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DIOPHANTINE PROOF OF NON-MONOGENEITY FOR TRIQUADRATIC NUMBER FIELDS WITH ODD DISCRIMINANT

FRANÇOIS E. TANOÉ* **and KOUASSI VINCENT KOUAKOU**

UFR Mathématiques et Informatique
Université Félix Houphouet BOIGNY
22 BP 582 Abidjan 22
Côte d'Ivoire
e-mail: aziz_marie@yahoo.fr

UFR Sciences Fondamentales Appliquées
Université NANGUI-ABROGOUA
02 BP 801 Abidjan 02
Côte d'Ivoire
e-mail: kouakouassivincent@gmail.com

Abstract

Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ be a triquadratic number field, whose discriminant is odd, that is to say, such that $(dm, dn, d'm'n'l) \equiv (1, 1, 1)$

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*Corresponding author

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$(\mathrm{mod}\ 4)$. We show that K_3 is not monogenous, that is to say that its ring of integers; which, as we know, is a free $\mathbb{Z}$-module of rank 8, does not admit a power basis of the type $\{1, \theta, \ldots, \theta^7\}$, or in an equivalent way, there exists no integer θ of K_3 such that $\mathrm{discr}(\theta) = D_{K_3/\mathbb{Q}} = (dmnl)^4$. To do this, we solve the monogenicity equation of K_3, obtained from its transformed Chatelain basis, using only diophantine reasoning, applied in particular to a system of three equations of Pell-Fermat, of the form: $X_i^2 - BY_i^2 = \pm 4,\ \ i = 1, 2, 3$.

1. Introduction

1.1. The problem of monogeneity

Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ be a triquadratic number field of odd discriminant, i.e., such that $(dm, dn, d'm'n'l) \equiv (1, 1, 1)\ (\mathrm{mod}\ 4)$. Then the discriminant of K_3 on $\mathbb{Q}$ is: $D_{K_3/\mathbb{Q}} = (dmnl)^4$ cf. [1]. Using only Diophantine methods, we want to solve the problem of non-existence of a power basis of the type $\{1, \theta, \ldots, \theta^7\}$, for the ring of integers $\mathbb{Z}_{K_3}$ which as we know is a $\mathbb{Z}$-module free of rank 8.

The classical method consists in solving an equivalent problem by solving in unknown $\theta \in \mathbb{Z}_{K_3}$, the classical monogeneity equation below, where $\sigma_i \in Gal(K_3\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$:

$$\Delta(\theta) = discr(1, \theta, \ldots, \theta^7) = \prod_{0 \le i < j \le 7} (\sigma_i(\theta) - \sigma_j(\theta))^2 = D_{K_3/\mathbb{Q}} = (dmnl)^4. \quad (1)$$

The Galois group $Gal(K_3\mathbb{Q})$ on an α-basis of Chatelain $\{\alpha_i : 0 \le i \le 7\}$ of K_3 (cf. [1, 5]) obtained via the α-matrix of Galois whose general term is:

$$a_{ji} = \frac{\sigma_i(\alpha_j)}{\alpha_j} = \pm 1,\ 0 \le i,\ j \le 7$$

and which is as follows:

$$M_3 = \begin{array}{c} \begin{matrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \end{array} \begin{matrix} \alpha_i \\ 1 \\ \sqrt{dm} \\ \sqrt{dn} \\ \lambda_d s(d)\sqrt{mn} \\ \sqrt{d'm'n'l} \\ \lambda_{d'm'} s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} \\ \lambda_{d'n'} s(d'n')\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} \\ \lambda_{dm'n'} s(dm'n')\sqrt{\frac{m}{m'}\frac{n}{n'}d'l} \end{matrix}.$$

Let $\mathfrak{B}_{K_3} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7\}$ be the $\mathbb{Z}$-basis of Chatelain of $\mathbb{Z}_{K_3}$ (cf. [5]), then the unknown $\theta \in \mathbb{Z}_{K_3}$ is written $\theta = \sum_{i=0}^{7} x_i \varepsilon_i$, where $x_i \in \mathbb{Z}$, and:

$$\Delta(\theta) = \prod_{0 \le i < j \le 7} (\sigma_i(\theta) - \sigma_j(\theta))^2 = I(x_1, \ldots, x_7)^2 D_{K_3/\mathbb{Q}}, \tag{2}$$

where I is a homogeneous form of $\mathbb{Z}[X_1, \ldots, X_7]$ of degree 28, called index form attached to the basis $\mathfrak{B}_{K_3}$. Then the resolution of the equation of monogeneity

$$\Delta(\theta) = \pm D_{K_3/\mathbb{Q}},$$

returns from a diophantine point of view to solve in unknowns $(x_1, \ldots, x_7) \in \mathbb{Z}^7$, the following equation of monogeneity:

$$I(x_1, \ldots, x_7) = \pm 1. \tag{3}$$

Remark 1.1. $\mathbb{Z}_{K_3}$ admits always a basis such that, at a permutation close, its first term is equal to 1 (since in $\mathfrak{B}_{K_3}$ we have $\sum_{i=0}^{7} \varepsilon_i = 1$).

Thus the variable x_0 disappears itself by difference when we solve the equation

$$\Delta(\theta) = \pm D_{K_3/\mathbb{Q}}.$$

It is a property which will be kept independently of the choice of the basis.

This is why we can put $x_0 = 0$, without affecting the generality of our resolution.

In the case where the problem is solvable, and we want to rewrite the general solution taking into account x_0, it will suffice to introduce the coordinate x_0 nearby ε_0. The remaining found coordinates x_i, $i \neq 0$ associated to ε_i, $i \neq 0$.

This kind of problem has been solved for fields of small degrees, especially for biquadratic fields, among others by [2, 6].

This work is about of the solution of the problem of monogeneity of the fields of 8 degree, with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. We find results demonstrated between 2002 and 2006 in [9, 7, 10, 8, 11] given in special cases and general.

However, in our methodology, unlike the previous authors, we use different methods, namely purely diophantine to solve equation (3), using modular calculations in $\mathbb{Z}/4\mathbb{Z}$.

The principle of the demonstration is as follows.

In a first step, see Definition 1.1, after having agreed on a canonical writing for K_3, we generally construct an integer basis for any triquadratic field $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ using the works of D. Chatelain cf. [1] on *n*-quadratic fields, which we apply to degree 8 cf. [5]. We transform this basis of Chatelain, in a basis better adapted to the problem of the monogeneity, by scaling said basis. This will allow us to write much more simply the equation of monogeneity (3), which finally splits into a system of seven quadratic normative equations (S_1) (note that for the practical resolution we will only use the first three of these equations (10). In general, this type of Pell-Fermat system either does not admit solutions, or when it admits, we get a unique solution cf. [14, 3].

We establish, much as we did for the biquadratic case cf. [2], Lemmas 1.1 and 1.2 which contain linear constraints between the variables d, m, n, d', m', n' and l of the field K_3. These conditions will be quite strong to conclude that the system

resulting from that of monogeneity (10) is not solvable.

In general, the problem of monogeneity can arise in any degree n and on all kinds of algebraic extensions: Galoisian cyclic or non-cyclic, non-Galoisian, relative etc.

Note that for the number fields, this property for a field to be monogenous or not, has an arithmetical importance, because it facilitates among other things, the factorization of the prime ideals $p\mathbb{Z}$ extended to $\mathbb{Z}_K$. So if θ_0 is found monogenous and $F_{\theta_0}(X)$ the irreducible polynomial of the monogeneous element θ_0 supposed to exist, then $p\mathbb{Z}_K = \prod_{i=1}^{r}(F_i(\theta_0)^{e_i}$ is the decomposition into prime ideals of the ideal $p\mathbb{Z}_K$, where $F_{\theta_0}(X) = \prod_{i=1}^{r}(F_i(X)^{e_i}$ is the decomposition in irreducible factors of $F_{\theta_0}(X)$ considered in $\mathbb{F}_p[X]$.

This result from a cryptographic point of view can be interesting since the security of many cryptographic models, are based on decompositions of certain quantities, which can only be done in exponential time.

1.2. Definitions - notations - conventions for $\mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$

Consider a triquadratic number field $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ with odd discriminant, i.e., such that $(dm, dn, d'm'n'l) \equiv (1, 1, 1) \pmod 4$.

