{ab} and its diagonal terms are: $\eta_{aa} = (+1, -1, -1, -1)$. Latin indices are lowered with η_{ab} and raised with the inverse tensor η^{ab}.

In the neighbourhood of a given point, the local coordinates, with respect to the local orthonormal frame attached to this point, are given by the 1-forms: $\omega^a = h_\alpha^a \, dx^\alpha$, which satisfy the structure equations:

$$d\omega^a + \omega^a_{\ .b} \wedge \omega^b = \sum{}^a, \tag{1.1}$$

where: $\omega^a_{\ .b} = \omega^a_{\ .b\gamma} dx^\gamma$ are the connexion 1-forms and $\sum^a$ is the torsion

2-form. We shall also write $\omega^{a}_{.\,b} = \omega^{a}_{.\,bc}\omega^{c} \leftrightarrow \omega^{a}_{.\,bc} = \omega^{a}_{.\,b\gamma}h^{\gamma}_{c}$. The connexion 1-forms are related to the connexion coefficients by

$$\omega^{a}_{.\,b\gamma} = \Gamma^{\alpha}_{.\beta\gamma}h^{a}_{\alpha}h^{\beta}_{b} + h^{a}_{\delta}\partial_{\gamma}h^{\delta}_{b}.$$

The connexion coefficients are the sum of two terms:

$$\Gamma^{\alpha}_{.\beta\gamma} = \tilde{\Gamma}^{\alpha}_{.\beta\gamma} + \overline{S}^{\alpha}_{.\beta\gamma},$$

where the first term on the right is the Christoffel symbol and the second is the contorsion tensor. The contorsion is anti symmetric with respect to the two first indices: $\overline{S}_{\alpha\beta\gamma} + \overline{S}_{\beta\alpha\gamma} = 0$. The torsion tensor is:

$$S^{\alpha}_{.\beta\gamma} = \frac{1}{2}(\Gamma^{\alpha}_{.\beta\gamma} - \Gamma^{\alpha}_{.\gamma\beta}) = \frac{1}{2}(\overline{S}^{\alpha}_{.\beta\gamma} - \overline{S}^{\alpha}_{.\gamma\beta}),$$

and inversely: $\overline{S}^{\alpha}_{.\beta\gamma} = S^{\alpha}_{.\beta\gamma} - S^{\alpha}_{\beta.\gamma} - S^{\alpha}_{\gamma.\beta}$; $S^{\alpha}_{\beta\cdot\gamma} = g^{\alpha\delta}S_{\beta\delta\gamma}$.

The torsion 2-form is: $\sum^{a} = \sum^{a}_{.\,bc\cdot}\omega^{b} \wedge \omega^{c} = -\,h^{a}_{\alpha}S^{\alpha}_{.\beta\gamma}dx^{\beta} \wedge dx^{\gamma}$.

We set: $\tilde{\Gamma}^{a}_{.\,b\gamma} = \tilde{\Gamma}^{\alpha}_{.\beta\gamma}h^{a}_{\alpha}h^{\beta}_{b} + h^{a}_{\delta}\partial_{\gamma}h^{\delta}_{b}$.

The curvature 2-form is defined by

$$\Omega^{a}_{.\,b} = d\omega^{a}_{.\,b} + \omega^{a}_{.\,c} \wedge \omega^{c}_{.\,b} = R^{a}_{.\,bcd}\omega^{c} \wedge \omega^{d}.$$

2. Outline

In this note, the electromagnetic field, which will be noted A^{α}, is studied in static four dimensional space-times of the form: $V = t \otimes H^{3}$, $V = t \otimes M$. In H^{3}, elementary rigid motions, which are not symmetries, are either rotations around an axis or transvections (which generalize translations) along a geodesic called base geodesic. Each element of the covering group is the product of a rotation and a transvection having the same axis. Isotropic spaces are symmetric spaces and have no torsion. This applies to the spatial part only, but in the following calculations, the space-time torsion is set to $S^{\alpha}_{.\beta\gamma} = 0$. However, in Appendices B and C torsion has been re-introduced to get more general relations.

Cosmology studies the evolution of universes whose unperturbed metric is spatially isotropic and of the form [5]: $ds^2 = a^2(\eta)[d\eta^2 + \gamma_{ij}dx^i dx^j]$, where η is called the conformal time and the indices i, j correspond to spatial coordinates. For these cosmological metrics, the equations of motion of scalar and vector fields show that the technique of variable separation can be used (separation of the conformal time from the spatial coordinates) if these fields are expanded on the eigenvectors of the 3-dimensional spatial Laplacian operator ${}^3\Delta$. This justifies the form of the space-times V chosen above.

The most direct approach would be to proceed as in [1], that is to say: find solutions of the electromagnetic field equations in $V = t \otimes H^3$ which are also eigenvectors of one generator of Γ, and then impose the periodicity conditions corresponding to the other elements of the covering group. As in [1], it is difficult to avoid using numerical method, which is not satisfactory. Appendix A shows why cylindrical coordinates are well suited to the search of solutions for vector and spinor fields.

Therefore we shall proceed as follows.

Given a set of tensor fields satisfying some relations between them, it was shown in [6] that there exist a spinor field ψ such that they can be written as $\bar{\psi}\gamma^h\psi$ with: $\gamma^h = \gamma^{\alpha_1}\gamma^{\alpha_2} \ldots \gamma^{\alpha_h}$, where all the γ^{α_i} $(1 \le i \le h)$ are different. In the following, we shall assume that scalar fields and vector fields can be written respectively (up to a multiplicative constant) as product of spinor fields of the form: $\bar{\psi}\varphi$ and $\bar{\psi}\gamma^{\alpha}\varphi$. One can show that interaction tensors of the form $\bar{\psi}\gamma^h\varphi$ satisfy also the relations (1.3) and

(1.2) of [6]. In [6], the construction does not require that the spinor satisfies the Dirac equation. In Sections 3 and 4, it is assumed that ψ and φ satisfy the second order Dirac equation (Appendix C), and we shall look at the constraints on the spinors resulting from the scalar and electromagnetic field equations and show that such a construction is possible by comparing the number of constraints and the number of degrees of freedom.

If the scalar field $S = \overline{\psi}\varphi$ is an eigenvector of the Laplacian in H^3 the spinors ψ and φ must satisfy some constraints. This is shown in Section 3. The vector field $a^\alpha = \overline{\psi}\gamma^\alpha\varphi$ can not represent directly the electromagnetic field A^α. A gauge change must be applied in order to satisfy the field equations and the Lorenz condition $D_\alpha A^\alpha = 0$. In Section 4, we show that if: $A^\alpha = a^\alpha + g^{\alpha\beta}\partial_\beta f$ with $f = \mu\overline{\psi}\varphi$ and if $\overline{\psi}\varphi$ is an eigenfunction of the 3-dimensional space Laplacian, then the electromagnetic field equations and the Lorenz condition can be satisfied and that they represent 4 constraints. In Section 4, it is shown that the total number of constraints is less than the number of degrees of freedom, therefore it is possible to find ψ and φ such that the field equations are satisfied.

If the scalar field S (eigenvector of the spatial Laplacian) satisfies the periodicity conditions in H^3 for some frequencies (eigenvalues), then this must also be true for the spinors ψ and φ. Therefore, this will also be true for the electromagnetic field A^α, with the same frequencies, and conversely, showing that the scalar and the electromagnetic fields have the same spectra if $M = H^3/\Gamma$.

3. The Scalar Field

We first consider a free scalar field S whose Lagrangian is

$$L = \partial_\alpha S^+ g^{\alpha\beta} \partial_\beta S - m^2 S^+ S.$$

The Euler-Lagrange equations are: $\Delta S = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta S) = -m^2 S.$

We assume that S can be written as the product of two spinors

$$S = \bar{\psi}\varphi, \tag{3.1}$$

where ψ and φ satisfy the second order Dirac equation (C.16) with masses m_1 and m_2, respectively.

Note that (3.1) is not the only scalar that one can build from two spinor fields. For instance we could have considered $S = a\bar{\psi}\varphi + bD_\alpha\psi$ $g^{\alpha\beta} D_\beta \varphi$, where a, b are constant parameters, but this brings no simplification and no cancellation in the calculations.

Equations (B.4) and (C.16) give

$$\Delta S = D_\alpha (g^{\alpha\beta} D_\beta S) = [\frac{R}{2} - m_1{}^2 - m_2{}^2]\bar{\psi}\varphi + 2g^{\alpha\beta}\overline{D_\alpha\psi} D_\beta \varphi. \tag{3.2}$$

If $\psi, \varphi \sim e^{i\omega t}$, S is independent of t. Then we chose $\psi \sim e^{\pm i\omega t} \varphi \sim e^{\mp i\omega t}$, then

$$^3\Delta(\bar{\psi}\varphi) = -4\omega^2 \bar{\psi}\varphi + [m_1{}^2 + m_2{}^2 - \frac{R}{2}]\bar{\psi}\varphi - 2g^{\alpha\beta}\overline{D_\alpha\psi} D_\beta\varphi, \tag{3.3}$$

where $^3\Delta$ is the 3-dimensional Laplacian operator for M or H^3.

