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CONTROLLABILITY OF NON-DENSELY DEFINED IMPULSIVE NEUTRAL INTEGRODIFFERENTIAL INCLUSIONS WITH INFINITE DELAYS

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Abstract

In this paper, we investigate the controllability for a class of abstract impulsive neutral integrodifferential inclusions with infinite delays where the linear paper is non-densely defined and satisfies the Hille-Yosida condition. The approach used is the fixed point theorem for multivalued maps due to Dhage.

1. Introduction

In recent years, the theory of impulsive integrodifferential equations or inclusions has become an active area of investigation due to their applications in the

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fields such as mechanics, electrical engineering, medicine biology, ecology and so on. One can see the monograph of Chang et al. [1] and the papers of Park and Balachandtran et al. [2], Laskshmikantham et al. [3] and the survey papers of Bainov [4] and the references therein.

It is well known that the issue of controllability plays an important role in control theory and engineering [5-7] because they have close connections to pole assignment, structural de-composition, quadratic optimal control, observer design etc. In recent years, the problem of controllability for various kinds of impulsive neutral differential equations or inclusions with infinite delays has been extensively studied by many authors using different approaches. For example, Kavitha and Mallika Arjunan et al. [8] study the controllability of non-densely defined impulsive neutral functional differential systems with infinite delay in Banach space by using the Shauder fixed point theorem combined with the operator semigroups, Park and Balachandtran et al. [9] consider the controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces by using Shauder fixed point theorem. More recently Hu and Liu et al. [10] proved the existence results of impulsive partial neutral integrodifferential inclusions with infinite delay by using another nonlinear alternative of Leray-Shauder typer for multivalued maps due to D. O'Regan. Motivated by the previous mentioned paper, we prove the controllability of non-densely defined impulsive neutral integrodifferential inclusions with infinite delay. Our approach will be based on a fixed point theorem for multivalued maps due to Dhage.

In this paper, we shall study a class of non-densely defined impulsive neutral functional integrodifferential inclusions with infinite delay in Banach spaces described in the form

$$\begin{cases} \frac{d}{dt}[x(t) - p(t, x_t)] \in Ax(t) + Bu(t) + f(t, x_t) + \int_0^t G(t, s, x_s))ds, \\ t \in J - \{t_1, \dots, t_m\}, \text{ where } J = [0, b], \\ \Delta x(t_k) = I_k(x(t_k^-))), \quad k = 1, \dots, m, \\ x_0 = \phi(t) \in \mathcal{B}_h, \end{cases} \tag{1.1}$$

where the state variable $x(\cdot)$ takes in Banach space X with the norm $|\cdot|$ and the control function $u(\cdot)$ is given in $L^2(J, U)$, the Banach space of admissible control functions with U a Banach space. B is a bounded linear operator from U to X, the unbounded linear operator A is not defined densely on X, that is $\overline{D(A)} \neq X$. $f : J \times \mathcal{B}_h \to X$, $p : J \times \mathcal{B}_h \to X$, $G : J \times J \times \mathcal{B}_h \to P(X)$, $P(X)$ denotes the class of all nonempty subsets of X. $I_k : X \to \overline{D(A)}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)\,(k = 1, \ldots, m)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$. Here $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The histories $x_t : (-\infty, 0) \to X$, defined by $x_t(\theta) = x(t + \theta)$, $\theta \leq 0$, belongs to some abstract phase space $\mathcal{B}_h$.

This paper has three sections. In the next section, we introduce some notations and necessary preliminaries. In Section 3, we prove the controllability results of mild solutions of system (1.1).

2. Preliminaries

In this section, we shall introduce some basic definitions and Lemmas which are used through-out this paper.

At first, we will employ the abstract phase space $\mathcal{B}_h$ which is similar to that used in [8]. Assume that $h : (-\infty, 0) \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)\,dt < +\infty$. For any $a > 0$, we define

$$\mathcal{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable }\},$$

and equip the space $\mathcal{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s \in [-a,0]} |\psi(s)|, \quad \forall \psi \in \mathcal{B}.$$

Let us define

$$\mathcal{B}_h = \{\psi : (-\infty, 0] \to X \text{ such that for any } c > 0,\ \psi|_{[-c,0]} \in \mathcal{B} \text{ and}$$

$$\int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < +\infty\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds, \quad \forall \psi \in \mathcal{B}_h,$$

then it is clear that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space

$\mathcal{B}'_h = \{x : (-\infty, b] \to X$ such that $x_k \in C(J_k, X)$ and there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k) = x(t_k^-)$, $x_0 = \phi \in B_h$, $k = 0, 1, \ldots, m\}$, where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$. Set $\|\cdot\|'_h$ be a seminorm in $\mathcal{B}'_h$ defined by

$$\|x\|'_h = \|\phi\|_{\mathcal{B}_h} + \sup\{|x(s)| : s \in [0, b]\},\ x \in \mathcal{B}'_h.$$

Let $P(X)$ denote the class of all nonempty subsets of X. Let $P_{bd,cl}(X)$, $P_{cp,cv}(X)$, $P_{bd,cl,cv}(X)$ and $P_{cd}(X)$ denote, respectively, the family of all nonempty bounded-closed, compact-convex, bounded-closed-convex and compact-acyclic (see [11]) subset of X. For $x \in X$ and $Y, Z \in P_{bd,cl}(X)$, we define by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$, $\rho(Y, Z) = \sup_{a\in Y} D(a, Z)$ and the Hausdorff metric $H : P_{bd,cl}(X) \times P_{bd,cl}(X) \to R^+$ by $H(A, B) = \max\ \{\rho(A, B), \rho(B, A)\}$.

G is called upper semicontinuous (shortly u.s.c) on X, if for each $x_* \in X$, the $G(x_*)$ is nonempty, closed subset of X, and if for each open of V of X containing $G(x_*)$, there exists an open neighborhood N of x_* such that $G(N) \subseteq V$. G is said to be completely continuous if $G(N)$ is relatively compact, for every bounded subset $V \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

A point $x_0 \in X$ is called a fixed point of the multivalued map G if $x_0 \in G(x_0)$.

A multivalued map $G : J \to P_{bd, cl, cv}(X)$ is said to be measurable if for each $x \in X$, the function $t \mapsto D(x, G(t))$ is a measurable function on J. For more detail on the multivalued maps, see the books of Deimling [12].

Definition 2.1. Let $F : X \to P_{bd, cl}(X)$ be a multivalued map. Then F is called a multivalued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$ we have

$$H(F(x), F(y)) \leq k\|x - y\|.$$

The constant k is called a contraction constant of F.