The case where the discriminant is even, corresponds to the other two remaining cases, namely $(dm, dn, d'm'n'l) \equiv (1, 1, 2 \text{ or } 3)$ and $(1, 2, 3,) \pmod 4$. This case will be treated by the method described here but in another article. It should be known that there is only one triquadratic field depending on this second case, which is monogenous, it is the cyclotomic field $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\sqrt{-3}, \sqrt{2}, \sqrt{-1})$ cf. [12] where a method similar to this one was applied afterwards.

Let us give some writing conventions and remarks.

1.2.1. Writing conventions for the fields $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l'})$

(1) Let $\mathbb{Q}(\sqrt{dm})$, $\mathbb{Q}(\sqrt{dn})$, $\mathbb{Q}(\sqrt{d'm'n'l'})$ be three quadratic subfields of K_3, two by two distinct such that: $(m, n) = 1$, $(dmn, l) = 1$, $d' = (d, d'm'n'l)$, $m' = (m, d'm'n'l)$ and $n' = (n, d'm'n'l)$.

Then, according to the definition of Chatelain cf. [1], the seven quadratic subfields of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l'})$ are:

$$k_1 = \mathbb{Q}(\sqrt{dm}), \quad k_2 = \mathbb{Q}(\sqrt{dn}), \quad k_3 = \mathbb{Q}(\sqrt{mn}), \quad k_4 = \mathbb{Q}(\sqrt{d'm'n'l'}),$$

$$k_5 = \mathbb{Q}\left(\sqrt{\frac{dm}{d'm'}n'l}\right), \quad k_6 = \mathbb{Q}\left(\sqrt{\frac{dn}{d'n'}m'l}\right) \quad \text{and} \quad k_7 = \mathbb{Q}\left(\sqrt{\frac{mn}{m'n'}d'l}\right).$$

We deduce the seven biquadratic subfields of K_3.

(a)

$$K_{3,1} = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}) = \mathbb{Q}(\sqrt{md}, \sqrt{mn}) = \mathbb{Q}(\sqrt{nd}, \sqrt{nm}),$$

(b)

$$K_{3,2} = \mathbb{Q}\left(\sqrt{(d'm')\frac{dm}{d'm'}}, \sqrt{(d'm')n'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{dm}{d'm'}\right)d'm'}, \sqrt{\left(\frac{dm}{d'm'}\right)n'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{(n'l)d'm'}, \sqrt{(n'l)\frac{dm}{d'm'}}\right),$$

(c)

$$K_{3,3} = \mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}m'\right)d'\frac{m}{m'}}, \sqrt{\left(\frac{d}{d'}m'\right)\frac{n}{n'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}d'\right)\frac{d}{d'}m'}, \sqrt{\left(\frac{m}{m'}d'\right)\frac{n}{n'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{n}{n'}l\right)\frac{d}{d'}m}, \sqrt{\left(\frac{n}{n'}l\right)\frac{m}{m'}d'}\right),$$

(d)

$$K_{3,4} = \mathbb{Q}\left(\sqrt{(d'n')\frac{dn}{d'n'}}, \sqrt{(d'n')m'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{dn}{d'n'}\right)d'n'}, \sqrt{\left(\frac{dn}{d'n'}\right)m'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{(m'l)d'n'}, \sqrt{(m'l)\frac{dn}{d'n'}}\right),$$

(e)

$$K_{3,5} = \mathbb{Q}\left(\sqrt{\left(d'\frac{n}{n'}\right)\frac{d}{d'}n'}, \sqrt{\left(d'\frac{n}{n'}\right)\frac{m}{m'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}n'\right)d'\frac{n}{n'}}, \sqrt{\left(\frac{d}{d'}n'\right)\frac{m}{m'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}l\right)\frac{d}{d'}n'}, \sqrt{\left(\frac{m}{m'}l\right)d'\frac{n}{n'}}\right),$$

(f)

$$K_{3,6} = \mathbb{Q}\left(\sqrt{(m'n')\frac{mn}{m'n'}}, \sqrt{(m'n')d'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{mn}{m'n'}\right)m'n'}, \sqrt{\left(\frac{mn}{m'n'}\right)d'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}l\right)\frac{m}{m'}n'}, \sqrt{\left(\frac{d}{d'}l\right)m'\frac{n}{n'}}\right)$$

and

(g)

$$K_{3,7} = \mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}n'\right)m'\frac{n}{n'}}, \sqrt{\left(\frac{m}{m'}n'\right)\frac{d}{d'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{n}{n'}m'\right)\frac{m}{m'}n'}, \sqrt{\left(\frac{n}{n'}m'\right)\frac{d}{d'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{(d'l)m'n'}, \sqrt{(d'l)\frac{mn}{m'n'}}\right).$$

(2) Each of these seven biquadratic subfields $K_{3,i} = \mathbb{Q}\left(\sqrt{d_i m_i}, d_i n_i\right)$ can be written in its canonical form cf. [2], which means that $d_i m_i \equiv d_i n_i \pmod 4$, $0 < d_i$, possibly even, $n_i < m_i$ odd (and when $d_i m_i \equiv d_i n_i \equiv 1 \pmod 4$, we take $d_i < \inf\ (|m_i|, |n_i|)$.

Remark 1.2. In all the following we note (cf. [5] and [13]), and we have the following formula, concerning in particular the function

$$\gamma_a : \ 2\mathbb{Z}+1 \ \rightarrow \ \mathbb{Z}$$

$$a \ \mapsto \ \frac{a-\lambda_a}{4}.$$

(1) $s(a)$ the sign of a, $a \in \mathbb{Z}^\star$.

(2) Let $a \equiv 1 \pmod 2$, we write $\lambda_a \in \{-1; 1\}$, such that $a \equiv \lambda_a \pmod 4$, then

$$a = \lambda_a + 4\gamma_a.$$

(3) Let us note that $a \equiv 1 \pmod 2 \Rightarrow \lambda_a a \equiv 1 \pmod 4$.

(4) $\forall a, b \in 2\mathbb{Z}+1$, then $\lambda_{ab} = \lambda_a\lambda_b$. In particular $\lambda_{a^2} = 1$ et $\lambda_{a^2 b} = \lambda_b$.

(5) $\forall a, b \in 2\mathbb{Z}+1$, $\lambda_{ab} = 1 \Leftrightarrow \lambda_a = \lambda_b$.

(6) In particular the following equalities hold:

$$\lambda_{dm} = \lambda_{dn} = \lambda_{mn} = \lambda_{d'm'n'l} = \lambda_{\frac{dm}{d'm'}n'l} = \lambda_{\frac{d}{d'}m'\frac{n}{n'}l} = \lambda_{d'\frac{m}{m'}\frac{n}{n'}l} = 1,$$

and will allow useful factorizations via point (5).

(7) Let a and b be odd, then:

$$\gamma_{ab} \equiv \lambda_a\gamma_b + \lambda_b\gamma_a \pmod 4.$$

Moreover, $\forall c \in \mathbb{Z}$, we have:

$$2c(\lambda_a \pm \lambda_b) \equiv 0 \pmod 4.$$

All these formulas will be used extensively in the demonstrations.

1.2.2. Chatelain's writing of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$

Definition 1.1. Let us take $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ with $(dm, dn, d'm'n'l) \equiv (1, 1, 1) \pmod 4$, then we give a canonical writing of K_3 as follows:

(i) We first choose $K_{3,2} = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}\right)$ written in biquadratic canonical form such that $0 < d = \inf(d_i)$ and with maximal m among the eligible m_i choices and maximum $n < m$ among the remaining n_i.

(ii) Then choose $\mathbb{Q}\left(\sqrt{d'm'n'l}\right)$ among the four remaining quadratic fields such as:

$$d'm'n'l = \inf\left\{d'm'n'l, \frac{dm}{d'm'}n'l, \frac{d}{d'}m'\frac{n}{n'}l, d'\frac{m}{m'}\frac{n}{n'}l\right\}$$

and

$$s(d') = s(m') = s(n').$$

This last important condition is always possible, leaving us to change the sign of l. This has no effect on the field $\mathbb{Q}\left(\sqrt{d'm'n'l}\right)$. Indeed for $d'm'n'l$ chosen one can write:

- $d'm'n'l = d'm'(-n')(-l)$ if $s(d') = s(m') \neq s(n')$,
- $d'm'n'l = (-d')m'n'(-l)$ if $s(d') \neq s(m') = s(n')$ and
- $d'm'n'l = d'(-m')n'(-l)$ if $s(d') = s(n') \neq s(m')$.