We are interested by scalar fields which are eigenfunctions of the Laplacian on the manifold M : $^3\Delta f = -\lambda f$, then one has the condition

$$2g^{\alpha\beta}\overline{D_\alpha\psi} D_\beta\varphi \sim \bar{\psi}\varphi. \tag{3.4}$$

Remark. In [1] and in many publications, the eigenvalue problem is written as: $^3\Delta f = -(1 + \beta^2)f$ which is convenient if spherical coordinates are used.

If M is a manifold without border, then for any function h (sufficiently well behaved):

$$\int_M ({}^3\Delta f)^+ h dV = -\lambda^+ \int_M f^+ h dV = \int_M \partial_\alpha(\sqrt{g} g^{\alpha\beta}\partial_\beta f^+) h d^3x$$

$$= \int_M \partial_\alpha(\sqrt{g} g^{\alpha\beta}\partial_\beta f^+ h) d^3x - \int_M g^{\alpha\beta}\partial_\beta f^+ \partial_\alpha h \sqrt{g} d^3x.$$

The first integral is zero since M is assumed without border. If we set $h = Ct$, then for $\lambda \neq 0$, one has the constraint $\int_M f dV = 0$.

Is this condition compatible with the hypothesis (3.1)? Let us consider: $\overline{\psi} D^2 \varphi$, where $D = \gamma^\alpha D_\alpha$ is defined in Appendix C. Using (B.3):

$$\overline{\psi} D^2 \varphi = \overline{\psi}\gamma^\beta D_\beta D\varphi = D_\beta(\overline{\psi}\gamma^\beta D\varphi) - \overline{D_\beta \psi}\gamma^\beta D\varphi.$$

The first term of the right member is a divergence, therefore, if $\partial M = 0$: $\int \overline{\psi} D^2 \varphi dV = -\int \overline{D\psi} D\varphi dV$. If φ is a solution of the second order Dirac equation $\int \overline{D\psi} D\varphi dV = {m_2}^2 \int \overline{\psi}\varphi dV$.

If $\psi \in D_\pm$ (see Appendix C) and $\varphi \in D_\mp$, then $\int_M \overline{\psi}\varphi dV = 0$ which is consistent with the above constraint.

4. The Vector and Electromagnetic Fields

The equation of motion of a vector field a^α with mass m can be deduced from the Lagrangian $L_V = g^{\alpha\beta} g_{\mu\nu} D_\alpha a^\mu D_\beta a^\nu - m^2 a_\mu a^\mu + L_S$, where L_S represents source terms. There are other possible Lagrangians. The Euler-Lagrange equations are

$$g^{\alpha\beta} D_\alpha D_\beta a^\mu - R^\mu_{\ .\beta} a^\beta = -m^2 a^\mu + J^\mu, \tag{4.1}$$

where J^{μ} represents source terms. In the following we consider only free fields, then $J^{\mu} = 0$. The Lagrangian of the electromagnetic field A^{α} is

$$L_A = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + L_S,$$

where $F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} = D_{\alpha} A_{\beta} - D_{\beta} A_{\alpha}$ (the last equality is true when there is no torsion). The corresponding equations of motion are

$$g^{\alpha\beta} D_{\alpha} D_{\beta} A^{\mu} - R^{\mu}_{.\beta} A^{\beta} - g^{\alpha\mu} D_{\alpha} D_{\gamma} A^{\gamma} = J^{\mu} (= 0). \tag{4.2}$$

The gauge freedom is used to impose the Lorenz condition $D_{\gamma} A^{\gamma} = 0$.

In the differential form formalism the Laplacian operator is defined as $\Delta \simeq d\delta + \delta d$, where: δ is the adjoint of the differentiation operator d. Applying this definition to the 1-form $\omega = a_{\alpha} dx^{\alpha}$ gives

$$-\Delta a^{\alpha} = g^{\gamma\beta} \widetilde{D}_{\gamma} \widetilde{D}_{\beta} a^{\alpha} - \widetilde{R}^{\alpha}_{.\beta} a^{\beta},$$

where: $\widetilde{D}_{\beta}$ and $\widetilde{R}^{\alpha}_{.\beta}$ are the covariant derivative and the Ricci tensor computed with the Christoffel symbols only instead of the connexion coefficients, as if there were no torsion, although this expression is valid also with torsion. Note that: $\Delta d = d\Delta$ and $\Delta\delta = \delta\Delta$, two properties which will be used later.

We assume that one can write $a^{\alpha} = \overline{\psi}\gamma^{\alpha}\varphi$ where, as for the scalar field, ψ and φ satisfy the second order Dirac equation with masses m_1 and m_2, respectively. Then, using (B.4) and (C.16), one has:

$$\Delta a^{\mu} = \left[\frac{R}{2} - m_1^2 - m_2^2 \right] \overline{\psi}\gamma^{\mu}\varphi - R^{\mu}_{.\nu} \overline{\psi}\gamma^{\nu}\varphi + 2g^{\alpha\beta}\, \overline{D_{\alpha}\psi}\, \gamma^{\mu} D_{\beta}\varphi. \tag{4.3}$$

If one wants $a^{\alpha} = \overline{\psi}\gamma^{\alpha}\varphi$ to represent the electromagnetic field, the Lorenz condition must be satisfied. Using the same calculation as for (B.7), we have:

$$D_\alpha a^\alpha = D_\alpha(\bar{\psi}\gamma^\alpha\varphi) = \overline{D\psi}\,\varphi + \bar{\psi}D\varphi.$$

If, as in the scalar case, $\psi \in D_\pm$ and $\varphi \in D_\mp$, then

$$D_\alpha a^\alpha = \mp i(m_1 + m_2)\bar{\psi}\varphi. \tag{4.4}$$

It is now assumed that the electromagnetic field A^α is equivalent to the vector field a^α up to a gauge change: $A^\alpha = a^\alpha + g^{\alpha\beta}\partial_\beta f$ whence

$$D_\alpha A^\alpha = D_\alpha a^\alpha + \Delta f = \mp i(m_1 + m_2)\bar{\psi}\varphi + \Delta f.$$

If: $f = \mu\bar{\psi}\varphi$ where μ is a constant coefficient, the Lorenz condition looks like an eigenvalue condition. Let us assume that $\Delta(\bar{\psi}\varphi) = \lambda\bar{\psi}\varphi$, then the Lorenz condition is satisfied if $\mu\lambda = \pm i(m_1 + m_2)$, and the electromagnetic field is

$$A^\alpha = \bar{\psi}\gamma^\alpha\varphi + g^{\alpha\beta}\mu\partial_\beta(\bar{\psi}\varphi). \tag{4.5}$$

The free electromagnetic field equation is then such that: $\Delta a^\mu + \Delta(g^{\mu\beta}\partial_\beta f) = 0$. As seen above the Laplacian of a gradient field is equal to the gradient of the Laplacian, then using (4.3), the field equations are

$$\Delta A^\mu = [\frac{R}{2} - m_1^2 - m_2^2]\bar{\psi}\gamma^\mu\varphi - R^\mu_{.\nu}\bar{\psi}\gamma^\nu\varphi$$
$$+ 2g^{\alpha\beta}\overline{D_\alpha\psi}\,\gamma^\mu D_\beta\varphi + g^{\mu\beta}\mu\partial_\beta(\Delta(\bar{\psi}\varphi)) = J^\mu. \tag{4.6}$$

These equations represents 4 constraints. Are they independent and are they compatible with the Lorenz condition? Since the divergence operator commutes with the Laplacian $(\Delta\delta = \delta\Delta)$, we expect $D_\mu J^\mu = D_\mu\Delta A^\mu = \Delta D_\mu A^\mu = 0$. This is now checked directly.