Theorem 2.1. *Let X be a Banach space, $\Psi_1 : X \to P_{bd, cl, cv}(X)$ and $\Psi_2 : X \to P_{cp, cv}(X)$ be two multivalued maps satisfying*:

(i) Ψ_1 *is a contraction with a contraction constant* k, *and*

(ii) Ψ_2 *is completely continuous*.

Then, either

(1) the operator inclusion $x \in \Psi_1 x + \Psi_2 x$ has a solution, or

(2) the set $\varepsilon = \{x \in X : x \in l_1\Psi_1 x + l_1\Psi_2 x\}$ is unbounded for $l_1 \in (0, 1)$.

Theorem 2.2 [13]**.** *Let X be a Banach space, G be an L^1-Carathéodory multivalued map with $S_{G, \phi} \neq \emptyset$ where $S_{G, \phi} := \{g \in L^1(J, X) : g(t) \in G(t, \phi)$ a.e. $t \in J\}$, for each fixed $\phi \in \mathcal{B}_h$, and K be a linear continuous map from $L^1(J, X)$ to $C(J, X)$. Then the operator $K \circ S_{G, \phi} : C(J, X) \to P_{cp, cv}(C(J, X))$ is a closed graph operator in $C(J, X) \times C(J, X)$.*

Definition 2.2 [14]**.** An integrated semigroup $\{T_0(t)\}_{t \geq 0}$ is called locally

Lipschitz continuous if, for all $\delta > 0$, there exists constant γ such that

$$|T_0(t) - T_0(s)| \le \gamma |t - s|, \quad t, s \in [0, \delta].$$

Definition 2.3 [15]**.** We say that the linear operator A satisfies the Hille-Yosida condition if there exists $\overline{M} \ge 0$ and $\omega \in R$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in N, \lambda > \omega\} \le \overline{M}.$$

Theorem 2.3 [14]**.** *The following assertions are equivalent*:

(i) *A is the generator of a non-degenerate*, *locally Lipschitz continuous integrated semigroup*;

(ii) *A satisfies the Hille-Yosida condition*.

Definition 2.4. A function $x : (-\infty, b) \to X$ is called a mild solution of (1.1) if the following holds: $x_0 = \phi(t) \in \mathcal{B}_h$ on $(-\infty, 0)$, $\Delta x(t_k) = I_k(x(t_k^-)))$, $k = 1, 2, \dots, m$; the restriction of $x(\cdot)$ to the interval $[0, b] - \{t_1, \dots, t_m\}$ is continuous; for each $t \in [0, b]$, the function $AT_0(t - s)p(s, x_s)$, $s \in [0, t)$ is integrable such that

$$\begin{aligned} x(t) = {} & \phi(0) - p(0, \phi) + p(t, x_t) + A\int_0^t x(s)ds \\ & + \int_0^t [Bu(s) + f(s, x_s) + \int_0^s g(s, \tau, x_\tau)d\tau]ds \\ & - \sum_{0<t_k<t} I_k(x(t_k^-)), \end{aligned}$$

where $g \in S_{G,x} = \{g \in L^1(J, X) : g(t) \in G(t, s, x_s),$ for a.e. $t \in J\}$.

Let A_0 be the part of A on $\overline{D(A)}$ define by

$$\begin{aligned} & D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ & A_0 x = Ax. \end{aligned}$$

Then A_0 generates a strongly continuous semigroup T_0 on $\overline{D(A)}$ (see Pazy [16] for semigroup theory for semigroup theory) and the general solution in Definition 2.4 (if it exists) is given by

$$x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_0(t)[\phi(0) - p(0, \phi)] + p(t, x_t) + \int_0^t AT_0(t-s)p(s, x_s)ds \\ \quad + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)B(\lambda)[Bu(s) + f(s, x_s) + \int_0^s g(s, \tau, x_\tau)d\tau]ds \\ \quad + \sum_{0<t_k<t} T_0(t-t_k)I_k(x(t_k^-)), \qquad t \in J, \end{cases}$$

where $B(\lambda) = \lambda R(\lambda, A)$.

Remark 2.1. We should point out here that, from Definition 2.4, it is not difficult to verify that if x is an integral solution of (1.1) on $(-\infty, b]$, then for all $t \in J$, $x(t) - p(t, x_t) \in \overline{D(A)}$. In particular, $\phi(t) - p(0, \phi) \in \overline{D(A)}$.

Definition 2.5. The system (1.1) is said to be controllable on the interval if for every initial function $\phi \in \mathcal{B}_h$ with $\phi(t) - p(0, \phi) \in \overline{D(A)}$ and $x_1 \in \overline{D(A)}$, there exists a control $u \in L^2(J, U)$ such that the integral solution $x(\cdot)$ of (1.1) satisfies $x(b) = x_1$.

Lemma 2.1 [5]. *Assume* $x \in \mathcal{B}'_h$, *then for* $t \in J$, $x_t \in \mathcal{B}_h$. *Moreover*,

$$l|x(t)| \leq \|x_t\|_{\mathcal{B}_h} \leq \|\phi\|_{\mathcal{B}_h} + l \sup_{s \in [0, t]} |x(s)|,$$

where $l = \int_{-\infty}^0 h(t)dt < +\infty$.

In order to establish our result we consider the following assumption in the sequel:

(H_1) A satisfies the Hille-Yosida condition.

Let $B(\lambda) = \lambda R(\lambda, A) = \lambda R(\lambda, A) = \lambda(\lambda I - A)^{-1}$. Then for all $x \in \overline{D(A)}$, $B(\lambda)x \to x$ as $\lambda \to \infty$. Also from Definition 2.3 (with $n = 1$), it is easy to see that

$$\lim_{\lambda\to+\infty}|B(\lambda)x| \le \overline{M}|x|, \text{ because } |B(\lambda)| = |\lambda(\lambda I - A)^{-1}| \le \frac{\overline{M}\lambda}{\lambda-\omega}.$$

Thus $\lim_{\lambda\to+\infty}|B(\lambda)| \le \overline{M}$.

(H_2) The operator A generates a strongly continuous semigroup $T_0(t)$ in $\overline{D(A)}$ and there exists $M_1 \ge 1$, such that

$$\|T_0(t)\| \le M_1, \quad \forall t \in J.$$

(H_3) The operator $B : U \to X$ is bounded and linear. The linear operator $W : L^2(J, U) \to \overline{D(A)}$ defined by

$$Wu = \lim_{\lambda\to+\infty}\int_0^b T_0(b-s)B(\lambda)Bu(s)\,ds,$$

has an induced inverse operator $W^{-1} : \overline{D(A)} \to L^2(J, U)$ and there exists a positive constant M_2 such that $\|BW^{-1}\| \le M_2$.