We will note again and for the rest, this pre-writing of Chatelain for K_3 (leaving to explain the integers d, m, n, d', m', n' and l the moment came).

These constraints will have their importance in the following when it comes to solving the system (S_1).

We recall that in the following cf. [5]:

$$s_1 = \lambda_d s(d),\ s_2 = \lambda_{d'm'} s(d'm'),\ s_3 = \lambda_{d'n'} s(d'n'),\ s_4 = \lambda_{dm'n'} s(dm'n').$$

According to Definition 1.1(ii) we have:

Proposition 1.1. *Let* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *written in canonical form. Then*

$$s(d) = 1 \text{ and } s_1 = \lambda_d,\ s_2 = \lambda_{d'm'},\ s_3 = \lambda_{d'n'},\ s_4 = \lambda_{dm'n'}.$$

In the following, K_3 is supposed to be written in canonical form.

Let us recall the following results cf. [5] too.

Theorem 1.1. *Let* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *such that* $(dm, dn, d'm'n'l) \equiv (1, 1, 1)\ (mod\ 4)$. *Then* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *is one of Chatelain's writing of* K_3, *and*:

(*a*) *The Chatelain* β*-basis of* K_3 *is given by*:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \sqrt{d'm'n'l}, s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}, s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\},$$

with the s_i *defined in Proposition* 1.1.

(b) *The* $\mathbb{Z}$*-basis of Chatelain,* $\mathfrak{B}_{K_3}$ *of* $\mathbb{Z}_{K_3}$, *consists of*:

$$\varepsilon_0 = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn} + \sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$
$$\left. + s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right)$$

and its seven other conjugates.

(c)

$$D_{K_3/\mathbb{Q}} = (dmnl)^4 = dm \times dn \times mn \times d'm'n'l \times \frac{d}{d'}\frac{m}{m'}n'l \times \frac{d}{d'}\frac{n}{n'}m'l \times \frac{m}{m'}\frac{n}{n'}d'l.$$

1.3. Change of bases

We can (cf. [5]) make the $\mathbb{Z}$-transformations of the matrix M of $\mathfrak{B}_{K_3} = \{\varepsilon_i : 0 \le i \le 7\}$ relative to β into a triangular matrix lower M' of $\mathfrak{B}'_{K_3} = \{\varepsilon'_i : 0 \le i \le 7\}$ which is another basis of $\mathbb{Z}_{K_3}$. We get $\mathfrak{B}'_{K_3}$ from elementary operations that respect $\mathbb{Z}$. The goal is to minimize the number of square roots in the new matrix, making the system (S_1) be easier to handle. We obtain the following result:

Theorem 1.2. *The following family* $\mathfrak{B}'_{K_3} = \{\varepsilon'_i : 0 \le i \le 7\}$ *is a new basis of integers of* K_3.

$$\varepsilon'_0 = \varepsilon_0 = \frac{1}{8}\Bigg(1 + \sqrt{dm} + \sqrt{dn} + \lambda_d s(d)\sqrt{mn} + \sqrt{d'm'n'l} + \lambda_{d'm'}s(d'm')\sqrt{\frac{dm}{d'm'}n'l} + \lambda_{d'n'}s(d'n')\sqrt{\frac{dn}{d'n'}m'l} + \lambda_d\lambda_{m'n'}s(d)s(m'n')\sqrt{\frac{mn}{m'n'}d'l}\Bigg),$$

$$\varepsilon'_1 = -\varepsilon_0 + \varepsilon_1 = \frac{1}{8}\Bigg(-2\sqrt{dm} - 2s_1\sqrt{mn} - 2s_2\sqrt{\frac{dm}{d'm'}n'l} - 2s_4\sqrt{\frac{mn}{m'n'}d'l}\Bigg),$$

$$\varepsilon'_2 = -\varepsilon_1 + \varepsilon_3 = \frac{1}{8}\Bigg(-2\sqrt{dn} + 2s_1\sqrt{mn} - 2s_3\sqrt{\frac{dn}{d'n'}m'l} + 2s_4\sqrt{\frac{mn}{m'n'}d'l}\Bigg),$$

$$\varepsilon'_3 = -\varepsilon_0 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 = \frac{1}{8}\Big(-4s_1\sqrt{mn} - 4s_4\sqrt{\frac{mn}{m'n'}d'l}\Big),$$

$$\varepsilon'_4 = -\varepsilon_2 + \varepsilon_6 = \frac{1}{8}\Bigg(-2\sqrt{d'm'n'l} - 2s_2\sqrt{\frac{dm}{d'm'}n'l} + 2s_3\sqrt{\frac{dn}{d'n'}m'l} + 2s_4\sqrt{\frac{mn}{m'n'}d'l}\Bigg),$$

$$\varepsilon'_5 = \varepsilon_2 - \varepsilon_3 - \varepsilon_6 + \varepsilon_7 = \frac{1}{8}\left(4s_2\sqrt{\frac{dm}{d'm'}n'l} - 4s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_6 = -\varepsilon_1 + \varepsilon_3 + \varepsilon_5 - \varepsilon_7 = \frac{1}{8}\left(-4s_3\sqrt{\frac{dn}{d'n'}m'l} + 4s_4\sqrt{\frac{mn}{m'n'}d'l}\right) \quad \textit{and}$$

$$\varepsilon'_7 = -\varepsilon_0 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 = \frac{1}{8}\left(-8s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

We will use this staggered basis to solve the problem of monogeneity.

1.3.1. Monogeneity equations

On this new scaled basis $\mathfrak{B}'_{K_3}$ of $\mathbb{Z}_{K_3}$, let us take $\theta \in \mathbb{Z}_{K_3}$, then there exist $l_0, l_1, l_2, l_3, l_4, l_5, l_6, l_7 \in \mathbb{Z}$ such that:

$$\theta = l_0\varepsilon'_0 + \ldots + l_7\varepsilon'_7. \tag{4}$$

The equation of monogeneity (3) is written:

$$I(l_1, \ldots, l_7) = \pm 1. \tag{5}$$

For the actual calculation, we come back to the Chatelain β-basis of K_3:

$$\begin{aligned}\theta = {} & \frac{l_0}{8} + \frac{1}{8}(l_0 - 2l_1)\sqrt{dm} + \frac{1}{8}(l_0 - 2l_2)\sqrt{dn} + \frac{1}{8}(l_0 - 2l_1 + 2l_2 - 4l_3)s_1\sqrt{mn} \\ & + \frac{1}{8}(l_0 - 2l_4)\sqrt{d'm'n'l} + \frac{1}{8}(l_0 - 2l_1 - 2l_4 + 4l_5)s_2\sqrt{\frac{dm}{d'm'}n'l} \\ & + \frac{1}{8}(l_0 - 2l_2 + 2l_4 - 4l_6)s_3\sqrt{\frac{dn}{d'n'}m'l} \\ & + \frac{1}{8}(l_0 - 2l_1 + 2l_2 - 4l_3 + 2l_4 - 4l_5 + 4l_6 - 8l_7)s_4\sqrt{\frac{mn}{m'n'}d'l}.\end{aligned}$$

Let $a_i \in \mathbb{Z}$, $i = 0, \ldots, 7$, such that:

$$\begin{cases} a_0 = l_0;\ a_1 = l_0 - 2l_1;\ a_2 = l_0 - 2l_2;\ a_4 = l_0 - 2l_4; \\ a_3 = l_0 - 2l_1 + 2l_2 - 4l_3;\ a_5 = l_0 - 2l_1 - 2l_4 + 4l_5;\ a_6 = l_0 - 2l_2 + 2l_4 - 4l_6; \\ a_7 = l_0 - 2l_1 + 2l_2 + 2l_4 - 4l_3 - 4l_5 + 4l_6 - 8l_7. \end{cases} \tag{6}$$

Note that conversely for these same a_i :

$$\begin{cases} l_0 = a_0;\ l_1 = \dfrac{a_0 - a_1}{2};\ l_2 = \dfrac{a_0 - a_2}{2};\ l_4 = \dfrac{a_0 - a_4}{2}; \\ l_3 = \dfrac{a_0 + a_1 - a_2 - a_3}{4};\ l_5 = \dfrac{a_0 - a_1 - a_4 + a_5}{4};\ l_6 = \dfrac{a_0 + a_2 - a_4 - a_6}{4}; \\ l_7 = \dfrac{a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7}{8}. \end{cases} \quad (7)$$