We first compute $D_\mu J_U^\mu$, where $J_U{}^\mu = g^{\alpha\beta}\overline{D_\alpha\psi}\,\gamma^\mu D_\beta\varphi$. Using (B.7) and (B.9), one has

$$D_\mu J_U{}^\mu = g^{\alpha\beta}[\overline{D_T D_\alpha \psi} D_\beta \varphi + \overline{D_\alpha \psi} D_T D_\beta \varphi],$$

$$D_\mu J_U{}^\mu = \frac{1}{4} g^{\alpha\beta} R_{cd\gamma\alpha}(\bar{\psi}\gamma^{dc}\gamma^\gamma D_\beta \varphi + \overline{D_\beta \psi}\gamma^\gamma \gamma^{cd}\varphi)$$

$$+ g^{\alpha\beta}[\overline{D_\alpha D \psi} D_\beta \varphi + \overline{D_\alpha \psi} D_\beta D \varphi].$$

With $\gamma^{cd}\gamma^e = \eta^{de}\gamma^c - \eta^{ce}\gamma^d + \gamma^{cde}$ and $\gamma^e\gamma^{cd} = \eta^{ce}\gamma^d - \eta^{de}\gamma^c + \gamma^{ecd}$ and using the Bianchi identities of the first kind in the first bracket, it remains

$$D_\mu J_U{}^\mu = \frac{1}{2} R_c^\beta (\bar{\psi}\gamma^c D_\beta \varphi + \overline{D_\beta \psi}\gamma^c \varphi) + g^{\alpha\beta}[\overline{D_\alpha D \psi} D_\beta \varphi + \overline{D_\alpha \psi} D_\beta D \varphi].$$

Finally, with (B.3)

$$D_\mu J_U{}^\mu = \frac{1}{2} R_c^\beta D_\beta (\bar{\psi}\gamma^c \varphi) + g^{\alpha\beta}[\overline{D_\alpha D \psi} D_\beta \varphi + \overline{D_\alpha \psi} D_\beta D \varphi],$$

where, according to our hypotheses $R^\mu_{\cdot\nu} \sim \delta^\mu_\nu$.

Replacing this term in the divergence of (4.6) and using (4.4) gives

$$D_\mu J^\mu = [\frac{R}{2} - m_1^2 - m_2^2](\mp) i (m_1 + m_2) \bar{\psi}\varphi$$

$$+ 2g^{\alpha\beta}[\overline{D_\alpha D \psi} D_\beta \varphi + \overline{D_\alpha \psi} D_\beta D \varphi] + \mu D_\mu (g^{\mu\beta}\partial_\beta(\Delta(\bar{\psi}\varphi))).$$

With $\psi \in D_\pm$ and $\varphi \in D_\mp$ and with

$$D_\mu (g^{\mu\beta}\partial_\beta(\Delta(\bar{\psi}\varphi))) = \lambda D_\mu (g^{\mu\beta}\partial_\beta(\bar{\psi}\varphi)) = \lambda\Delta(\bar{\psi}\varphi),$$

we obtain

$$D_\mu J^\mu = (\mp) i (m_1 + m_2)([\frac{R}{2} - m_1{}^2 - m_2^2]\bar{\psi}\varphi + 2g^{\alpha\beta}\overline{D_\alpha \psi} D_\beta \varphi - \Delta(\bar{\psi}\varphi)),$$

which, with condition (3.2) gives $D_\mu J^\mu = 0$.

Therefore, the four Equations (4.6) are compatible with the Lorenz condition and are in fact three independent equations only.

The constraints used are now recalled. The spinor fields ψ and φ are such that $S = \bar{\psi}\varphi$, and are solutions of the Dirac equation. This is $1 + 4 \times 2$ constraints. Then the condition (3.4) is necessary if S is an eigenfunction of the spatial Laplacian. The vector (4.5) must satisfy the four Equations (4.6) or three of them plus the Lorenz condition, that is four conditions. The total amounts to 14 conditions, the two spinors representing 16 real functions.

5. Appendix A: Coordinates and Transport of Local Frames

5.1. Hyperbolic spaces

The hyperbolic n dimensional space H^n is defined as the upper part of the sphere of radius $\sqrt{|K|}$ in the Minkowski space M^{n+1}. More precisely, if $\{x^\alpha\}$ are Cartesian coordinates in M^{n+1} with origin O_M, H^n is the surface defined by:

$$\sum_{\alpha=0}^{n-1} x^\alpha x^\alpha - x^n x^n = K,$$

where $K < 0$ and: $x^n \geq \sqrt{|K|}$. We set $R = \sqrt{|K|}$.

The correspondance between the spherical coordinates (χ, θ, φ) of H^3, where (θ, φ) are the usual polar angles with respect to some local orthonormal frame *Oxyz* and the coordinates of M^4 is

$$x^0 = Rsh\chi c, \quad c = \cos(\theta), \quad s = \sin(\theta),$$

$$x^1 = Rsh\chi sc_\varphi; \quad c_\varphi = \cos(\varphi), \quad s_\varphi = \sin(\varphi), \tag{A.1}$$

$$x^2 = Rsh\chi ss_\varphi, \quad x^3 = Rch\chi,$$

$$\chi \geq 0, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

Then the linear element of H^3 is

$$\begin{aligned} ds^2 &= (dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2 \\ &= R^2\left[d\chi^2 + sh^2\chi((d\theta)^2 + s^2(d\varphi)^2)\right]. \end{aligned} \tag{A.2}$$

The coordinates (χ, θ, φ) are the Riemann normal (spherical) coordinates with origin at $\chi = 0$, which corresponds to the point $(0, 0, 0, R)$ in M^4. The curvature tensor is $R_{\alpha\beta\gamma\delta} = -\frac{1}{R^2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$, the Ricci tensor is $R_{\alpha\beta} = -\frac{2}{R^2}g_{\alpha\beta}$, and the scalar curvature $R_H = -\frac{6}{R^2}$. R is a scale factor, in the following it is set to 1.

H^n, which is a space of constant curvature, is a symmetric space. A transvection in a symmetric space is an isometry which generalizes the notion of translation in Euclidean space. It is defined as the product of two successive symmetries with respect to two different points *A* and *B*. The geodesic going through these two points is invariant and is called the base geodesic. In H^3 one can perform a rotation around this base geodesic, it commutes with the transvection and the base geodesic is invariant. The base geodesic of a transvection γ is given by the intersection of the invariant plane, associated to the real eigenvalues of the $SO(3, 1)$ element representing γ in M^4, with H^3.

The elements of the covering group Γ are screw motions. A screw motion is the product of a transvection by a rotation around the base geodesic of the transvection.

Since H^3 is isotropic, it is always possible to chose a particular generator of Γ such that its base geodesic defines the *Oz* axis of the spherical coordinates. This particular generator is named γ^0 in the present note.

The Laplacian operator acting on a form ω is defined as $\Delta \simeq d\delta + \delta d$, where δ is the adjoint of the differentiation operator d. ω of degree 0 corresponds to the case of a scalar field, and ω of degree 1 to the case of a vector field. The Laplacian operator commutes with all the isometries. It is then possible to find a common set of eigenvectors to Δ and one of the generator of Γ (only one, because the generators of Γ do not, a priori, commute between themselves). The action of rotations around the *Oz* and *Ox* axis, and transvections along the *Oz* axis, on the vectors of local frames, has been described in ([7], Appendix A).

If we call L the length of the transvection, which is twice the distance between A and B, a point p whose spherical coordinates are (χ, θ, φ) is transformed, by a transvection along *Oz*, into a point q of coordinates $(\chi_q, \theta_q, \varphi_q)$ given by

$$\begin{aligned} ch(\chi_q) &= ch(\chi)ch(L) + sh(\chi)sh(L)c, \\ c_q &= (ch(\chi)sh(L) + c\,sh(\chi)ch(L)) / sh(\chi_q), \\ \varphi_q &= \varphi, \end{aligned} \tag{A.3}$$

where $c = \cos(\theta)$ (as in (A.1)) and $c_q = \cos(\theta_q)$.

Depending on the problem it could be more convenient to use cylindrical co-ordinates as described in [1]. The axis of these cylindrical coordinates is chosen to be the same as the polar axis of the spherical coordinates. These coordinates are named: z which is defined as the distance between the origin of the coordinates and the orthogonal projection of a point on the geodesic *Oz*, ρ the radius equal to the distance between the point and its projection, φ the cylindrical angle, the same angle as for spherical coordinates.

The correspondence between the cylindrical coordinates of H^3 and those of the Minkowski space M^4 is given by: $x = (sh\rho c_\varphi, sh\rho s_\varphi, ch\rho\, shz, ch\rho chz)$, where: c_φ and s_φ are defined above in (A.1).

The metric is:

$$ds^2 = d\rho^2 + sh^2(\rho)(d\varphi)^2 + ch^2(\rho)(dz)^2.$$

A rotation by an angle α around the *Oz* axis changes only the azimuthal angle: $\varphi \to \varphi + \alpha$. A transvection along the *Oz* axis of length $L = 2AB$ changes only the *z* coordinate: $z \to z + L$. The action of a screw motion, along the *Oz* axis, is given in M^4 coordinates by:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{bmatrix} c_\alpha & -s_\alpha & 0 & 0 \\ s_\alpha & c_\alpha & 0 & 0 \\ 0 & 0 & chL & shL \\ 0 & 0 & shL & chL \end{bmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

where $c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$.