(H_4) (i) The function $p : J \times \mathcal{B}_h \to X$ is continuous and there exists constants $L_2 > 0, L_3 > 0$, such that the function $A^k p$ satisfies the Lipschitz conditions:

$$\|A^k p(t_1, \phi_1) - A^k p(t_2, \phi_2)\| \le L_2(|t_1 - t_2| + \|\phi_1 - \phi_2\|_{\mathcal{B}_h}),$$

$$k = 0, 1,\ t_1, t_2 \in J,\ \phi_1, \phi_2 \in \mathcal{B}_h.$$

and

$$\|AT_0(t_1 - s)p(s, \phi) - AT_0(t_2 - s)p(s, \phi)\| \le L_3(|t_1 - t_2|), \quad t_1, t_2 \in J.$$

(ii) There exists a constant k_1, k_2 and k_3 such that

$$\|AT_0(t-s)p(s, \phi)\| \le k_1\|\phi\|_{\mathcal{B}_h} + k_2, \quad t \in J,\ \phi \in \mathcal{B}_h.$$

and

$$\|p(t, \phi)\| \le k_3(\|\phi\|_{\mathcal{B}_h} + 1), \quad t \in J,\ \phi \in \mathcal{B}_h.$$

(H_5) $I_k \in C(X, X)$ and there exist constants $c_k \geq 0$, $k = 1, \ldots, m$ and continuous nondecreasing functions $L_k : [0, +\infty) \to (0, +\infty)$ such that

$$\|I_k(x) - I_k(y)\| \leq c_k \|x - y\|, \quad \forall x, y \in X.$$

and

$$\|I_k(x)\| \leq L_k(x), \; k = 1, 2, \ldots, m, \quad x, y \in X.$$

(H_6) (i) $f : J \times \mathcal{B}_h \to X; (t, \phi) \to f(t, \phi)$ is strongly measurable with respect to t for each $\phi \in \mathcal{B}_h$, and continuous with respect to ϕ for a.e. $t \in J$, and f satisfies the Lipschitz conditions, that is, there exists a constant $L_4 > 0$ such that

$$\|f(t_1, \phi_1) - f(t_2, \phi_2)\| \leq L_4(|t_1 - t_2| + \|\phi_1 - \phi_2\|_{\mathcal{B}_h}),$$

$$t_1, t_2 \in J, \phi_1, \phi_2 \in \mathcal{B}_h.$$

(ii) There exists a continuous nondecreasing function $\alpha : R_+ \to (0, +\infty)$ such that

$$\|f(t, \phi)\| \leq \alpha(\|\phi\|_{\mathcal{B}_h}), \quad \text{for a.e. } t \in J.$$

(H_7) The multivalued map $G : J \times J \times \mathcal{B}_h \to P_{bd, cl, cv}(X)$ satisfies the following conditions:

(i) $G : J \times J \times B_h \to P_{bd, cl, cv}(X); (t, s, \phi) \to G(t, s, \phi)$ is strongly measurable with respect to t, s for each $\phi \in \mathcal{B}_h$, and u.s.c with respect to ϕ for a.e. $(t, s) \in J \times J$.

(ii) There exists a constant L_1 such that

$$\|G(t, \phi_1) - G(t, \phi_2)\| \leq L_1(|t_1 - t_2| + \|\phi_1 - \phi_2\|_{\mathcal{B}_h}), \; t_1, t_2 \in J, \phi_1, \phi_2 \in \mathcal{B}_h.$$

(iii) There exists a positive function $p_1 \in L^1(J, [0, +\infty))$ such that

$$\|G(t, s, \phi)\| := \sup\{\|v\| : v \in G(t, s, \phi)\} \leq p_1(t)\Theta(\|\phi\|_{\mathcal{B}_h}), \; a.e. \; t \in J, \phi \in \mathcal{B}_h,$$

where $\Theta : [0, \infty) \to (0, \infty)$ a continuous nondecreasing function and there exists a constant d with

$$\frac{(1-k_3-bk_1-b^2M_1MM_2k_1)d}{N_1+K_0} > 1, \tag{1}$$

where

$$K_0 = b\overline{M}M_1\alpha(d) + b\overline{M}M_1\Theta(d)\int_0^b p_1(s)\,ds + M_1\sum_{k=1}^{m} L_k(l^{-1}d)$$

$$+ bM_1\overline{M}M_2\{bk_1d + b\overline{M}M_1\alpha(d)$$

$$+ b\overline{M}M_1\Theta(d)\int_0^b p_1(s)\,ds + M_1\sum_{k=1}^{m} L_k(l^{-1}d\},$$

and

$$N_1 = M_1k_3(\|\phi\|_{\mathcal{B}_h}+1) + k_3bk_2 + bM_1\overline{M}M_2\{|x_1| + M_1\|\phi(0)\|$$

$$+ M_1k_3(\|\phi\|_{\mathcal{B}_h}+1) + k_3(\|y_b+\hat{\phi}_b\|_{\mathcal{B}_h}+1) + bk_2\}.$$

Remark 2.2. The construction of the operator W and its inverse is studied by Quinn and Carmichael [17].

3. Controllability Result

Theorem 3.1. *Suppose that* (H_1) - (H_7) *are satisfied and*

$$C_0 = [M_1M_2\overline{M}b(L_2 + M_1bL_2 + M_1\overline{M}bL_4 + M_1\overline{M}b^2L_1 + M_1\sum_{k=1}^{m} c_k)$$

$$+ M_1\sum_{k=1}^{m} c_k] < 1. \tag{2}$$

Let $\phi \in \mathcal{B}_h$ *with* $\phi(t) - p(0, \phi) \in \overline{D(A)}$. *Then the system* (1.1) *is controllable on J.*

Proof. Using hypothesis (H_3) for an arbitrary function $x(\cdot)$, define the control process

$$u(t) = W^{-1}[x_1 - T_0(b)[\phi(0) - p(0, \phi)] - p(b, x_b)$$

$$- \int_0^b AT_0(b-s)p(s, x_s)ds$$

$$- \lim_{\lambda\to\infty} \int_0^b T_0(b-s)B(\lambda)[f(s, x_s) + \int_0^s g(s, \tau, x_\tau)d\tau]ds$$

$$- \sum_{k=1}^{m} T_0(b-t_k)I_k(x(t_k^-))](t),$$

where $g \in S_{G,x}$.