So that:

$$\theta = \frac{1}{8}a_0 + \frac{1}{8}a_1\sqrt{dm} + \frac{1}{8}a_2\sqrt{dn} + \frac{1}{8}a_3 s_1\sqrt{mn} + \frac{1}{8}a_4\sqrt{d'm'n'l}$$

$$+ \frac{1}{8}a_5 s_2\sqrt{\frac{dm}{d'm'}n'l} + \frac{1}{8}a_6 s_3\sqrt{\frac{dn}{d'n'}m'l} + \frac{1}{8}a_7 s_4\sqrt{\frac{mn}{m'n'}d'l}.$$

As a result:

$$I^2(l_1, \ldots, l_7) = I^2(a_1, \ldots, a_7).$$

So that, solve (5) is equivalent to solve:

$$I(a_1, \ldots, a_7) = \pm 1. \quad (8)$$

1.3.2. Calculation of $\Delta(\theta) = discr(\theta)$

Remarks 1.1. (1) We calculate $\Delta(\theta) = discr(\theta) = \prod_{0 \le i < j \le 7} (\sigma_i(\theta) - \sigma_j(\theta))^2$ in terms of $I(a_1, \ldots, a_7)$, the variable a_0 disappearing, cf. Remark 1.1. So we can take $a_0 = l_0 = 0$, in (6), without affecting the generality of our resolution. So in the following we will have:

$$\theta = \frac{1}{8}a_1\sqrt{dm} + \frac{1}{8}a_2\sqrt{dn} + \frac{1}{8}a_3 s_1\sqrt{mn} + \frac{1}{8}a_4\sqrt{d'm'n'l}$$

$$+ \frac{1}{8}a_5 s_2\sqrt{\frac{dm}{d'm'}n'l} + \frac{1}{8}a_6 s_3\sqrt{\frac{dn}{d'n'}m'l} + \frac{1}{8}a_7 s_4\sqrt{\frac{mn}{m'n'}d'l}$$

with:

$$\begin{cases} a_1 = -2l_1;\ a_2 = -2l_2;\ a_3 = -2l_1 + 2l_2 - 4l_3;\ a_4 = -2l_4; \\ a_5 = -2l_1 - 2l_4 + 4l_5;\ a_6 = -2l_2 + 2l_4 - 4l_6; \\ a_7 = -2l_1 + 2l_2 + 2l_4 - 4l_3 - 4l_5 + 4l_6 - 8l_7. \end{cases}$$

If later we want to give the general solution θ including $a_0 = l_0$, we will use formulas (7), (6) and (4).

(2) For the calculations of $\Delta(\theta)$, let us make the following groupings and define by the same the following pairs of $\mathbb{Z}^2 : (A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1), (I_1, J_1), (K_1, L_1), (M_1, N_1)$, as well as the numbers of $K_3 : \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ and θ_7 as follows:

- $\Delta_1 = (\sigma_0(\theta) - \sigma_2(\theta)) \times (\sigma_4(\theta) - \sigma_5(\theta)) \times (\sigma_1(\theta) - \sigma_3(\theta)) \times (\sigma_5(\theta) - \sigma_7(\theta))$

$$= \left(\frac{n}{n'}\right)^2 \times N_{k_1/\mathbb{Q}} \times \left(\frac{\frac{a_2^2 dn' + a_3^2 mn' - a_6^2 \frac{d}{d'} m'l - a_7^2 \frac{m}{m'} d'l}{8} + \frac{a_2 a_3 s_1 n' - a_6 a_7 s_3 s_4 l}{4} \sqrt{dm}}{2} \right)$$

$$= \left(\frac{n}{n'}\right)^2 \times N_{k_1/\mathbb{Q}} \left(\frac{A_1 + C_1\sqrt{dm}}{2} \right) = \left(\frac{n}{n'}\right)^2 \times N_{k_1/\mathbb{Q}}(\theta_1);$$

- $\Delta_2 = (\sigma_0(\theta) - \sigma_6(\theta)) \times (\sigma_2(\theta) - \sigma_4(\theta)) \times (\sigma_1(\theta) - \sigma_7(\theta)) \times (\sigma_3(\theta) - \sigma_5(\theta))$

$$= (n')^2 \times N_{k_1/\mathbb{Q}} \times \left(\frac{\frac{a_2^2 d \frac{n}{n'} + a_3^2 m \frac{n}{n'} - a_4^2 d'm'l - a_5^2 \frac{dm}{d'm'} l}{8} + \frac{a_2 a_3 s_1 \frac{n}{n'} - a_4 a_5 s_3 l}{4} \sqrt{dm}}{2} \right)$$

$$= (n')^2 \times N_{k_1/\mathbb{Q}} \left(\frac{B_1 + D_1\sqrt{dm}}{2} \right) = (n')^2 \times N_{k_1/\mathbb{Q}}(\theta_2);$$

• $\Delta_3 = (\sigma_0(\theta) - \sigma_4(\theta)) \times (\sigma_2(\theta) - \sigma_6(\theta)) \times (\sigma_1(\theta) - \sigma_5(\theta)) \times (\sigma_3(\theta) - \sigma_7(\theta))$

$= (l)^2 \times N_{k_1/\mathbb{Q}}$

$$\times \left(\frac{\frac{a_4^2 d'm'n' + a_5^2 \frac{dm}{d'm'} n' - a_6^2 \frac{dn}{d'n'} m' - a_7^2 \frac{mn}{m'n'} d'}{8} + \frac{a_4 a_5 s_2 n' - a_6 a_7 s_3 s_4 \frac{n}{n'}}{4} \sqrt{dm}}{2} \right)$$

$$= (l)^2 \times N_{k_1/\mathbb{Q}} \left(\frac{E_1 + F_1 \sqrt{dm}}{2} \right) = (l)^2 \times N_{k_1/\mathbb{Q}}(\theta_3);$$

•$\Delta_4 = (\sigma_0(\theta) - \sigma_1(\theta)) \times (\sigma_4(\theta) - \sigma_5(\theta)) \times (\sigma_2(\theta) - \sigma_3(\theta)) \times (\sigma_6(\theta) - \sigma_7(\theta))$

$= (\frac{m}{m'})^2 \times N_{k_2/\mathbb{Q}}$

$$\times \left(\frac{\frac{a_1^2 dm' + a_3^2 m'n - a_5^2 \frac{d}{d'} n'l - a_7^2 \frac{n}{n'} d'l}{8} + \frac{a_1 a_3 s_1 m' - a_5 a_7 s_2 s_4 l}{4} \sqrt{dn}}{2} \right)$$

$$= (\frac{m}{m'})^2 \times N_{k_2/\mathbb{Q}} \left(\frac{G_1 + H_1 \sqrt{dn}}{2} \right) = (\frac{m}{m'})^2 \times N_{k_2/\mathbb{Q}}(\theta_4);$$

• $\Delta_5 = (\sigma_0(\theta) - \sigma_5(\theta)) \times (\sigma_1(\theta) - \sigma_4(\theta)) \times (\sigma_2(\theta) - \sigma_7(\theta)) \times (\sigma_3(\theta) - \sigma_6(\theta))$

$= (m')^2 \times N_{k_2/\mathbb{Q}}$

$$\times \left(\frac{\frac{a_1^2 d \frac{m}{m'} + a_3^2 \frac{m}{m'} n - a_4^2 d'n'l - a_6^2 \frac{dn}{d'n'} l}{8} + \frac{a_1 a_3 s_1 \frac{m}{m'} - a_4 a_6 s_3 l}{4} \sqrt{dn}}{2} \right)$$

$$= (m')^2 \times N_{k_2/\mathbb{Q}} \left(\frac{I_1 + J_1 \sqrt{dn}}{2} \right) = (m')^2 \times N_{k_2/\mathbb{Q}}(\theta_5);$$

$$\bullet\Delta_6 = (\sigma_0(\theta)-\sigma_7(\theta))\times(\sigma_1(\theta)-\sigma_6(\theta))\times(\sigma_2(\theta)-\sigma_5(\theta))\times(\sigma_3(\theta)-\sigma_4(\theta))$$

$$= (d')^2 \times N_{k_3/\mathbb{Q}}$$

$$\times\left(\frac{\frac{a_1^2}{d'}m+a_2^2\frac{d}{d'}n-a_4^2m'n'l-a_7^2\frac{mn}{m'n'}l}{8}+\frac{a_1a_2\frac{d}{d'}-a_4a_7s_4l}{4}\sqrt{mn}}{2}\right)$$

$$= (d')^2 \times N_{k_3/\mathbb{Q}}\left(\frac{K_1+L_1\sqrt{mn}}{2}\right) = (d')^2 \times N_{k_3/\mathbb{Q}}(\theta_6);$$

$$\bullet\Delta_7 = (\sigma_0(\theta)-\sigma_3(\theta))\times(\sigma_4(\theta)-\sigma_7(\theta))\times(\sigma_5(\theta)-\sigma_6(\theta))\times(\sigma_1(\theta)-\sigma_2(\theta))$$

$$= \left(\frac{d}{d'}\right)^2 \times N_{k_3/\mathbb{Q}}$$

$$\times\left(\frac{\frac{a_1^2d'm+a_2^2d'n-a_5^2\frac{m}{m'}n'l-a_6^2\frac{n}{n'}m'l}{8}+\frac{a_1a_2d'-a_5a_6s_2s_3l}{4}\sqrt{mn}}{2}\right);$$

$$= \left(\frac{d}{d'}\right)^2 \times N_{k_3/\mathbb{Q}}\left(\frac{M_1+N_1\sqrt{mn}}{2}\right) = \left(\frac{d}{d'}\right)^2 \times N_{k_3/\mathbb{Q}}(\theta_7).$$