The vectors e_α of the natural local frames are given in M^4 by

$$e_1 = (ch\rho c_\varphi, ch\rho s_\varphi, sh\rho shz, sh\rho chz),$$

$$e_2 = (-sh\rho s_\varphi, sh\rho c_\varphi, 0, 0),$$

$$e_3 = (0, 0, ch\rho chz, ch\rho shz).$$

It is now easy to find how such screw motions along the *Oz* axis act. Using cylindrical coordinates, a screw motion along the *Oz* axis transforms the local frame basis vectors as

$$\gamma^0(h_\alpha(x)) = h_a(\gamma^0 x). \qquad \text{(A.4)}$$

In the case of the spherical coordinates, the transported frame would be equivalent to the local frame up to a rotation since rotations and transvections are isometries of H^n. The advantage of using cylindrical coordinates is the following. A vector field can be written as $V(x) = V^a(x)h_\alpha(x)$. If this field is periodic under the action of γ^0, only the components have to be periodic, there is no additional rotation to take into account. These components can be expanded on fonctions of the form $V^a \sim e^{i(\nu\varphi + kz)}$ with the constraint $\nu\alpha + kL = 2\pi m$, where m is a relative integer, as for the scalar case ([1]).

Likewise, spinors are defined with respect to (space-time) local orthonormal frames, therefore, using cylindrical coordinates, γ^0 periodicity conditions apply only to the spinor components.

5.2. Spherical spaces

The spherical n-dimensional space S^n is defined as the sphere of radius R in the Euclidean space E^{n+1}. For S^3 in E^4 : $x^0 = R\sin\chi c$

$$x^1 = R\sin\chi s c_\varphi, \quad x^2 = R\sin\chi s s_\varphi, \quad x^3 = R\cos\chi, \tag{A.5}$$

$$\chi,\ \theta \in [0,\ \pi], \quad \varphi \in [0,\ 2\pi].$$

Then the linear element of S^3 is

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= R^2\left[d\chi^2 + \sin^2\chi((d\theta)^2 + s^2(d\varphi)^2)\right]. \tag{A.6}$$

As above R is set to 1. With the same notations as in the hyperbolic case, a point of coordinates $(\chi,\ \theta,\ \varphi)$ is transformed, by a transvection along Oz, into a point whose coordinates are given by:

$$\cos(\chi_q) = \cos(\chi)\cos(L) - \sin(\chi)\sin(L)c,$$

$$c_q = (\cos(\chi)\sin(L) + c\sin(\chi)\cos(L)) / \sin(\chi_q), \tag{A.7}$$

$$\varphi_q = \varphi.$$

The correspondence between the cylindrical coordinates of S^3, which are defined as in the hyperbolic case, and those of the Euclidean space E^4 is given by $x = (s_\rho c_\varphi,\ s_\rho s_\varphi,\ c_\rho s_z,\ c_\rho c_z)$, where c_φ and s_φ are defined in (A.1),

$$c_\rho = \cos(\rho),\ s_\rho = \sin(\rho),\ c_z = \cos(z),\ s_z = \sin(z).$$

The metric is:

$$ds^2 = d\rho^2 + s_\rho^2(d\varphi)^2 + c_\rho^2(dz)^2.$$

In E^4, a screw motion whose base geodesic is Oz has the form:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{bmatrix} c_\alpha & -s_\alpha & 0 & 0 \\ s_\alpha & c_\alpha & 0 & 0 \\ 0 & 0 & c_L & s_L \\ 0 & 0 & -s_L & c_L \end{bmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

where: $c_L = \cos(L),\ s_L = \sin(L)$.

The vectors of the natural local frames are given in E^4 by:

$$e_1 = (c_\rho c_\varphi,\ c_\rho s_\varphi,\ -s_\rho s_z,\ -s_\rho c_z),$$

$$e_2 = (-s_\rho s_\varphi,\ s_\rho c_\varphi,\ 0,\ 0),$$

$$e_3 = (0,\ 0,\ c_\rho c_z,\ -c_\rho s_z),$$

which can be used to show that (A.4) is also true in the spherical case.

6. Appendix B: Tensors Built from Spinors, Derivatives and Laplacian

The calculations of this appendix are performed in the more general case of spaces with torsion and with coupling to gauge fields W_α, although this additional environment will not be used in the main section of this note.

We consider tensors of the form: $\overline{\psi}\gamma^h T_x \varphi$, where: $\gamma^h = \gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_h}$, in which all the γ^{α_i} $(1 \leq i \leq h)$ are different (appear only once), and where T_x belongs to a representation of the Lie algebra of the gauge group.

We set:

$$D_\alpha = \partial_\alpha + \Gamma_\alpha + W_\alpha - S_\alpha, \tag{B.1}$$

where: $\Gamma_\alpha = \Gamma_{cd\alpha}\dfrac{\gamma^c\gamma^d}{4}$, $S_\alpha = S^{\gamma}_{.\alpha\gamma}$ and $W_\alpha = W_\alpha^x T_x$. W_α^x is the gauge field. It is chosen real, and for a unitary group with real parameters: ${T_x}^+ = -T_x$.

The tensor derivative is:

$$D_\beta(\overline{\psi}\gamma^h T_x\varphi) = \partial_\beta(\overline{\psi}\gamma^h T_x\varphi) + \Gamma^{\gamma_p}_{.\,\varphi_p\beta}(\overline{\psi}\gamma^{\gamma_1}\cdots\gamma^{\varphi_p}\cdots\gamma^{\gamma_h}T_x\varphi),$$

$$\begin{aligned} D_\beta(\overline{\psi}\gamma^h T_x\varphi) = {}& \overline{D_\beta\psi}\,\gamma^h T_x\varphi + \overline{\psi}\gamma^h T_x D_\beta\varphi \\ & + \Gamma^{\gamma_p}_{.\,\varphi_p\beta}(\overline{\psi}\gamma^{\gamma_1}\cdots\gamma^{\varphi_p}\cdots\gamma^{\gamma_h}T_x\varphi) + \frac{1}{4}\Gamma_{cd\beta}\,\overline{\psi}\,[\gamma^c\gamma^d,\, \gamma^h]\,T_x\varphi \\ & + 2S_\beta\,(\overline{\psi}\gamma^h T_x\varphi) - W_\beta^y\overline{\psi}\gamma^h({T_y}^+T_x + T_xT_y)\varphi. \end{aligned} \tag{B.2}$$

In the following discussion $\{h\}$ means a fixed set of indices $\{\alpha_1, \ldots, \alpha_h\}$ all different, as above.

If $c, d \in \{h\}$, or if: $c, d \notin \{h\}$, then: $[\gamma^c\gamma^d, \gamma^h] = 0$. If $c = \alpha_j \in \{h\}$, $d \notin \{h\} \to [\gamma^c\gamma^d, \gamma^h] = -2\eta^{cc}\gamma^{\alpha_1}\cdots\gamma^{\alpha_{j-1}}\gamma^d\gamma^{\alpha_{j+1}}\cdots\gamma^{\alpha_h}$. If $d = \alpha_j \in \{h\}$, $c \notin \{h\} \to [\gamma^c\gamma^d, \gamma^h] = 2\eta^{dd}\gamma^{\alpha_1}\cdots\gamma^{\alpha_{j-1}}\gamma^c\gamma^{\alpha_{j+1}}\cdots\gamma^{\alpha_h}$. Recalling that spinors are defines with respect to orthonormal frames, and therefore that $\Gamma_{cd\alpha} = -\Gamma_{dc\alpha}$ these contributions cancel with the third term of the right member of (B.2), and it remains

$$D_\beta(\bar{\psi}\gamma^h T_x \varphi) + C^z_{.xy} W^y_\beta (\bar{\psi}\gamma^h T_z \varphi)$$

$$= \overline{D_\beta \psi}\, \gamma^h T_x \varphi + \bar{\psi}\gamma^h T_x D_\beta \varphi + 2S^\gamma_{.\beta\gamma}(\bar{\psi}\gamma^h T_x \varphi), \tag{B.3}$$

where $C^z_{.xy}$ are the structure constant of the gauge group: $[T_x, T_y] = C^z_{.xy} T_z$.

The relation (B.3) is very general, in the rest of this note, we shall not consider any more gauge fields, then (B.3) will be used with $T_x = I$.