We shall now show that when using this control the operator Γ defined by

$$(\Gamma x)(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_0(t)[\phi(0) - p(0, \phi)] + p(t, x_t) + \int_0^t AT_0(t-s)p(s, x_s)ds \\ \quad + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B(\lambda)[Bu(s) + f(s, x_s) + \int_0^s g(s, \tau, x_\tau)d\tau]ds \\ \quad + \sum_{0<t_k<t} T_0(t-t_k)I_k(x(t_k^-)), \qquad t \in J, \end{cases}$$

where $g \in S_{G,x} = \{g \in L^1(J, X) : g(t) \in G(t, s, x_s),$ for a.e. $t \in J\}$. It has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a integral solution of the system (1.1). Clearly, $x(b) = (\Gamma x)(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time b, provide we can obtain a fixed point of the operator Γ which implies that the system is controllable. For $\phi \in \mathcal{B}_h$, we define $\hat{\phi}$ by

$$\hat{\phi} = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_0(t)\phi(0), \quad t \in J, \end{cases}$$

then $\hat{\phi} \in \mathcal{B}'_h$. Let $x(t) = y(t) + \hat{\phi}(t), -\infty < t \le b$. It easy to see that y satisfies $y_0 = 0$ and

$$y(t) = -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds$$

$$+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s,\ y_s+\hat{\phi}_s)+\int_0^s g(s,\ \tau,\ y_\tau+\hat{\phi}_\tau)d\tau]ds$$

$$+\sum_{0<t_k<t} T_0(t-t_k)I_k(y(t_k^-)+\hat{\phi}(t_k^-))+\lim_{\lambda\to\infty}\int_0^t T_0(t-\eta)B(\lambda)BW^{-1}$$

$$[x_1-T_0(b)[\phi(0)-p(0,\ \phi)]-p(b,\ y_b+\hat{\phi}_b)-\int_0^b AT_0(b-s)p(s,\ y_s+\hat{\phi}_s)ds$$

$$-\lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)[f(s,\ y_s+\hat{\phi}_s)+\int_0^s g(s,\ \tau,\ y_\tau+\hat{\phi}_\tau)d\tau]ds$$

$$-\sum_{k=1}^m T_0(b-t_k)I_k(y(t_k^-)+\hat{\phi}(t_k^-))](\eta)d\eta,$$

if and only if x satisfies

$$x(t)=T_0(t)[\phi(0)-p(0,\ \phi)]+p(t,\ x_t)+\int_0^t AT_0(t-s)p(s,\ x_s)ds$$

$$+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s,\ x_s)+\int_0^s g(s,\ \tau,\ x_\tau)d\tau]ds$$

$$+\sum_{0<t_k<t} T_0(t-t_k)I_k(x(t_k^-))+\lim_{\lambda\to\infty}\int_0^t T_0(t-\eta)B(\lambda)BW^{-1}$$

$$[x_1-T_0(b)[\phi(0)-p(0,\ \phi)]-p(b,\ x_b)-\int_0^b AT_0(b-s)p(s,\ x_s)ds$$

$$-\lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)[f(s,\ x_s)+\int_0^s g(s,\ \tau,\ x_\tau)d\tau]ds$$

$$-\sum_{k=1}^m T_0(b-t_k)I_k(x(t_k^-))](\eta)d\eta,$$

and

$$x(t)=\phi(t),\quad t\in(-\infty,\ 0].$$

Define $\mathcal{B}''_h=\{y\in\mathcal{B}'_h : y_0=0\in\mathcal{B}_h\}$. For any $y\in\mathcal{B}''_h$,

$$\|y\|'_{\mathcal{B}_h} = \|y\|_{\mathcal{B}_h} + \sup\{|y(s)| : 0 \le s \le b\} = \sup\{|y(s)| : 0 \le s \le b\},$$

thus $(\mathcal{B}''_h, \|\cdot\|'_h)$ is a Banach space. Set

$$B_q = \{y \in \mathcal{B}''_h : \|y\|'_{\mathcal{B}_h} \le q\}$$

for some $q > 0$, then $\mathcal{B}_q$ for each q, is a bounded, closed convex set in $\mathcal{B}''_h$. Moreover, for any $y \in \mathcal{B}_q$, from Lemma 2.1, we have

$$\begin{aligned}
\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h} &\le \|y_t\|_{\mathcal{B}_h} + \|\hat{\phi}_t\|_{\mathcal{B}_h} \\
&\le l \sup_{s\in[0,t]} |y(s)| + \|y_0\|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} |\hat{\phi}(s)| + \|\hat{\phi}_0\|_{\mathcal{B}_h} \\
&\le l(q + M_1|\phi(0)|) + \|\phi\|_{\mathcal{B}_h} = q'. \qquad (3)
\end{aligned}$$

In view of Lemma 2.1, for each $t \in J$,

$$|y(t) + \hat{\phi}(t)| \le l^{-1}\|y_t - \hat{\phi}_t\|_{\mathcal{B}_h}.$$

For each $t \in J$, $y \in B_q$, we have (3) and (H_5)

$$\sup_{t\in J} |y(t) + \hat{\phi}(t)| \le l^{-1}\|y_t - \hat{\phi}_t\|_{\mathcal{B}_h} \le l^{-1}q',$$

$$\begin{aligned}
|I_k(y(t_k^-) + \hat{\phi}(t_k^-))| &\le L_k(|y(t_k^-) + \phi(t_k^-)|) \\
&\le L_k(\sup_{t\in J} |y(t) - \hat{\phi}(t)|) \\
&\le L_k(l^{-1}q'),\ k = 1, \dots, m.
\end{aligned}$$

Let the operator $\Psi : \mathcal{B}''_h \to \mathcal{B}''_h$ defined by Ψy the set of $\hat{\rho} \in \mathcal{B}''_h$ such that

$$\hat{\rho}(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds \\ + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds \\ + \sum_{0<t_k<t} T_0(t-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-)) + \lim_{\lambda\to\infty}\int_0^t T_0(t-\eta)B(\lambda)BW^{-1} \\ [x_1 - T_0(b)[\phi(0) - p(0, \phi)] - p(b, y_b + \hat{\phi}_b) - \int_0^b AT_0(b-s)p(s, y_s + \hat{\phi}_s)ds \\ - \lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds \\ - \sum_{k=1}^m T_0(b-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-))](\eta)d\eta, & t \in J. \end{cases}$$

We can see that if Ψ has a fixed point in $\mathcal{B}_h''$, then Γ has a fixed point in $\mathcal{B}_h'$ which is a solution of system (1.1).