It is clear that:

$$discr\ (\theta) = (\Delta_1\times\Delta_2\times\Delta_3\times\Delta_4\times\Delta_5\times\Delta_6\times\Delta_7)^2.$$

And that we have:

$$\prod_{0\le i<j\le 7}(\sigma_i(\theta)-\sigma_j(\theta))$$

$$= \left[\left(\frac{n}{n'}\right)^2 \times N_{k_1/\mathbb{Q}}(\theta_1)\right]\times\left[n'^2 \times N_{k_1/\mathbb{Q}}(\theta_2)\right]\times[l^2 \times N_{k_1/\mathbb{Q}}(\theta_3)]$$

$$\times\left[\left(\frac{m}{m'}\right)^2 \times N_{k_2/\mathbb{Q}}(\theta_4)\right]\times[m'^2 \times N_{k_2/\mathbb{Q}}(\theta_5)]$$

$$\times\left[d'^2 \times N_{k_3/\mathbb{Q}}(\theta_6)\right]\times\left[\left(\frac{d}{d'}\right)^2 \times N_{k_3/\mathbb{Q}}(\theta_7)\right]$$

$$= (dmnl)^2 \times N_{k_1/\mathbb{Q}}(\theta_1)\times N_{k_1/\mathbb{Q}}(\theta_2)\times N_{k_1/\mathbb{Q}}(\theta_3)\times N_{k_2/\mathbb{Q}}(\theta_4)$$

$$\times N_{k_2/\mathbb{Q}}(\theta_5)\times N_{k_3/\mathbb{Q}}(\theta_6)\times N_{k_3/\mathbb{Q}}(\theta_7).$$

Thus:

Proposition 1.2. *The powers of* $\theta \in \mathbb{Z}_{K_3}$ *form a basis of* $\mathbb{Z}_{K_3}$ *if and only if*

$$N_{k_1/\mathbb{Q}}(\theta_1)\times N_{k_1/\mathbb{Q}}(\theta_2)\times N_{k_1/\mathbb{Q}}(\theta_3)$$

$$\times N_{k_2/\mathbb{Q}}(\theta_4)\times N_{k_2/\mathbb{Q}}(\theta_5)\times N_{k_3/\mathbb{Q}}(\theta_6)\times N_{k_3/\mathbb{Q}}(\theta_7) = \pm 1. \tag{9}$$

Equation equivalent to (8): $I(a_1, \ldots, a_7) = \pm 1$.

And, we obtain the following system:

$$(S_1): \begin{cases} N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_1) = \epsilon_1, \\ N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_2) = \epsilon_2, \\ N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_3) = \epsilon_3, \\ N_{\mathbb{Q}(\sqrt{dn})/\mathbb{Q}}(\theta_4) = \epsilon_4, \\ N_{\mathbb{Q}(\sqrt{dn})/\mathbb{Q}}(\theta_5) = \epsilon_5, \\ N_{\mathbb{Q}(\sqrt{mn})/\mathbb{Q}}(\theta_6) = \epsilon_6, \\ N_{\mathbb{Q}(\sqrt{mn})/\mathbb{Q}}(\theta_7) = \epsilon_7, \end{cases}$$

where $\epsilon_k = \pm 1$, $k = 0, 1, \ldots, 7$.

1.3.3. System of monogeneity equations

Let us show that in the product $I(a_1, \ldots, a_7)$, each factor is in $\mathbb{Z}$ and consequently is necessarily equal to ± 1. This will give us the system (S_1).

● The detailed calculation hereafter show that the numbers $A_1, C_1, \ldots, M_1, N_1$ are in $\mathbb{Z}$.

As an example we write A_1 et C_1:

$$A_1 = \lambda_{dn'}(l_1l_4 - l_4^2) + 2(\lambda_{dn'}(l_1l_3 - l_1l_5 + l_1l_6 - l_2l_3 - l_2l_4 + l_2l_5 + l_4l_5 + l_3^2 - l_5^2)$$

$$+ \gamma_{dn'}l_2^2 + \gamma_{mn'}(l_1^2 + l_2^2) + \gamma_{\frac{d}{d'}m'l}(-l_2^2 + l_4^2) + \gamma_{\frac{m}{m'}d'l}(-l_1^2 - l_2^2 - l_4^2)) + 4(\lambda_{\frac{m}{m'}d'l}$$

$$(-l_1l_7 - l_2l_6 + l_2l_7 + l_4l_7 + l_5l_6 - l_6^2) + \gamma_{\frac{d}{d'}m'l}(l_2l_4) + \gamma_{\frac{m}{m'}d'l}(l_1l_2 + l_1l_4 - l_2l_4)) +$$

$$8(\gamma_{mn'}(l_1l_3 - l_2l_3) + \gamma_{\frac{d}{d'}m'l}(-l_2l_6 + l_4l_6 - l_6^2) + \gamma_{\frac{m}{m'}d'l}(-l_1l_7 + l_2l_7 + l_4l_7 + l_5l_6))$$

$$+ 16(-l_1l_7 + l_2l_7 + l_4l_7 + l_5l_6) + 32(-l_7^2 - l_5l_7 + l_6l_7)) \quad \text{and,}$$

$$C_1 = \lambda_{dn'}(l_1l_4 - l_4^2) + 2\lambda_{dn'}(-l_1l_6 + l_2l_3 - l_2l_5 + l_4l_5) + 4(\lambda_{\frac{m}{m'}d'l}(l_2l_6 - l_2l_7 + l_4l_7$$

$$- l_5l_6 + l_6^2) + \lambda_d\gamma_{n'}(l_1l_2 - l_2^2) + \lambda_{\frac{d}{d'}m'}\gamma_l(-l_1l_2 + l_1l_4 + l_2^2 - l_4^2)) + 8(-\lambda_{\frac{m}{m'}d'l}l_6l_7$$

$$+ \lambda_d\gamma_{n'}l_2l_3 + \lambda_{\frac{d}{d'}m'}\gamma_l(-l_1l_6 - l_2l_5 + l_4l_5)) + 16(\lambda_{\frac{d}{d'}m'}\gamma_l(l_2l_6 - l_2l_7 + l_4l_7 - l_5l_6$$

$$+ l_6^2)) - 32\lambda_{\frac{d}{d'}m'}\gamma_l l_6l_7.$$

- Moreover, the components of the following couples: (A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1), (I_1, J_1), (K_1, L_1), (M_1, N_1) have the same parity. Indeed:

$$A_1 - C_1 \equiv \lambda_{dn'}(l_1l_4 - l_4^2) - \lambda_{dn'}(l_1l_4 - l_4^2) \equiv 0 \pmod 2;$$

$$B_1 - D_1 \equiv \lambda_{d'm'l}(-l_1l_2 + l_2^2 - l_1l_4 - l_4^2) - \lambda_{d'm'l}(l_1l_2 - l_2^2 - l_1l_4 - l_4^2) \equiv 0 \pmod 2;$$

$$E_1 - F_1 \equiv \lambda_{d'm'n'}(l_1l_2 - l_2^2) - \lambda_{d'm'n'}(-l_1l_2 + l_2^2) \equiv 0 \pmod 2;$$

$$G_1 - H_1 \equiv \lambda_{dm'}(-l_2l_4 - l_4^2) - \lambda_{dm'}(l_2l_4 + l_4^2) \equiv 0 \pmod 2;$$

$$I_1 - J_1 \equiv \lambda_{d\frac{m}{m'}}(-l_1l_2 + l_1^2 + l_2l_4 - l_4^2) - \lambda_{d\frac{m}{m'}}(-l_1l_2 + l_1^2 - l_2l_4 + l_4^2) \equiv 0 \pmod 2;$$

$$K_1 - L_1 \equiv \lambda_{\frac{d}{d'}m}(l_1l_2 + l_1l_4 - l_2l_4 - l_4^2) - \lambda_{\frac{d}{d'}}(l_1l_2 - l_1l_4 + l_2l_4 + l_4^2) \equiv 0 \pmod 2;$$

$$M_1 - N_1 \equiv \lambda_{d'm}(-l_1l_4 + l_2l_4 - l_4^2) - \lambda_{d'}(l_1l_4 - l_2l_4 + l_4^2) \equiv 0 \pmod 2.$$