The Laplacian of the tensor field $\bar{\psi}\gamma^h\varphi$ involves the term $D_\alpha(g^{\alpha\beta} D_\beta(\bar{\psi}\gamma^h\varphi))$ which can be written (if there is no gauge field):

$$D_\alpha(g^{\alpha\beta} D_\beta(\bar{\psi}\gamma^h\varphi)) = g^{\alpha\beta}\left[\overline{D_\alpha D_\beta \psi - \Gamma^\gamma_{.\beta\alpha} D_\gamma \psi}\right]\gamma^h \varphi$$

$$+ \bar{\psi}\gamma^h g^{\alpha\beta}\left[D_\alpha D_\beta \varphi - \Gamma^\gamma_{.\beta\alpha} D_\gamma \varphi\right] \tag{B.4}$$

$$+ 2g^{\alpha\beta}\overline{D_\alpha\psi}\,\gamma^h D_\beta\varphi + 2D_\alpha(S^\alpha \bar{\psi}\gamma^h\varphi), \quad (W_\alpha = 0).$$

Let us consider the current:

$$J^\mu = g^{\alpha\beta}\overline{\psi_\alpha}\gamma^\mu \varphi_\beta, \tag{B.5}$$

where $\psi_\alpha = D_\alpha\psi$, $\varphi_\beta = D_\beta\varphi$ and let us calculate its divergence: $\mathrm{div}(J) = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g} J^\mu)$.

Using the identity:

$$\partial_\mu \gamma^\mu = [\gamma^\mu, \Gamma_\mu] - \frac{\partial_\mu\sqrt{g}}{\sqrt{g}}\gamma^\mu - 2S_\mu\gamma^\mu, \tag{B.6}$$

one obtains

$$\mathrm{div}(J) = \partial_\mu g^{\alpha\beta}\overline{\psi_\alpha}\gamma^\mu\varphi_\beta + g^{\alpha\beta}\,[\overline{D\psi_\alpha}\varphi_\beta + \overline{\psi_\alpha}D\varphi_\beta\,],$$

where

$$D = \gamma^\alpha(\partial_\alpha + \Gamma_\alpha + W_\alpha - S_\alpha).$$

Finally, with

$$\partial_\mu g^{\alpha\beta} = -\Gamma^{\alpha}_{.\,\delta\mu}g^{\delta\beta} - \Gamma^{\beta}_{.\,\delta\mu}g^{\delta\alpha},$$

$$\mathrm{div}(J) = g^{\alpha\beta}[\overline{D_T\psi_\alpha}\,\varphi_\beta + \overline{\psi_\alpha}D_T\varphi_\beta\,], \tag{B.7}$$

where we have set: $D_T\psi_\alpha = D\psi_\alpha - \Gamma^{\gamma}_{.\,\alpha\mu}\gamma^\mu\psi_\gamma$. Now, in (B.7), we would like to replace $D_T\psi_\alpha$ by $D_\alpha D\psi$

$$D_T\psi_\alpha = \gamma^\beta([D_\beta,\, D_\alpha\,]\psi + D_\alpha D_\beta\psi) - \gamma^\beta\Gamma^{\gamma}_{.\,\alpha\beta}D_\gamma\psi.$$

Using the relation (C.15) for the commutator and the identity

$$\partial_\alpha\gamma^\beta = [\gamma^\beta,\, \Gamma_\alpha\,] - \Gamma^{\beta}_{.\,\gamma\alpha}\gamma^\gamma, \tag{B.8}$$

we obtain

$$D_TD_\alpha\psi = D_\alpha D\psi + \gamma^\beta\Big(\frac{1}{4}R_{cd\beta\alpha}\gamma^{cd} + G_{\beta\alpha} + \partial_\alpha S_\beta - \partial_\beta S_\alpha\Big)\psi$$
$$+\, 2\gamma^\beta S^{\gamma}_{.\,\beta\alpha}D_\gamma\psi. \tag{B.9}$$

7. Appendix C: The Second Order Dirac Equation

Let ψ be a spinor. The second order Dirac equation is obtained by applying to it twice the Dirac operator. This formulation has some advantages and has been studied in [9]. This appendix summarizes some of its properties without entering into details.

Consider the Lagrangian

$$L = \overline{\gamma^{\alpha} D_{\alpha}\psi}\gamma^{\beta} D_{\beta}\psi - m^2\overline{\psi}\psi, \tag{C.1}$$

where $\gamma^{\alpha} = h^{\alpha}_{a}\gamma^{a}$, and the Dirac matrices γ^{a} are defined with respect to an orthonormal frame (in 4-dimensional Minkowski space). In the following we assume that the γ^{a} matrices are such that $\gamma^{0}(\gamma^{a})^{+}\gamma^{0} = \gamma^{a}$ and $\gamma^{0+} = \gamma^{0}$. One can find a description of Clifford algebra and their representations in [10]. The operator $D_{\alpha} = \partial_{\alpha} + \Gamma_{\alpha} + W_{\alpha} - S_{\alpha}$ has been introduced in (B.1). The sum $D = \gamma^{\alpha} D_{\alpha}$ is the Dirac operator.

The Euler-Lagrange equations give:

$$D^2\psi = h^{\alpha}_{a}\gamma^{a} D_{\alpha}\left(h^{\beta}_{b}\gamma^{b} D_{\beta}\right) = -\, m^2\psi. \tag{C.2}$$

From the Lagrangian we deduces the energy-momentum tensor

$$T^{a}_{\alpha} = \overline{D\psi}\gamma^{a} D_{\alpha}\psi + \overline{D_{\alpha}\psi}\gamma^{a} D\psi - L h^{a}_{\alpha}, \tag{C.3}$$

the spin tensor

$$S^{\alpha cd} = \frac{1}{4}[\overline{D\psi}\gamma^{\alpha}\gamma^{c}\gamma^{d}\psi - \overline{\psi}\gamma^{c}\gamma^{d}\gamma^{\alpha} D\psi], \tag{C.4}$$

and the current associated to the gauge invariance of the electromagnetic field $A^{\alpha}(W^{\alpha} = ieA^{\alpha})$:

$$J^{\alpha} = i[\overline{D\psi}\gamma^{\alpha}\psi - \overline{\psi}\gamma^{\alpha} D\psi] = -[\overline{\psi}\gamma^{\alpha} iD\psi + h.c.], \tag{C.5}$$

where *h.c.* means Hermitic conjugate. Using (B.7), one checks directly the conservation law

$$\frac{1}{\sqrt{g}}\partial_{\alpha}(\sqrt{g}J^{\alpha}) = 0.$$

We define

$$\gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}, \tag{C.6}$$

which satisfies

$$\left(\gamma^5\right)^2 = -1, \quad \left(\gamma^5\right)^+ = -\gamma^5. \tag{C.7}$$

The sets of solutions of the type: $iD\psi = \pm m\psi$ are called respectively, $D_\pm$. If $\psi \in D_\pm$, then: $i\gamma^5\psi \in D_\mp$.

Now we consider the charge conjugation operation and proceed as in the case of the linear Dirac equation by taking the complex conjugate of Equation (C.2) in order to reverse the sign of the electromagnetic charge, and set: $\psi_c = C\psi^*$, where C is a matrix operator. The charge conjugate spinor ψ_c is required to satisfy also the Dirac equation (C.2) with opposite charge, which leads to the condition:

$$C\gamma^{a*}C^{-1} = \pm\gamma^a. \tag{C.8}$$

The sign ambiguity comes from the fact that the matrices γ^a are in even number in Equation (C.2). Applying (C.8) to itself gives

$$C(C\gamma^{a*}C^{-1})^*C^{-1} = (CC^*)\gamma^a(CC^*)^{-1} = \gamma^a,$$

therefore $CC^* = kI$, where k is a complex number, since it commutes with all the Dirac matrices. Then: $C = k(C^*)^{-1} \Rightarrow C = k((kC^{*-1})^*)^{-1} = (k/k^*)C$, therefore k is real. At last, imposing that $(\psi_c)_c$ equals ψ up to a phase implies $k = \pm 1$. One can add a last contraint which is that the probability (see farther) is conserved, that is: $\psi_c^+\psi_c = \psi^+\psi$ whatever ψ is

$$(\psi^+\psi)^* = (\psi_c{}^+\psi_c)^* = (\psi_c{}^+\psi_c)^+ = \psi_c{}^+\psi_c = (C\psi^*)^+C\psi^* = (\psi^+(C^+C)^*\psi)^*,$$

and therefore $C^+C = I$.

In the Dirac representation of the γ^a matrices $C = \gamma^y$ is a solution of Equation (C.8) with the minus sign. We name it C_-.

$$C = i\gamma^5 C_- \tag{C.9}$$

is a solution of (C.8), with plus sign. It is named C_+. Both solution satisfy the relation $C^+C = I$. If $\psi \in D_\pm$, then $\psi_c = C_-\psi^* \in D_\pm$ and $\psi_c = C_+\psi^* \in D_\mp$. The Dirac representation has been chosen to find an explicit expression of C. For any other representation R equivalent to the Dirac representation, the γ^a matrices satisfy an equivalence relation of the type $\gamma_R^a = X\gamma_D^a X^{-1}$. Then the condition (C.8) is still true if C above is replaced by: XCX^{-1*}.