Now, we consider the following multi-valued operators Ψ_1 and Ψ_2 defined by

$$\Psi_1(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \lim_{\lambda\to\infty}\int_0^t T_0(t-\eta)B(\lambda)BW^{-1}[x_1 - T_0(b)[\phi(0) - p(0, \phi)] \\ - p(b, y_b + \hat{\phi}_b) - \int_0^b AT_0(b-s)p(s, y_s + \hat{\phi}_s)ds \\ - \lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds \\ - \sum_{k=1}^m T_0(b-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-))](\eta)d\eta \\ + \sum_{0<t_k<t} T_0(t-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-)), & t \in J \end{cases}$$

and

$$\Psi_2(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds \\ + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds, \; t \in J. \end{cases}$$

It is clear that

$$\Psi = \Psi_1 + \Psi_2.$$

The problem of finding mild solutions of (1.1) is reduced to find the solutions of the operator inclusion $x \in \Psi_1(x) + \Psi_2(x)$. In the follow, we show that the operators Ψ_1 and Ψ_2 satisfy the conditions of Theorem 2.1.

Step 1. Ψ_1 is a contraction.

Let $y_1, y_2 \in \mathcal{B}''_h$. By the assumption, we have

$$\|\Psi_1(y_1) - \Psi_1(y_2)\| \le \lim_{\lambda\to\infty}\int_0^t \|T_0(t-\eta)\|\|B(\lambda)\|$$

$$\|BW^{-1}\{[p(b, y_{2,b} + \hat{\phi}_b) - p(b, y_{1,b} + \hat{\phi}_b)]$$

$$+ \int_0^b AT_0(b-s)[p(s, y_{2,s} + \hat{\phi}_s) - p(s, y_{1,s} + \hat{\phi}_s)]ds$$

$$+ \lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)[f(s, y_{2,s} + \hat{\phi}_s) - f(s, y_{1,s} + \hat{\phi}_s)]ds$$

$$+ \lim_{\lambda\to\infty}\int_0^b T_0(b-s)B(\lambda)\int_0^s [g(s, \tau, y_{2,\tau} + \hat{\phi}_\tau)d\tau$$

$$- g(s, \tau, y_{1,\tau} + \hat{\phi}_\tau)d\tau]ds - \sum_{k=1}^m T_0(b-t_k)[I_k(y_2(t_k^-) + \hat{\phi}(t_k^-))$$

$$- I_k(y_1(t_k^-) + \hat{\phi}(t_k^-))](\eta)\}d\eta\| + \sum_{0<t_k<t}\|T_0(t-t_k)\|$$

$$\|[I_k(y_1(t_k^-) + \hat{\phi}(t_k^-)) - I_k(y_2(t_k^-) + \hat{\phi}(t_k^-))]\|$$

$$\le M_1 M_2 \overline{M}\int_0^b \{L_2\|y_{2,b} - y_{1,b}\|_{\mathcal{B}_h}$$

$$+ \int_0^b M_1 L_2 \|y_{2,s} - y_{1,s}\|_{\mathcal{B}_h} ds$$

$$+\int_0^b M_1\overline{M}L_4\|y_{2,s}-y_{1,s}\|ds$$

$$+M_1\overline{M}\int_0^b\int_0^b L_1\|y_{2,\tau}-y_{1,\tau}\|d\tau ds$$

$$+M_1\sum_{k=1}^{m}c_k\|(y_2(t_k^-)-y_1(t_k^-)\|\}d\eta$$

$$+M_1\sum_{k=1}^{m}c_k\|(y_1(t_k^-)-y_2(t_k^-)\|$$

$$\leq [M_1M_2\overline{M}b(L_2+M_1bL_2+M_1\overline{M}bL_4+M_1\overline{M}b^2L_1$$

$$+M_1\sum_{k=1}^{m}c_k)+M_1\sum_{k=1}^{m}c_k]\|y_1-y_2\|$$

$$\leq C_0\|y_1-y_2\|,$$

where C_0 is given in (2). hence, Ψ_1 is a contraction.

Step 2. Ψ_2 has compact, convex value and it is completely continuous. This will divided into the following claims.

Claim 1. $\Psi_2 y$ is convex for each $y\in\mathcal{B}_h''$.

In fact, if $\hat{\rho}_1,\hat{\rho}_2\in\Psi_2 y$, then, there exists $g_1, g_2\in S_{G,y}$ such that

$$\hat{\rho}_i=-T_0(t)p(0,\phi)+p(t,y_t+\hat{\phi}_t)+\int_0^t AT_0(t-s)p(s,y_s+\hat{\phi}_s)ds$$

$$+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s,y_s+\hat{\phi}_s)+\int_0^s g_i(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau]ds$$

$$i=1,2,\quad t\in J.$$

Let $\beta\in[0,1]$. Since the operators B and W^{-1} are linear, we have

$$(\beta\hat{\rho}_1+(1-\beta)\hat{\rho}_2)(t)$$

$$= -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds$$

$$+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s [\beta g_1(s, \tau, y_\tau + \hat{\phi}_\tau)$$

$$+ (1-\beta)g_2(s, \tau, y_\tau + \hat{\phi}_\tau)]d\tau]ds.$$

Since $S_{G, y}$ is convex (because G has convex values), we have

$$(\beta\hat{\rho}_1 + (1-\beta)\hat{\rho}_2)(t) \in \Psi_2.$$

Claim 2. $\Psi_2 y$ maps bounded sets into bounded sets in $\mathcal{B}_h''$.

Indeed, it is enough to show that there exists a positive constant Λ such that for $\overline{\rho} \in \Psi_2 y,\ y \in B_q = \{y \in \mathcal{B}_h'' : \|y\|'_{\mathcal{B}_h} \le q\}$, one have $\|\overline{\rho}\|'_{\mathcal{B}_h} \le \Lambda$.

If $\overline{\rho} \in \Psi_2 y$, then there exists $g \in S_{G, y}$ such that, for each $t \in J$

$$\overline{\rho}(t) = -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds$$

$$+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds. \quad (4)$$

Therefore, by hypotheses $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_4), (\mathrm{H}_6)$, and (H_7), we observe that, for each $t \in J$

$$\|\overline{\rho}(t)\| \le M_1 k_3(\|\phi\|_{B_h} + 1) + k_3(\|y_t + \hat{\phi}_t\|_{B_h} + 1) + b(k_1\|y_s + \hat{\phi}_s\|_{B_h} + k_2)$$

$$+ \overline{M}M_1\int_0^b \alpha_{q'}(s)ds + \overline{M}M_1\int_0^b\int_0^b p_1(s)\Theta(\|y_s + \hat{\phi}_s\|_{B_h})d\tau ds$$

$$:= \Lambda.$$

Then, for each $\overline{\rho} \in \Psi_2 y$, we have $\|\overline{\rho}\|'_{\mathcal{B}_h} \le \Lambda$.