Note that for our demonstration, we will use only the reduced modulo 4 computation of the values of the first three couples, whose expressions are:

Proposition 1.3. *We have the following relationships modulo* 4:

$$\begin{cases} A_1 \equiv \lambda_{dn'}(l_1l_4 - l_4^2) + 2\lambda_{dn'}(l_1l_3 - l_1l_5 + l_1l_6 - l_2l_3 - l_2l_4 + l_2l_5 + l_4l_5 + l_3^2 - l_5^2) \\ \quad + 2\gamma_{dn'}l_2^2 + 2\gamma_{mn'}(l_1^2 + l_2^2) + 2\gamma_{\frac{d}{d'}m'l}(-l_2^2 + l_4^2) + 2\gamma_{\frac{m}{m'}d'l}(-l_1^2 - l_2^2 - l_4^2) \pmod 4, \\ B_1 \equiv \lambda_{d\frac{n}{n'}}(-l_1l_2 + l_2^2 - l_1l_4 - l_4^2) + 2\lambda_{d\frac{n}{n'}}(l_1l_3 - l_1l_5 - l_4l_5 + l_3^2 - l_5^2) + 2\gamma_{d\frac{n}{n'}}l_2^2 \\ \quad + 2\gamma_{m\frac{n}{n'}}(l_1^2 + l_2^2) + 2\gamma_{d'm'l}(-l_4^2) + 2\gamma_{\frac{dm}{d'm'}l}(-l_1^2 - l_4^2) \pmod 4, \\ E_1 \equiv \lambda_{d'm'n'}(l_1l_2 - l_2^2) + 2\lambda_{d'm'n'}(l_1l_4 + l_1l_6 + l_2l_5) + 2\gamma_{d'm'n'}l_4^2 + 2\gamma_{\frac{dm}{d'm'}n'}(l_1^2 + l_4^2) \\ \quad + 2\gamma_{\frac{dn}{d'n'}m'}(-l_2^2 - l_4^2) + 2\gamma_{\frac{mn}{m'n'}d'}(-l_1^2 - l_2^2 - l_4^2) \pmod 4, \end{cases}$$

and

$$\begin{cases} C_1 \equiv \lambda_{dn'}(l_1l_4 - l_4^2) + 2\lambda_{dn'}(-l_1l_6 + l_2l_3 - l_2l_5 + l_4l_5) \pmod 4, \\ D_1 \equiv \lambda_{d'm'l}(l_1l_2 - l_2^2) + 2\lambda_{d'm'l}(l_2l_3 - l_4l_5) \pmod 4, \\ F_1 \equiv \lambda_{d'm'n'}(l_2^2 - l_1l_2) + 2\lambda_{d'm'n'}(-l_1l_6 - l_2l_5) \pmod 4. \end{cases}$$

1.3.4. Resolution of the system (S_1) - Modular Calculations - Lemmas

We are particularly interested in the system formed by the first three equations of (S_1).

We get:

$$(S_1'): \begin{cases} A_1^2 - C_1^2 dm = 4\epsilon_1, \\ B_1^2 - D_1^2 dm = 4\epsilon_2, \\ E_1^2 - F_1^2 dm = 4\epsilon_3, \end{cases} \tag{10}$$

where for recall the numbers

$$A_1 = \frac{a_2^2 dn' + a_3^2 mn' - a_6^2 \frac{d}{d'} m'l - a_7^2 \frac{m}{m'} d'l}{8},$$

$$C_1 = \frac{a_2 a_3 s_1 n' - a_6 a_7 s_3 s_4 l}{4},$$

$$B_1 = \frac{a_2^2 d \frac{n}{n'} + a_3^2 m \frac{n}{n'} - a_4^2 d'm'l - a_5^2 \frac{dm}{d'm'} l}{8},$$

$$D_1 = \frac{a_2 a_3 s_1 \frac{n}{n'} - a_4 a_5 s_2 l}{4},$$

$$E_1 = \frac{a_4^2 d'm'n' + a_5^2 \frac{dm}{d'm'} n' - a_6^2 \frac{dn}{d'n'} m' - a_7^2 \frac{mn}{m'n'} d'}{8},$$

$$F_1 = \frac{a_4 a_5 s_2 n' - a_6 a_7 s_3 s_4 \frac{n}{n'}}{4}$$

and the $a_1, \ldots, a_7$ are defined in Remarks 1.1.

In the next paragraph we give the following useful lemmas.

Let put:

$$\begin{cases} d_1 = \gcd(A_1, C_1) \geq 1, \\ d_2 = \gcd(B_1, D_1) \geq 1, \\ d_3 = \gcd(E_1, F_1) \geq 1. \end{cases} \tag{11}$$

It is clear that (S_1') is solvable $\Rightarrow d_1^2$, d_2^2 and d_3^2 divide $4 \Rightarrow d_1, d_2, d_3 \in \{1, 2\}$. We assume that (S_1') is solvable, so we have the following results.

Lemmas 1.1. *We have the following three lemmas*:

(a)

$$(S_0): \begin{cases} \frac{n}{n'} A_1 = n'B_1 + lE_1, \\ \frac{n}{n'} C_1 = n'D_1 + lF_1. \end{cases}$$

(b) *The following system* (S_1'') *is solvable* (*exactly* $(S_1') \Leftrightarrow (S_1'')$).

$$(S_1''): \begin{cases} 2n'l \times \left[\dfrac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3, \\ 2\dfrac{n}{n'}l \times \left[\dfrac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3, \\ 2n \times \left[\dfrac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3. \end{cases}$$

(c) $(d_1, d_2, d_3) = (2, 1, 1)$. *So that*

$$A_1 \equiv C_1 \equiv 0 \pmod 2,\ B_1 \equiv D_1 \equiv 1 \pmod 2 \text{ and } E_1 \equiv F_1 \equiv 1 \pmod 2.$$

Firstly; we demonstrate points (a) and (b).

Proof 1.1. (a) Let us establish the system (S_0):

Evident (by simple calculation).

(b) Let us show that (S_1') and (S_1'') sont équivalents when (S_0) is checked. Transforms (S_1') from the relations of (S_0).

We have:

$$A_1^2 - C_1^2dm = 4\epsilon_1 \Leftrightarrow (n'B_1 + lE_1)^2 - (n'D_1 + lF_1)^2dm = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1$$

$$\Leftrightarrow n'^2(B_1^2 - D_1^2dm) + l^2(E_1^2 - F_1^2dm) + 2n'l[E_1B_1 - F_1D_1dm] = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1$$

$$\Leftrightarrow n'^2 \times 4\epsilon_1 + l^2 \times 4\epsilon_3 + 2n'l[E_1B_1 - F_1D_1dm] = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1.$$

Which gives us:

$$2n'l \times \left[\frac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3.$$

Note that; $\left(\dfrac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1 \pmod 4$.

In the same way we have:

$B_1^2 - D_1^2 dm = 4\epsilon_2$ equals:

$$2\frac{n}{n'}l \times \left[\frac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3$$

and $\left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1 \pmod 4$.

Similarly; $E_1^2 - F_1^2 dm = 4\epsilon_1$ equals:

$$2n \times \left[\frac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3$$

and $\left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1 \pmod 4$.

The system (S_1') gives rise to the system (S_1'') below:

$$(S_1''): \begin{cases} 2n'l \times \left[\frac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3, \\ 2\frac{n}{n'}l \times \left[\frac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3, \\ 2n \times \left[\frac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3. \end{cases}$$

For (c),

Let us first establish that: $(d_1, d_2, d_3) = (2, 1, 1)$ or $(1, 1, 2)$ or $(1, 2, 1)$.