Now we look at the charge conjugation effect on the Energy-momentum tensor (C.3) and on the current (C.5). We have

$$C_\pm(\gamma^a(\partial_\beta + \Gamma_\beta + ieA_\beta - S_\beta)\psi)^* = \pm\gamma^a(\partial_\beta + \Gamma_\beta - ieA_\beta - S_\beta)\psi_c,$$

where, in this relation $\psi_c = C_\pm\psi^*$. As a consequence $C_\pm(D\psi)^* = \pm D\psi_c$ and

$$J_c^\alpha = -[\overline{\psi_c}\gamma^\alpha iD\psi_c + h.c.] = -[\overline{C_\pm\psi^*}\gamma^\alpha iD\psi_c + h.c.]$$

$$= -[\psi^{*+}C_\pm{}^+\gamma^0 i\gamma^\alpha(\pm)C_\pm(D\psi)^* + h.c.] = -(\pm)[\psi^{*+}\gamma^{0*}i(\gamma^\alpha D\psi)^* + h.c.]$$

$$= (\pm)[(\overline{\psi}\gamma^\alpha iD\psi)^* + h.c.] = (\mp)J^\alpha. \tag{C.10}$$

With the solution (C.9) the electromagnetic current is inverted as it should be.

Let us calculate the first term of the Energy-momentum tensor (C.3) for the charge conjugate spinor

$$\overline{D\psi_c}\gamma^a D_\alpha\psi_c = (\pm C_\pm D\psi^*)^+\gamma^0(\pm)C_\pm(\gamma^a D_\alpha\psi)^*$$

$$= (D\psi)^{+*}C_\pm^{-1}\gamma^0 C_\pm(\gamma^a D_\alpha\psi)^*$$

$$= \pm(D\psi)^{+*}\gamma^{0*}(\gamma^a D_\alpha\psi)^* = \pm(\overline{D\psi}\gamma^a D_\alpha\psi)^* = \pm(\overline{D\psi}\gamma^a D_\alpha\psi)^+,$$

the same treatment applies to L, and finally

$$T^{\alpha}_{\alpha}(\psi_c) = \pm T^{\alpha}_{\alpha}(\psi). \tag{C.11}$$

The solution $C = C_+$ keeps the energy positivity and reverse the electromagnetic current as desired.

As seen above, if $\psi \in D_\pm$, then $\psi_c = C_+\psi^* \in D_\mp$ and $J^\alpha(\psi_c) = \pm 2m\overline{\psi}\gamma^\alpha\psi = -J^\alpha(\psi)$. The 4-vector $\overline{\psi}\gamma^\alpha\psi$ is usually used to define the probability density current. Although it is invariant under charge conjugation, therefore keeping the positivity of the probability density, it is not a conserved current within the second order equation formalism, because, applying (B.7) gives: $D_\alpha(\overline{\psi}\gamma^\alpha\psi) = \overline{D\psi}\psi + \overline{\psi}D\psi$ (however, if $\psi \in D_\pm$, $D_\alpha(\overline{\psi}\gamma^\alpha\psi) = 0$).

The solution has been given in [9], for solutions of (C.2), by defining the probability current:

$$J^\alpha_P = \frac{1}{2}\left(\overline{\psi}\gamma^\alpha\psi + \frac{1}{m^2}\overline{D\psi}\gamma^\alpha D\psi\right), \tag{C.12}$$

which satisfies the conservation law $\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}J^\alpha_P) = 0$.

Applying the charge conjugation operation, we obtain

$$J^\alpha_P(\psi_c) = J^\alpha_P(\psi), \tag{C.13}$$

and if $\psi \in D_\pm$ the probability current is as usual: $J^\alpha_P = \overline{\psi}\gamma^\alpha\psi$.

The current (C.12) can be obtained by transforming the second degree equation (C.2) into a first degree system. For that purpose we set $\gamma^\alpha D_\alpha\psi = im\varphi$. Then Equation (C.2) can be put in the form

$$\gamma^\alpha D_\alpha\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = im\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix}.$$

The matrix in the right member is the Pauli matrix σ^x. This equation is re-written

$$(\sigma^x \otimes \gamma^\alpha) D_\alpha \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = im \begin{pmatrix} \psi \\ \varphi \end{pmatrix}.$$

Therefore Equation (C.2) is equivalent to the usual linear Dirac equation in $(n+2)$-dimensional Minkowski space with spinors not depending on the co-ordinates x^n, x^{n+1}. For the extension to higher dimensions of the standard 4-dimensional γ^a matrices see for instance [8]. Then the 6-dimensional vector $\overline{\Psi}\gamma^\alpha_{n+2}\Psi$, where: $\Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$ and: γ^α_{n+2} are the Dirac matrices in 6 dimensions, gives the current (C.12) for $\alpha < 4$. Its conservation law can be checked directly, and can be considered as a consequence of the conservation law of the 6-dimensional current.

Remark. As seen above, in the second order formalism, the solution $\psi_c = C_+\psi^*$ conserves the energy positivity, inverts the sign of the electromagnetic current, and keeps the positivity of the probability current. Applying the charge conjugation operation does not change the coupling to the connection, then an antiparticle is coupled to the gravitation field as a particle.

Equation (C.2) is now transformed in order to match Equation (B.4):

$$D^2\psi = h^\alpha_a \gamma^a D_\alpha \left(h^\beta_b \gamma^b D_\beta \psi \right)$$

$$= h^\alpha_a \gamma^a h^\beta_b \gamma^b D_\alpha D_\beta \psi + h^\alpha_a \gamma^a \left[D_\alpha, h^\beta_b \gamma^b \right] D_\beta \psi = -m^2 \psi.$$

The commutator reduces to

$$\left[D_\alpha, h^\beta_b \gamma^b \right] = (\partial_\alpha + \Gamma_\alpha + W_\alpha - S_\alpha) h^\beta_b \gamma^b - h^\beta_b \gamma^b D_\alpha$$

$$= \gamma^b \partial_\alpha h^\beta_b + \frac{1}{4} \Gamma_{cd\alpha} \left[\gamma^c \gamma^d, \gamma^b \right] h^\beta_b$$

and finally

$$\left[D_\alpha, h^\beta_b \gamma^b \right] = -\gamma^b h^\gamma_b \Gamma^\beta_{.\gamma\alpha}.$$

Writing $\gamma^{\alpha}\gamma^{\beta} = h^{\alpha}_{a}\gamma^{a}h^{\beta}_{b}\gamma^{b} = h^{\alpha}_{a}h^{\beta}_{b}\gamma^{a}\gamma^{b} = h^{\alpha}_{a}h^{\beta}_{b}(\eta^{ab} + \gamma^{ab}) = g^{\alpha\beta} + \gamma^{\alpha\beta}$, the second order equation becomes

$$g^{\alpha\beta}\left[D_{\alpha}D_{\beta}\psi - \Gamma^{\gamma}_{.\beta\alpha}D_{\gamma}\psi\right] + \frac{1}{2}\gamma^{\alpha\beta}[D_{\alpha}, D_{\beta}]\psi + \gamma^{\alpha\beta}S^{\gamma}_{.\alpha\beta}D_{\gamma}\psi = -m^{2}\psi, \quad \text{(C.14)}$$

where the commutator is

$$[D_{\alpha}, D_{\beta}] = \frac{1}{4}R_{cd\alpha\beta}\gamma^{cd} + G_{\alpha\beta} + (\partial_{\beta}S_{\alpha} - \partial_{\alpha}S_{\beta}), \quad \text{(C.15)}$$

and

$$G_{\alpha\beta} = \partial_{\alpha}W_{\beta} - \partial_{\beta}W_{\alpha} + [W_{\alpha}, W_{\beta}].$$

The second term of (C.14) implies a term: $R_{cdab}\gamma^{ab}\gamma^{cd}$, where

$$\gamma^{a}\gamma^{b}\gamma^{c}\gamma^{d} = \gamma^{abcd} + \eta^{ab}\gamma^{cd} - \eta^{ac}\gamma^{bd} + \eta^{ad}\gamma^{bc} + \eta^{bc}\gamma^{ad} - \eta^{bd}\gamma^{ac}$$
$$+ \eta^{cd}\gamma^{ab} + \eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}.$$

If the torsion is null, then $R_{cdab}\gamma^{abcd} = 0$ by the Bianchi identities of the first kind. Using the fact that the Ricci tensor is symmetric if the torsion is null, the second order Dirac equation is

$$g^{\alpha\beta}\left[D_{\alpha}D_{\beta}\psi - \Gamma^{\gamma}_{.\beta\alpha}D_{\gamma}\psi\right] - \frac{1}{4}R\psi + \frac{1}{2}G_{\alpha\beta}\gamma^{\alpha\beta}\psi = -m^{2}\psi, \quad \text{(C.16)}$$

$$(S^{\alpha}_{.\beta\gamma} = 0),$$

where R is the scalar curvature.