Claim 3. $\Psi_2 y$ maps bounded sets into equicontinuous sets of $\mathcal{B}_h''$.

Let $0 < \tau_1 < \tau_2 \leq b$. Then, we have for each and $y \in B_q$ and $\overline{\rho} \in \Psi_2 y$, there exists $g \in S_{G,y}$ such (4) holds. Therefore,

$$\begin{aligned}
\|\overline{\rho}(\tau_2) - \overline{\rho}(\tau_1)\| &\leq \|[T_0(\tau_1) - T_0(\tau_2)]p(0, \phi)\| + \|p(\tau_2, y_{\tau_2} + \hat{\phi}_{\tau_2}) - p(\tau_1, y_{\tau_1} + \hat{\phi}_{\tau_1})\| \\
&+ \left\|\int_0^{\tau_1} [AT_0(\tau_2 - s)p(s, y_s + \hat{\phi}_s) - AT_0(\tau_1 - s)p(s, y_s + \hat{\phi}_s)]ds\right\| \\
&+ \left\|\int_{\tau_1}^{\tau_2} AT_0(\tau_2 - s)p(s, y_s + \hat{\phi}_s)ds\right\| \\
&+ \left\|\lim_{\lambda\to\infty}\int_0^{\tau_1} [T_0(\tau_2 - s) - T_0(\tau_1 - s)]B(\lambda)f(s, y_s + \hat{\phi}_s)ds\right\| \\
&+ \left\|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} T_0(\tau_2 - s)B(\lambda)f(s, y_s + \hat{\phi}_s)ds\right\| \\
&+ \left\|\lim_{\lambda\to\infty}\int_0^{\tau_1}\int_0^b [T_0(\tau_2 - s) - T_0(\tau_1 - s)]B(\lambda)g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau ds\right\| \\
&+ \left\|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2}\int_0^b T_0(\tau_2 - s)B(\lambda)g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau ds\right\| \\
&\leq \|T_0(\tau_1) - T_0(\tau_2)\|\|p(0, \phi)\| \\
&+ L_2[|\tau_2 - \tau_1| + \|y_{\tau_2} - y_{\tau_1}\|_{B_h} + \|y_{\hat{\phi}_{\tau_2}} - y_{\hat{\phi}_{\tau_1}}\|_{B_h}] \\
&+ L_3\int_0^{\tau_1} |\tau_2 - \tau_1|ds + \int_{\tau_1}^{\tau_2}[k_1\|y_s + \hat{\phi}_s\| + k_2]ds \\
&+ \overline{M}\int_0^{\tau_1}\|T_0(\tau_2 - s) - T_0(\tau_1 - s)\|\alpha_{q'}(s)ds \\
&+ \overline{M}M_1\int_{\tau_1}^{\tau_2}\alpha_{q'}(s)ds \\
&+ \overline{M}b\int_0^{\tau_1}\|T_0(\tau_2 - s) - T_0(\tau_1 - s)\|p_1(s)\Theta(q')ds
\end{aligned}$$

$$+ M_1\overline{M}b\int_{\tau_1}^{\tau_2} p_1(s)\Theta(q')ds.$$

The right-hand side is independent of $y \in B_q$ and tends to zero as $\tau_2 \to \tau_1$, since the strong continuity of $T_0(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, the set $\{\Psi_2 y : y \in B_q\}$ is equicontinuous.

Claim 4. $\Psi_2 y$ is compact multi-valued map.

From the above claims, we see that family $\Psi_2 B_q$ is a uniformly bounded and equicontinuous collection. Therefore, it suffices to show by Arzelá-Ascoli theorem that Ψ_2 map B_q into a precompact set into $\mathcal{B}''_h$. That is for each fixed $t \in J$, the set $V(t) = \{\Psi_2 y(t) : y \in B_q\}$ is precompact in *X*.

Obviously, $V(0) = \{\Psi_2(0)\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$, define

$$\Psi_2^{\epsilon} y(t) = -T_0(t)p(0, \phi) + p(t, y_t + \hat{\phi}_t) + \int_0^{t-\epsilon} AT_0(t-s)p(s, y_s + \hat{\phi}_s)ds$$

$$+ \| \lim_{\lambda\to\infty} \int_0^{t-\epsilon} T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds.$$

Since T_0 is a strongly continuous operator, and from condition (H_6) and (H_7), the set $V_\epsilon(t) = \{\Psi_2^{\epsilon} y(t) : y \in B_q\}$ is precompact *X* for every ϵ, $0 < \epsilon < t$. Moreover,

$$\|\Psi_2 y(t) - \Psi_2^{\epsilon} y(t)\| \leq \int_{t-\epsilon}^{\epsilon} \|AT_0(t-s)p(s, y_s + \hat{\phi}_s)\|ds$$

$$+ \lim_{\lambda\to\infty} \int_{t-\epsilon}^{t} T_0(t-s)B(\lambda)\|f(s, y_s + \hat{\phi}_s)\|ds$$

$$+ \lim_{\lambda\to\infty} \int_{t-\epsilon}^{t} \|T_0(t-s)B(\lambda)g(s, \tau, y_\tau + \hat{\phi}_\tau)\|d\tau]ds.$$

Therefore,

$$\|\Psi_2 y(t) - \Psi_2^{\epsilon} y(t)\| \to 0, \quad a.s\ \epsilon \to 0^+,$$

and there are precompact sets arbitrary close to the set $\{\Psi_2 y(t) : y \in B_q\}$. Hence, the Arzelá-Ascoli shows that Ψ_2 is a compact multi-valued map.

Claim 5. $\Psi_2 y$ has a closed graph.