• Suppose $d_1 = 2$, then A_1 and C_1 are even.

It is clear that the case $d_2 = d_3 = 2$ is impossible because otherwise

$$\frac{E_1B_1 - F_1D_1dm}{4} \in \mathbb{Z} \Rightarrow 2n'l \times \left[\frac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv 0 \pmod 2,$$ which is absurd.

Similarly, if $(d_2, d_3) = (1, 2)$ or $(2, 1)$, then either B_1 and D_1 are even and E_1 and F_1 odd, or we have the opposite, in all cases as $\frac{n}{n'} A_1 = n'B_1 + lE_1$ and $\frac{n}{n'} C_1 = n'D_1 + lF_1$ it comes that A_1 and C_1 would be odd, which is absurd.

So the only possibility is $(d_1, d_2, d_3) = (2, 1, 1)$.

• Now suppose that $d_1 = 1$. So A_1 and C_1 are odd.

If we had $d_2 = d_3 = 2 \Rightarrow B_1, D_1, E_1, F_1$ would be even $\Rightarrow$ impossible, see above.

If we have $d_2 = d_3 = 1$, then B_1, D_1, E_1, F_1 would be odd, but then $(\frac{n}{n'} A_1 = n'B_1 + lE_1$ and $\frac{n}{n'} C_1 = n'D_1 + lF_1) \Rightarrow A_1$ and C_1 would be even. This is absurd. So $(d_1, d_2, d_3) = (1, 1, 2)$ or $(1, 2, 1)$.

In summary we have: $(d_1, d_2, d_3) = (2, 1, 1)$ or $(1, 1, 2)$ or $(1, 2, 1)$.

• To show that $(d_1, d_2, d_3) = (2, 1, 1)$, we use the computation of C_1, D_1, F_1 cf. Proposition 1.2, from which we deduce the value of $\frac{n}{n'} C_1 = n'D_1 + lF_1 \pmod 4$.

We get $\frac{n}{n'} C_1 \equiv n'D_1 + lF_1 \pmod 4 \Rightarrow l_1 l_4 - l_4^2 \equiv 0 \pmod 4$.

As a consequence $C_1 \equiv 0 \pmod 2$, so $A_1 \equiv 0 \pmod 2$, because they are of the same parity.

So $d_1 = 2$ and consequently $(d_1, d_2, d_3) = (2, 1, 1)$ as announced in Lemmas 1.1(c).

The system (S_1) being always supposed to be solvable, we also have the following Lemma:

Lemmas 1.2. (i) $l_1 \equiv l_4 \equiv 0 \pmod 2$; $l_3 \equiv 0 \pmod 4$ *and* $l_2 \equiv l_5 \equiv 1 \pmod 2$
(ii) *For the quantities* $A_1, B_1, E_1, C_1, D_1, F_1$, *we have*:

$$\begin{cases} C_1 \equiv -2\lambda_{dn'}l_2l_5 \pmod 4; \\ D_1 \equiv \lambda_{d'm'l}(l_1l_2 - 1) \pmod 4; \\ F_1 \equiv -\lambda_{d'm'n'}(l_1l_2 - 1) - 2\lambda_{d'm'n'}l_2l_5 \pmod 4; \end{cases}$$

and

$$\begin{cases} A_1 \equiv -2\lambda_{dn'} + 2\lambda_{dn'}l_2l_5 \pmod 4; \\ B_1 \equiv -\lambda_{d\frac{n}{n'}}(l_1l_2 - 1) + 2\lambda_{\frac{n}{n'}}(\gamma_d + \gamma_m) \pmod 4; \\ E_1 \equiv \lambda_{d'm'n'}(l_1l_2 - 1) + 2\lambda_{d'm'n'}l_2l_5 - 2\lambda_{nl}(\gamma_d + \gamma_m) \pmod 4. \end{cases}$$

Remark 1.3. Note that the point (i) of the lemma, is not sufficient to find a contradiction in computing modulo 4, the quantities (A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1), (I_1, J_1), (K_1, L_1), (M_1, N_1).

Proof 1.2. We had already calculated modulo 4, the quantities A_1, B_1, E_1 as well as C_1, D_1, F_1 cf. Lemmas 1.2. Recall also that we have shown cf. proof of Lemmas 1.1(c) that $l_1l_4 - l_4^2 \equiv 0 \pmod 4$.

From which we deduce that F_1 is odd that: $l_2 \equiv 1 \pmod 2$ and $l_1 \equiv 0 \pmod 2$ and that accordingly $l_4 \equiv 0 \pmod 2$.

Said congruences simplify a first time, considering among other things that $l_2^2 \equiv 1 \pmod 4$, we find:

$$\begin{cases} A_1 \equiv 2\lambda_{dn'}(-l_2l_3 + l_2l_5 + l_3^2 - l_5^2) + 2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l}) \pmod 4, \\ B_1 \equiv \lambda_{d\frac{n}{n'}}(-l_1l_2 + 1) + 2\lambda_{d\frac{n}{n'}}(l_3^2 - l_5^2) + 2(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) \pmod 4, \\ E_1 \equiv \lambda_{d'm'n'}(l_1l_2 - 1) + 2\lambda_{d'm'n'}(l_2l_5) - 2(\gamma_{\frac{dn}{d'n}m'} + \gamma_{\frac{mn}{m'n'}d'}) \pmod 4 \end{cases}$$

and

$$\begin{cases} C_1 \equiv 2\lambda_{dn'}(l_2l_3 - l_2l_5) \pmod 4, \\ D_1 \equiv \lambda_{d'm'l}(l_1l_2 - 1) + 2\lambda_{d'm'l}(l_2l_3) \pmod 4, \\ F_1 \equiv \lambda_{d'm'n'}(1 - l_1l_2) + 2\lambda_{d'm'n'}(-l_2l_5) \pmod 4. \end{cases}$$

• Now we consider: $\frac{n}{n'}A_1 \equiv n'B_1 + lE_1 \pmod 4$, (see Lemmas 1.1(a)) considering

that $\lambda_{dn} = \lambda_{d'm'n'l} = 1$, gives the reduction:

$$-l_2 l_3 \equiv -2\lambda_{\frac{n}{n'}}(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l})$$

$$+2(\lambda_{n'}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) - \lambda_l(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'})) \pmod 4.$$

But the two quantities:

$$2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l}) \quad \text{and} \quad 2(\lambda_{n'}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) - \lambda_l(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'}))$$

are $\equiv 0 \pmod 4$.

Indeed, let us use the form of Remark 1.1.

• $2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l}) \equiv 2(\lambda_d\gamma_{n'} + \lambda_{n'}\gamma_d + \lambda_m\gamma_{n'} + \lambda_{n'}\gamma_m$

$$- \lambda_{\frac{d}{d'}m'}\gamma_l - \lambda_l\gamma_{\frac{d}{d'}m'} - \lambda_{\frac{m}{m'}d'}\gamma_l - \lambda_l\gamma_{\frac{m}{m'}d'}) \pmod 4$$

$$\equiv 2(\gamma_{n'}(\lambda_d + \lambda_m) + \lambda_{n'}(\gamma_d + \gamma_m) - \gamma_l(\lambda_{\frac{d}{d'}m'} + \lambda_{\frac{m}{m'}d'}) - \lambda_l(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'})) \pmod 4$$

$$\equiv 2(\lambda_{n'}(\gamma_d + \gamma_m) - \lambda_l(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'})) \pmod 4$$

$$\equiv 2(\lambda_{n'}(\lambda_{d'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{d}{d'}}\gamma_d + \lambda_{m'}\gamma_{\frac{m}{m'}} + \lambda_{\frac{m}{m'}}\gamma_{m'}) - \lambda_l(\lambda_{\frac{d}{d'}}\gamma_{m'} + \lambda_{m'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{m}{m'}}\gamma_{d'}$$

$$+ \lambda_{d'}\gamma_{\frac{m}{m'}})) \pmod 4$$

$$\equiv 2(\lambda_{d'n'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{d}{d'}n'}\gamma_{d'} + \lambda_{m'n'}\gamma_{\frac{m}{m'}} + \lambda_{\frac{m}{m'}n'}\gamma_{m'} - (\lambda_{\frac{d}{d'}l}\gamma_{m'} + \lambda_{m'l}\gamma_{\frac{d}{d'}} + \lambda_{\frac{m}{m'}}\gamma_{d'}$$

$$+ \lambda_{d'l}\gamma_{\frac{m}{m'}})) \pmod 4$$

$$\equiv 2(\gamma_{\frac{d}{d'}}(\lambda_{d'n'} - \lambda_{m'l}) + \gamma_{d'}(\lambda_{\frac{d}{d'}n'} - \lambda_{\frac{m}{m'}l}) + \gamma_{\frac{m}{m'}}(\lambda_{m'n'} - \lambda_{d'l}) + \gamma_{m'}(\lambda_{\frac{m}{m'}n'} - \lambda_{\frac{d}{d'}l}))$$

$\pmod 4 \equiv 0 \pmod 4$.