8. Appendix D: Solutions to the Free Electromagnetic Field Equations in $t \otimes H^3$

This appendix looks for solutions to the electromagnetic field equations in $V = t \otimes H^3$ which can be used as a basis to expand eigenvectors . A subclass of them can be interpreted as the equivalent of the plane wave solutions in Minkowski space, in the neighbourhood of one geodesic. For local plane wave solutions, see also [13]. We shall use cylindrical coordinates (see Appendix A and [1]). The space-time metric used is:

$$ds^2 = d\rho^2 + sh^2(\rho)d\varphi^2 + ch^2(\rho)dz^2 - dt^2.$$

The symmetry axis of the cylindrical coordinates (Oz axis) is chosen to be the base geodesic of one generator element of the covering group Γ called γ^0. In [1], the scalar functions which are eigenvectors of the Laplacian and γ^0 at the same time are of the form

$$\varphi_{\mu\nu} \sim I_{\mu,\,\nu}(\rho)\exp(i(\mu_z z + \nu\varphi)). \tag{D.1}$$

These functions are invariant with respect to γ^0 if $\mu_z L + \nu\alpha = 2\pi m$, where m is a relative integer, L is the length of the transvection and α is the rotation angle around the base geodesic.

The coordinates of a vector are written on the local natural base as

$$\overrightarrow{v} = v^\rho\overrightarrow{e_\rho} + v^\varphi\overrightarrow{e_\varphi} + v^z\overrightarrow{e_z}.$$

It will be convenient to use orthonormal local frames defined with respect to the natural frames by $\overrightarrow{h_a} = h_\alpha^\alpha\overrightarrow{e_\alpha}$ and write $\overrightarrow{v} = v^a\overrightarrow{h_a}$, $v^\alpha = h_a^\alpha v^a$ but in order to avoid any confusion between the component names and to which local frame they correspond, we rename them $\overrightarrow{v} = u^a\overrightarrow{h_a}$.

With the above metric we have $u^1 = v^{a=1} = v^{\alpha=1}$, $u^2 = v^{a=2} = sh\rho v^{\alpha=2}$, $u^3 = v^{a=3} = ch\rho v^{\alpha=3}$.

All the Christoffel symbols having an index equal to 0 are null and the other are equal to those of the space metric alone. The non zero Christoffel symbols are:

$$\Gamma_{22}^1 = \Gamma_{33}^1 = -\,sh\rho ch\rho, \quad \Gamma_{12}^2 = ch\rho/sh\rho, \quad \Gamma_{13}^3 = sh\rho/ch\rho. \tag{D.2}$$

The non zero coefficients of the first structure equations are

$$\omega^1 = d\rho, \quad \omega^2 = sh\rho d\varphi, \quad \omega^3 = ch\rho dz, \tag{D.3}$$

$$\omega^2_{.1} = ch\rho d\varphi, \quad \omega^3_{.1} = sh\rho dz,$$

which satisfy (1.1).

The following equations will be used

$$D^2 v^a = D_\beta (g^{\beta\gamma} D_\gamma v)^a = g^{\beta\gamma}\Big(\partial_\beta (D_\gamma v)^a + \omega^a_{.b\beta} D_\gamma v^b - \Gamma^\delta_{.\gamma\beta} D_\delta v^a\Big), \tag{D.4}$$

where $(D_\gamma v)^a = \partial_\gamma v^a + \omega^a_{.b\gamma} v^b$,

$$D_\alpha v^\alpha = \frac{1}{sh\rho ch\rho}\partial_\rho (sh\rho ch\rho u^1) + \frac{1}{sh\rho}\partial_\varphi (u^2) + \frac{1}{ch\rho}\partial_z (u^3) + \partial_0 (v^0). \tag{D.5}$$

The electromagnetic field equations are (4.1)

$$\Delta A^\beta \equiv D_\alpha (g^{\alpha\gamma} D_\gamma A^\beta) - R^\beta_\alpha A^\alpha = J^\beta, \tag{D.6}$$

here $J^\beta = 0$ since we consider free fields, with the Lorenz condition $D_\alpha A^\alpha = 0$.

Using the cylindrical coordinates, (D.4) becomes

$$(D^2 v)^{a=1} = \Delta(v^{a=1}) - 2\frac{ch\rho}{sh^2\rho}\partial_\varphi u^2 - 2\frac{sh\rho}{ch^2\rho}\partial_z u^3 - \left(2 + \frac{1}{sh^2\rho ch^2\rho}\right)u^1, \tag{D.7a}$$

where $\Delta(v^{a=1})$ means the Laplacian operator applied the component $u^1 = v^{a=1}$ as if it were a scalar function

$$(D^2 v)^{a=2} = \Delta(v^{a=2}) + 2\frac{ch\rho}{sh^2\rho}\partial_\varphi u^1 - \frac{ch^2\rho}{sh^2\rho}u^2, \tag{D.7b}$$

$$(D^2 v)^{a=3} = \Delta(v^{a=3}) + \frac{2sh\rho}{ch^2\rho}\partial_z u^1 - \frac{sh^2\rho}{ch^2\rho}u^3, \tag{D.7c}$$

$$(\Delta v)^{\alpha=0} = {}_3\Delta(v^{\alpha=0}) - \partial_{00}v^0, \tag{D.7d}$$

where ${}_3\Delta$ is the spatial Laplacian.

Now, by analogy with the plane wave solutions of the Euclidean space, we consider waves propagating along the *Oz* axis, and try solutions of the form

$$A^1 = \overline{A}^1(\rho)\cos(\nu\varphi + \tau)e^{i(\omega t - kz)}, \quad A^2 = -\overline{A}^2(\rho)\sin(\nu\varphi + \tau)e^{i(\omega t - kz)}$$

$$A^3 = \overline{A}^3(\rho)\cos(\nu\varphi + \tau)e^{i(\omega t - kz)}, \tag{D.8}$$

where τ is a phase and, a priori, $k = k(\rho)$. In this appendix ω has not exactly the same meaning as in Section 3 (it is up to a factor 2). Since the Lorenz condition does not fix completely the gauge, we chose: $A^0 = 0$.

In the Euclidean case, the equivalent of Equation (D.7c) is homogeneous and $A^3 = 0$ is a solution. Setting $\overline{A}^1 = \overline{A}^2$ gives solutions of the form: $\overline{A}^2 = J_{\nu-1}(\sqrt{\omega^2 - k^2}\,\rho)$, where $J_{\nu-1}$ is the cylindrical Bessel function of index $\nu - 1$. When $\omega = |k|$ and $\nu = 1$, one obtains the standard plane wave solution propagating along the *Oz* axis with electric field aligned with the *Ox* axis if $\tau = 0$ and aligned with the *Oy* axis if $\tau = -\pi/2$.

With the hypothesis (D.8), the Lorenz condition (D.5) is

$$\frac{1}{sh\rho ch\rho}\partial_\rho(sh\rho ch\rho\overline{A}^1) - i\overline{A}^1\partial_\rho kz - \frac{\nu}{sh\rho}\overline{A}^2 - \frac{ik}{ch\rho}\overline{A}^3 = 0, \tag{D.9}$$

and the z dependence implies $\partial_\rho k = 0$.

For H^3 : $R_\alpha^\beta = -2\delta_\alpha^\beta \to R_a^b = -2\delta_a^b$ for the spatial components of the Ricci tensor ($R = 1$). Using expressions (D.7) in (D.6) gives the following equations:

$$\frac{1}{sh\rho ch\rho}\partial_\rho\left(sh\rho ch\rho\partial_\rho\overline{A}^1\right)-\left(1+\frac{1}{sh^2\rho ch^2\rho}+\frac{\nu^2}{sh^2\rho}\right)\overline{A}^1-\frac{k^2}{ch^2\rho}\overline{A}^1$$

$$+\,2ik\frac{sh\rho}{ch^2\rho}\overline{A}^3+2\nu\frac{ch\rho}{sh^2\rho}\overline{A}^2=-(1+\omega^2)\overline{A}^1, \qquad \text{(D.10a)}$$

$$\frac{1}{sh\rho ch\rho}\partial_\rho\left(sh\rho ch\rho\partial_\rho\overline{A}^2\right)-\left(\frac{\nu^2+1}{sh^2\rho}\right)\overline{A}^2-\frac{k^2}{ch^2\rho}\overline{A}^2$$

$$+\,2\nu\frac{ch\rho}{sh^2\rho}\overline{A}^1=-(1+\omega^2)\overline{A}^2, \qquad \text{(D.10b)}$$

$$\frac{1}{sh\rho ch\rho}\partial_\rho\left(sh\rho ch\rho\partial_\rho\overline{A}^3\right)+\left(\frac{1}{ch^2\rho}-\frac{\nu^2}{sh^2\rho}\right)\overline{A}^3-\frac{k^2}{ch^2\rho}\overline{A}^3-2ik\frac{sh\rho}{ch^2\rho}\overline{A}^1$$

$$=-(1+\omega^2)\overline{A}^3. \qquad \text{(D.10c)}$$

There are 3 unknown functions and 4 equations including the Lorenz condition (D.9). First consider Equation (D.10b). If we had $ch\rho\overline{A}^1=\overline{A}^2$, it would become homogeneous

$$\frac{1}{sh\rho ch\rho}\partial_\rho\left(sh\rho ch\rho\partial_\rho\overline{A}^2\right)-\frac{(\nu-1)^2}{sh^2\rho}\overline{A}^2-\frac{k^2}{ch^2\rho}\overline{A}^2=-(1+\omega^2)\overline{A}^2,$$

which is the defining equation of the scalar field Laplacian eigenfunction radial part (Equation (14) in [1]), then $\overline{A}^2\sim I_{k,\nu-1}(\rho)$ would be a solution.