Let $y^n \to y^*$, $\overline{\rho}_n \to \Psi_2 y^n$ and $\overline{\rho}_n \to \overline{\rho}_*$. We aim to show that $\overline{\rho}_* \in \Psi_2 y^*$. Indeed, $\overline{\rho}_n \in \Psi_2 y^n$ means that there exists $g_n \in S_{G, y^n}$ such that

$$\overline{\rho}_n(t) = -T_0(t)p(0, \phi) + p(t, y_t^n + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s^n + \hat{\phi}_s)ds$$
$$+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B(\lambda)[f(s, y_s^n + \hat{\phi}_s) + \int_0^s g_n(s, \tau, y_\tau^n + \hat{\phi}_\tau)d\tau]ds,$$
$$t \in J.$$

We need to prove that there exists $g_* \in S_{G, y^*}$ such that

$$\overline{\rho}_*(t) = -T_0(t)p(0, \phi) + p(t, y_t^* + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s, y_s^* + \hat{\phi}_s)ds$$
$$+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B(\lambda)[f(s, y_s^* + \hat{\phi}_s) + \int_0^s g_*(s, \tau, y_\tau^* + \hat{\phi}_\tau)d\tau]ds,$$
$$t \in J.$$

Since p and f is continuous, we get

$$\| [\overline{\rho}_n(t) + T_0(t)p(0, \phi) - p(t, y_t^n + \hat{\phi}_t)$$
$$- \int_0^t AT_0(t-s)p(s, y_s^n + \hat{\phi}_s)ds$$
$$- \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B(\lambda)f(s, y_s^n + \hat{\phi}_s)ds]$$
$$- [\overline{\rho}_*(t) + T_0(t)p(0, \phi) - p(t, y_t^* + \hat{\phi}_t)$$
$$- \int_0^t AT_0(t-s)p(s, y_s^* + \hat{\phi}_s)ds$$

$$-\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)f(s,\ y_s^* + \hat{\phi}_s)ds]\| \to 0,\ as\ n \to \infty.$$

Consider the linear continuous operator $K : L^1(J,\ X) \to C(J,\ X)$ define by

$$Kg(t) = \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)\int_0^s g(s,\ \tau,\ y_\tau + \hat{\phi}_\tau)d\tau ds.$$

From Theorem 2.2, it follows that $K \circ S_G$ is a closed graph operator, and

$$\overline{\rho}_n(t) + T_0(t)p(0,\ \phi) - p(t,\ y_t^n + \hat{\phi}_t) - \int_0^t AT_0(t-s)p(s,\ y_s^n + \hat{\phi}_s)ds$$

$$-\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)f(s,\ y_s^n + \hat{\phi}_s)ds \in K \circ S_{G,\ y^n}.$$

Since $y^n \to y^*$ and $\overline{\rho}_n \to \overline{\rho}_*$, it follows from Theorem 2.2 that, there exists an $g_* \in S_{G,\ y^*}$, such that

$$\overline{\rho}_*(t) = -T_0(t)p(0,\ \phi) + p(t,\ y_t^* + \hat{\phi}_t) + \int_0^t AT_0(t-s)p(s,\ y_s^* + \hat{\phi}_s)ds$$

$$+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)B(\lambda)[f(s,\ y_s^* + \hat{\phi}_s) + \int_0^s g_*(s,\ \tau,\ y_\tau^* + \hat{\phi}_\tau)d\tau]ds.$$

Therefore, $\Psi_2 y$ is completely continuous multi-valued map, u.s.c. with convex closed, compact values.

Step 3. A priori estimate.

Now it remains to show that the set

$$\varepsilon = \{y \in \mathcal{B}_h'' : y \in l_1\Psi_1 y + l\Psi_2 y, \text{ for some } 0 < l_1 < 1\}$$

is bounded.

Let $y \in \varepsilon$, then there exist $g \in S_{G,\ y}$ such that

$$y(t) = -l_1 T_0(t)p(0,\ \phi) + l_1 p(t,\ y_t + \hat{\phi}_t) + l_1\int_0^t AT_0(t-s)p(s,\ y_s + \hat{\phi}_s)ds$$

$$+ l_1 \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds$$

$$+ l_1 \sum_{0<t_k<t} T_0(t-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-)) + l_1 \lim_{\lambda\to\infty} \int_0^t T_0(t-\eta)B(\lambda)BW^{-1}$$

$$[x_1 - T_0(b)[\phi(0) - p(0, \phi)] - p(b, y_b + \hat{\phi}_b) - \int_0^b AT_0(b-s)p(s, y_s + \hat{\phi}_s)ds$$

$$- \lim_{\lambda\to\infty} \int_0^b T_0(b-s)B(\lambda)[f(s, y_s + \hat{\phi}_s) + \int_0^s g(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau]ds$$

$$- \sum_{k=1}^m T_0(b-t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-))](\eta)d\eta$$

for some $0 < l_1 < 1$. Then, by the assumptions, we have

$$\|y(t)\| \leq M_1k_3(\|\phi\|_{\mathcal{B}_h} + 1) + k_3(\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h} + 1) + b(k_1\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h} + k_2)$$

$$+ \overline{M}M_1\int_0^b \alpha(\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h})ds + b\overline{M}M_1\int_0^t p_1(s)\Theta(\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h})ds$$

$$+ M_1\sum_{k=1}^m L_k(l^{-1}\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h}) + bM_1\overline{M}M_2\{|x_1| + M_1\|\phi(0)\| + M_1k_3(\|\phi\|_{\mathcal{B}_h} + 1)$$

$$+ k_3(\|y_b + \hat{\phi}_b\|_{\mathcal{B}_h} + 1) + b(k_1\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h} + k_2) + \overline{M}M_1\int_0^b \alpha(\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h})ds$$

$$+ b\overline{M}M_1\int_0^b p_1(s)\Theta(\|y_s + \hat{\phi}_s\|_{\mathcal{B}_h})ds + M_1\sum_{k=1}^m L_k(l^{-1}\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h})\}.$$

Let

$$\mu(t) = l \sup_{s\in[0,t]}|y(s)| + \|y_0\|_{\mathcal{B}_h} + l \sup_{s\in[0,t]}|\hat{\phi}(s)| + \|\hat{\phi}_0\|_{\mathcal{B}_h},\ t \in [0, b],$$

where

$$\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h} \leq \|y_t\|_{\mathcal{B}_h} + \|\hat{\phi}_t\|_{\mathcal{B}_h}$$

$$\leq l \sup_{s\in[0,t]} |y(s)| + \sup\{\|\hat{\phi}(s)\|_{\mathcal{B}_h} : 0 \leq s \leq b\}.$$

Therefore, we get

$$\begin{aligned}\mu(t) &\leq M_1 k_3(\|\phi\|_{\mathcal{B}_h} + 1) + k_3(\mu(t) + 1) + b(k_1\mu(t) + k_2) \\ &+ \overline{M}M_1 \int_0^b \alpha(\mu(s))\,ds + b\overline{M}M_1 \int_0^t p_1(s)\Theta(\mu(s))\,ds \\ &+ M_1 \sum_{k=1}^m L_k(l^{-1}\mu(t)) + bM_1\overline{M}M_2\{|x_1| + M_1\|\phi(0)\| + M_1 k_3(\|\phi\|_{\mathcal{B}_h} + 1) \\ &+ k_3(\|y_b + \hat{\phi}_b\|_{\mathcal{B}_h} + 1) + b(k_1\mu(t) + k_2) + \overline{M}M_1 \int_0^b \alpha(\mu(s))\,ds \\ &+ b\overline{M}M_1 \int_0^b p_1(s)\Theta(\mu(s))\,ds + M_1 \sum_{k=1}^m L_k(l^{-1}\mu(t))\}.\end{aligned}$$