Similarly, we have:

$$2(\gamma_{d\frac{n}{n'}}+\gamma_{m\frac{n}{n'}})\equiv 2\lambda_{\frac{n}{n'}}(\gamma_d+\gamma_m))\ (\mathrm{mod}\ 4)$$

$$\equiv 2\lambda_{\frac{n}{n'}}(\gamma_d+\gamma_m))\ (\mathrm{mod}\ 4)$$

$$\equiv 2\lambda_{\frac{n}{n'}}(\lambda_{\frac{d}{d'}}\gamma_{d'}+\lambda_{d'}\gamma_{\frac{d}{d'}}+\lambda_{\frac{m}{m'}}\gamma_{m'}+\lambda_{m'}\gamma_{\frac{m}{m'}})\ (\mathrm{mod}\ 4)$$

and

$$2(\gamma_{\frac{dn}{d'n'}m'}+\gamma_{\frac{mn}{m'n'}d'})\equiv 2\lambda_{\frac{n}{n'}}(\gamma_{\frac{d}{d'}m'}+\gamma_{\frac{m}{m'}d'}))\ (\mathrm{mod}\ 4)$$

$$\equiv 2\lambda_{\frac{n}{n'}}(\lambda_{\frac{m}{m'}}\gamma_{d'}+\lambda_{m'}\gamma_{\frac{d}{d'}}+\lambda_{\frac{d}{d'}}\gamma_{m'}+\lambda_{d'}\gamma_{\frac{m}{m'}})\ (\mathrm{mod}\ 4).$$

So

$$2(\lambda_{n'}(\gamma_{d\frac{n}{n'}}+\gamma_{m\frac{m}{m'}})-\lambda_l(\gamma_{\frac{dn}{d'n'}m'}+\gamma_{\frac{mn}{m'n'}d'}))$$

$$\equiv 2(\lambda_{n'}\lambda_{\frac{n}{n'}}(\gamma_d+\gamma_m)-\lambda_l\lambda_{\frac{n}{n'}}(\gamma_{\frac{d}{d'}m'}+\gamma_{\frac{m}{m'}d'}))$$

$$\equiv 2(\gamma_{d'}(\lambda_{\frac{d}{d'}n}-\lambda_{\frac{m}{m'}\frac{n}{n'}l})+\gamma_{\frac{d}{d'}}(\lambda_{d'n}-\lambda_{m'\frac{n}{n'}l})+\gamma_{m'}(\lambda_{\frac{m}{m'}n}-\lambda_{\frac{d}{d'}\frac{n}{n'}l})$$

$$+\gamma_{\frac{m}{m'}}(\lambda_{m'n}-\lambda_{\frac{d}{d'}\frac{n}{n'}l})\equiv 0\ (\mathrm{mod}\ 4).$$

This implies that $-l_2l_3\equiv 0\ (\mathrm{mod}\ 4)\Rightarrow l_3\equiv 0\ (\mathrm{mod}\ 4)$:

But then $l_5\equiv 1\ (\mathrm{mod}\ 2)$ because otherwise we would have $d_1=4$, which would be absurd.

The congruences are re-written in finality as follows, noting the fact that $2(\gamma_{\frac{dn}{d'n'}m'}+\gamma_{\frac{mn}{m'n'}d'})\equiv 2\lambda_{n'l}(\gamma_{d\frac{n}{n'}}+\gamma_{m\frac{n}{n'}})\equiv 2\lambda_{nl}(\gamma_d+\gamma_m)\ (\mathrm{mod}\ 4)$ what makes it possible to make appear this quantity in the expression of E_1 at the level of the Lemma.

And we have as well as announced:

$$\begin{cases} C_1\equiv -2\lambda_{dn'}l_2l_5\ (\mathrm{mod}\ 4),\\ D_1\equiv \lambda_{d'm'l}(l_1l_2-1)\ (\mathrm{mod}\ 4),\\ F_1\equiv -\lambda_{d'm'n'}(l_1l_2-1)-2\lambda_{d'm'n'}l_2l_5\ (\mathrm{mod}\ 4)\end{cases}$$

and

$$\begin{cases} A_1 \equiv -2\lambda_{dn'} + 2\lambda_{dn'}l_2l_5 \pmod 4, \\ B_1 \equiv -\lambda_{d\frac{n}{n'}}(l_1l_2 - 1) + 2(\gamma_d + \gamma_m) \pmod 4, \\ E_1 \equiv \lambda_{d'm'n'}(l_1l_2 - 1) + 2\lambda_{d'm'n'}l_2l_5 - 2\lambda_{nl}(\gamma_d + \gamma_m) \pmod 4. \end{cases}$$

2. Non-monogeneity of the Fields $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ with odd Discriminant

We now state the main theorem of this article.

Theorem 2.1. *Let be a triquadratic field* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *with odd discriminant, i.e., such that* $(dm, dn, d'm'n'l) \equiv (1, 1, 1) \pmod 4$. *Then* K_3 *is not monogenous, i.e., the system* (S_1) *associated with the equation of monogeneity* (8) *of* K_3 *is not solvable.*

Proof 2.1. To demonstrate this theorem, it suffices to show that the system (S_1) admits no solution. Indeed the conditions of Lemmas 1.1 are strong enough to show that the system (S_1) is not solvable because (S_1'') (i.e., (S_1')) is not. We will show that in the first equation of (S_1'), cf. Lemmas 1.1(b), we have:

$$E_1B_1 - F_1D_1dm \equiv 0 \pmod 4 \Rightarrow \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv 0 \pmod 2,$$

which is absurd since all summed numbers are odd.

To show this, let us calculate $(E_1B_1 - F_1D_1dm) \pmod 4$.

Using the last Lemmas 1.2(ii), we have:

$$\begin{aligned} E_1B_1 - F_1D_1dm \equiv E_1B_1 - F_1D_1 \equiv (&\lambda_{d'm'n'}(l_1l_2 - 1) + 2\lambda_{d'm'n'}l_2l_5 \\ &-2\lambda_{nl}(\gamma_d + \gamma_m)) \times (-\lambda_{d\frac{n}{n'}}(l_1l_2 - 1) + 2(\gamma_d + \gamma_m)) \\ &-(-\lambda_{d'm'n'}(l_1l_2 - 1) - 2\lambda_{d'm'n'}l_2l_5) \times (\lambda_{d'm'l}(l_1l_2 - 1)) \pmod 4. \end{aligned}$$

So that

$$E_1B_1 - F_1D_1dm \equiv (\lambda_{n'l} - \lambda_{\frac{d}{d'}m'n}) + 2(l_1l_2 - 1)(\gamma_d + \gamma_m)(\lambda_{d'm'n'} + \lambda_{dn'l})$$

$$-2\lambda_{\frac{d}{d'}m'n}l_2l_5(l_1l_2 - 1)(\lambda_{n'l} - \lambda_{\frac{d}{d'}m'n}) \equiv 0 \pmod 4.$$

Conclusion 2.1. If $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ with $(dm, dn, d'm'n'l) \equiv (1, 1, 1) \pmod 4$, then the monogeneity equation does not admit solutions in $\mathbb{Z}_{K_3}$. It means that:

"**Any triquadratic number field with odd discriminant is monogenous.**"

We find the result of Y. Motoda and T. Nakahara [7], they used the ramification of 2. G. Nyul [9] also got this result by using the index form and the indices of sub-groups.

This method of demonstration should be able to apply a priori when the discriminant is even; that is to say the two remaining cases: $(dm, dn, d'm'n'l) \equiv (1, 1, 2 \text{ or } 3)$ and $\equiv (1, 2, 3) \pmod 4$. Indeed in each of these cases, similar lemmas to those used here have been established (cf. [4]).

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