We assume that

$$\overline{A}^1=f(\rho)/ch\rho. \qquad \text{(D.11)}$$

The Lorenz condition (D.9) is then

$$ik\overline{A}^3=\partial_\rho f+\frac{ch\rho}{sh\rho}f-\nu\frac{ch\rho}{sh\rho}\overline{A}^2, \qquad \text{(D.12)}$$

which can be used in (D.10a). Combining with (D.10b) one obtains

$$\overline{A}^2 \pm f \sim I_{k,\,\nu\mp 1}. \tag{D.13}$$

If $\nu = 0$, the only possibility is $f \sim I_{k,1}$.

This solution satisfies Equations (D.10a), (D.10b) and the Lorenz condition defines $\overline{A}^3$. It remains to check the compatibility with Equation (D.10c). The Laplacian operator (see Appendix A and Section 4) $\Delta = d\delta + \delta d$ commutes with δ, then if the Lorenz condition is satisfied one has: $\delta\Delta\omega = \Delta\delta\omega = 0$ which shows that the Equations (D.10) are not independent, and that (D.10c) is automatically satisfied. This can be checked directly. After some algebra we obtain $\partial_z\,(\text{D.10c}) = 0$, and since $(\text{D.10c}) \sim e^{-ikz}$ this means that (D.10c) = 0.

The equations of the electromagnetic field with the Lorenz condition are satisfied by the hypothesis (D.8) and the solution (D.11), (D.13). The general solution for a field propagating along the *Oz* axis is:

$$A^2 = \sum\nolimits_{\nu}[c^{\nu}I_{k,\,\nu-1} + d^{\nu}I_{k,\,\nu+1}](-\sin(\nu\varphi + \tau_{\nu}))e^{i(\omega t - kz)}, \tag{D.14a}$$

$$A^1 = \frac{1}{ch\rho}\sum\nolimits_{\nu}[c^{\nu}I_{k,\,\nu-1} - d^{\nu}I_{k,\,\nu+1}]\cos(\nu\varphi + \tau_{\nu})e^{i(\omega t - kz)}, \tag{D.14b}$$

$$-\partial_z A^3 = \sum\nolimits_{\nu}[c^{\nu}T^3_{\nu,-1} - d^{\nu}T^3_{\nu,1}]\cos(\nu\varphi + \tau_{\nu})e^{i(\omega t - kz)}, \tag{D.14c}$$

where, with $\varepsilon = \pm 1$; $T^3_{\nu,\varepsilon} = \partial_{\rho}I_{k,\,\nu+\varepsilon} + \dfrac{ch\rho}{sh\rho}I_{k,\,\nu+\varepsilon}(1 + \varepsilon\nu)$.

Each single ν mode is an eigenvector common to the Laplacian and γ^0 if: $kL + \nu\alpha = 2\pi m$, where m is a relative integer, for a given value of $\beta = \omega$, and can be used to find solutions to the free electromagnetic field equations in compact hyperbolic manifolds as expansion over these modes.

In Euclidean space, the case $\nu = 1$ corresponds to plane waves propagating in the *Oz* direction with linear polarization. With the gauge $A^0 = 0$, the electric field is co-linear to $\vec{A}$ which is then contained in the constant phase surfaces $z = Ct$, in other words: $\overline{A}^3 = 0$. In H^3 this is impossible, because if $\overline{A}^3 = 0$, then Equation (D.7c) implies $\partial_z u^1 = 0$, and ∂_z(D.7a) would give $\partial_z u^2 = 0$. This is not a propagating wave. Therefore, the vector $\vec{A}$ can not be tangent to the constant phase surface $z = Ct$ everywhere.

Using the recurrence relations for the cylindrical function derivatives [1], one gets if $\nu = 1$:

$$T^3_{1,-1} = \frac{(k^2 - 1 - \beta^2)}{2} \operatorname{Re} I_{ik+1,1} + k \operatorname{Im} I_{ik+1,1},$$

with:
$$4 \operatorname{Im} I_{ik+1,1} = -\, k \frac{sh\rho}{ch\rho} I_{k,2},$$

and:
$$T^3_{1,1} = 4 \operatorname{Re} I_{ik+1,1} = 4 \frac{sh\rho}{ch\rho}\left[I_{k,0} - \frac{(k^2 - 1 - \beta^2)}{8} I_{k,2} \right].$$

The choice $k^2 - 1 - \beta^2 = 0$ and $d^1 = 0$ gives

$$A^2 = -\, c^1 I_{k,0} \sin(\varphi + \tau_1) e^{i(\omega t - kz)},$$

$$A^1 = \frac{c^1}{ch\rho} I_{k,0} \cos(\varphi + \tau_1) e^{i(\omega t - kz)}, \tag{D.15}$$

$$-\,\partial_z A^3 = ikA^3 = -\, c^1 \frac{k^2 sh\rho}{4ch\rho} I_{k,2} \cos(\varphi + \tau_1) e^{i(\omega t - kz)}.$$

For $\rho \to 0$, $ikA^3 \sim O(sh^3\rho)$ which means that in the vicinity of the *Oz* axis the vector $\vec{A}$, and therefore the electric field, are tangent to constant phase planes. It can be checked that near the *Oz* axis the magnetic field is also tangent to these constant phase plane. In conclusion, the solution

(D.14) looks like an Euclidean plane wave if $\nu = 1$ in the neighbourhood of the geodesic *Oz*.

In order to get more insight, we can look at the behaviour of solution (D.15) when $\rho \to \infty$. The cylindrical radial functions $I_{k,\nu} \to Q_\nu \cos(\beta\rho + \kappa_\nu)/\sqrt{sh\rho ch\rho}$, where Q_ν and κ_ν are respectively an amplitude and a phase. A^1 becomes negligible with respect to A^2 and A^3. The potential vector is perpendicular to the radial vector $\overrightarrow{h_1}$ and the solution looks like a elliptical polarisation in the radial direction.

This can be compared with the solution obtained in spherical coordinates. The electromagnetic potential is expanded on the orthonormal set of vector spherical harmonics [11]:

$$\overrightarrow{A} = f_1(\chi)\overrightarrow{V_1}(\theta, \varphi) + f_2(\chi)\overrightarrow{V_2}(\theta, \varphi) + f_3(\chi)\overrightarrow{V_3}(\theta, \varphi).$$

where $\overrightarrow{V_1}(\theta, \varphi) = Y_l^m(\theta, \varphi)\overrightarrow{h_1}$ (here $\overrightarrow{h_1}$ is the radial vector attached to the spherical coordinates), and $\overrightarrow{V_2}(\theta, \varphi)$ and $\overrightarrow{V_3}(\theta, \varphi)$ are the other vector spherical harmonics. For each pair (l, m), the field equations are satisfied by (up to a constant factor):

$$f_1 = \phi_\beta^l / sh\chi, \quad f_2 = \frac{1}{\sqrt{l(l+1)}sh\chi}\partial_\chi(sh^2\chi f_1), \quad f_3 = \phi_\beta^l,$$

where $\phi_\beta^l(\chi) = \frac{1}{\sqrt{sh\chi}} B_\lambda^\mu(\chi)$ and $B_\lambda^\mu(\chi)$ are the Legendre functions with parameters

$$\lambda = -\frac{1}{2} + i\beta \quad \mu = -\frac{1}{2} - l.$$

When $\chi \to \infty$ the radial function behaves as $\phi_\beta^l \sim \cos(\beta\chi + \kappa_l)/sh\chi$, then f_1 becomes negligible compared with f_2, f_3, and the electromagnetic field is orthogonal to the radius direction, as in the cylindrical case. This is consistent since the plane $z = Ct$ is asymptotically tangent to the cone of summit O and angle $\cos(\theta) = thz$.

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