Consider the norm of the function $\mu(t)$, $\|\mu\| = \sup\{\mu(t) : 0 \leq t \leq b\}$, therefore, we have

$$\begin{aligned}\|\mu\| &\leq M_1 k_3(\|\phi\|_{\mathcal{B}_h} + 1) + k_3(\|\mu\| + 1) + b(k_1\|\mu\| + k_2) \\ &+ b\overline{M}M_1\alpha(\|\mu\|) + b\overline{M}M_1\Theta(\|\mu\|)\int_0^b p_1(s)\,ds + M_1 \sum_{k=1}^m L_k(l^{-1}\|\mu\|) \\ &+ bM_1\overline{M}M_2\{|x_1| + M_1\|\phi(0)\| + M_1 k_3(\|\phi\|_{\mathcal{B}_h} + 1) \\ &+ k_3(\|y_b + \hat{\phi}_b\|_{\mathcal{B}_h} + 1) + b(k_1\|\mu\| + k_2) + b\overline{M}M_1\alpha(\|\mu\|) \\ &+ b\overline{M}M_1\Theta(\|\mu\|)\int_0^b p_1(s)\,ds + M_1 \sum_{k=1}^m L_k(l^{-1}\|\mu\|\}.\end{aligned}$$

Thus, we obtain

$$\frac{(1 - k_3 - bk_1 - b^2 M_1 M M_2 k_1)\|\mu\|}{N_1 + N_2} \leq 1,$$

where N_1 are given in (H_7) and

$$N_2 = b\overline{M}M_1\alpha(\|\mu\|) + b\overline{M}M_1\Theta(\|\mu\|)\int_0^b p_1(s)\,ds + M_1\sum_{k=1}^{m} L_k(l^{-1}\|\mu\|)$$

$$+ bM_1\overline{M}M_2\{bk_1\|\mu\| + b\overline{M}M_1\alpha(\|\mu\|)$$

$$+ b\overline{M}M_1\Theta(\|\mu\|)\int_0^b p_1(s)\,ds + M_1\sum_{k=1}^{m} L_k(l^{-1}\|\mu\|\}.$$

Then by (1), there exists d such that $\|\mu\| \neq d$. Set $\varepsilon = \{y \in \mathcal{B}_h'' : \|y\|'_{\mathcal{B}_h} < d\}$, this indicate that the set ε is bounded. As a consequence of Theorem 2.1, we deduce that $\Psi_1 + \Psi_2$ has a fixed point which is a mild solution of the system (1.1). Thus the system (1.1) is controllable.

4. Example

As an application of Theorem 3.1, we consider the impulsive neutral partial integrodifferential inclusions of the following form:

$$\begin{cases} \dfrac{\partial}{\partial t}[z(t, s) - \displaystyle\int_{-\infty}^{t}\int_0^{\pi} a(s-t, \xi, x)d\xi ds] \in a(t, s)\dfrac{\partial^2}{\partial x^2} z(t, x) + k_0(x)z(t, s) \\ \qquad + \displaystyle\int_0^t \int_{-\infty}^{s} k(s-\tau)Q(z(\tau, x))d\tau ds + Bu(t), \\ \qquad x \in [0, \pi],\ t \in [0, b],\ t \neq t_k, \\ \Delta z(t_k, x) = z(t_k^+, x) - z(t_k^-, x) = I_k(z(t_k^-, x)), \quad k = 1, \ldots, m, \\ z(t, 0) = z(t, \pi) = 0,\ t \geq 0, \\ z(t, x) = \varphi(t, \pi) = 0,\ t \in [-\infty, 0),\ x \in [0, \pi]. \end{cases} \tag{4.1}$$

where $J = [0, b], k = 1, \ldots, m,$ $z(t_k^+, x) = \lim_{h \to 0} z(t_k + h, x),$ $z(t_k^-, x) = \lim_{h \to 0^-} z(t_k + h, x)$, $Q : J \times R \to R$ is given functions. We assume that for each $t \in J$, $Q(t, \cdot)$ is lower semi-continuous.

Let $X = L^2[0, \pi]$ be endowed with usual norm $|\cdot|_{L^2}$. Define $A : X \to X$ by $Av = -a(t, x)v''$ with the domain

$D(A) = \{v(\cdot) \in X : v, v'$ are absolutely continuous; $v'' \in X, v(0) = v(\pi) = 0\}$.

We have

$\overline{D(A)} = \{v(\cdot) \in X : v, v'$ are absolutely continuous; $v'' \in X, v(0) = v(\pi) = 0\} \neq X$.

It is well known that (see [18]) that A satisfies the following properties.

(i) $(0, \infty) \subset \rho(A)$;

(ii) $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \lambda > 0.$

This implies that the operator A satisfies the Hille-Yosida condition (with $\overline{M} = 1$ and $v = 0$). Assume that $B : U \to X, U \in J$ is a bounded linear operator and the operator

$$Wu = \int_0^b T(b - s)Bu(s)\,ds$$

has a bounded invertible operator $W^{-1} : \overline{D(A)} \to L^2(J, U)$. Example with $W : L^2(J, U) \to X$ such that W^{-1} exists and is bounded as discussed in [17].

Let $h(s) = e^{2s}, s < 0,$ then $l = \int_{-\infty}^0 h(s)\,ds = \frac{1}{2}$ and defined

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{\theta \in [s, 0]} |\varphi(\theta)|_{L^2}\,ds.$$

Hence for $(t, \varphi) \in [0, b] \times \mathcal{B}_h$, where $\varphi(\theta)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$. Now define

$$z(t)(x) = z(t, x),$$

$$p(t, \varphi)(x) = \int_{-\infty}^0 \int_0^{\varphi} a(s - t, \xi, x)\varphi(s, \xi)\,d\xi ds,$$

$$f(t, \varphi)(x) = k_0(x)\varphi(t, x),$$

and

$$\int_0^t G(t, s, x_s) ds = \int_0^t \int_{-\infty}^0 k(s-\theta) Q(\varphi(\theta)(x)) d\theta ds.$$

Then, (4.1) can be rewritten as the abstract form as the system (1.1).

Thus, under appropriate conditions on the functions f, p, G and I_k as those in (H_2)-(H_7), then the system (4.1) is controllable on J.

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