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CONTINUED FRACTION APPROXIMATION FOR THE WALLIS RATIO

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Abstract

In this paper, we establish a quicker approximation of the Wallis ratio based on continued fraction. This approximation is fast in comparison with the recently discovered asymptotic series. We also establish the double-side inequalities related to the approximation. Finally, we also give some numerical computations to demonstrate the superiority of our approximation over the classical ones.

Keywords and phrases: Wallis ratio, approximations, multiple-correction method.

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1. Introduction and Motivation

The factorial function $n!$ and the gamma function $\Gamma(x)$ are important components in the researches of pure mathematics, statistics and other branches of science. The gamma function can be regarded as an extension of the factorial function. When n is larger, a general method in the researches is to find approximations of the two functions. A well-known formula approximating to the ratio of factorial function is the Wallis formula. The Wallis ratio is defined as

$$W(n) = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)},$$

where $\Gamma(x)$ is the gamma function which has attracted the attention of many researchers [1-8] and can be defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \mathrm{Re}\,(x) > 0.$$

The study and applications of $W(n)$ have a long history, many remarkable results and applications for the Wallis ration can be found in the literature [9-16] and references cited therein.

For every natural number n, Chen and Qi [9] presented the following inequalities for the Wallis ratio:

$$\frac{1}{\sqrt{\pi\left(n + \frac{4}{\pi} - 1\right)}} \leq \frac{(2n-1)!!}{(2n)!!} \leq \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}}, \tag{1.1}$$

where the constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ are the best possible.

Guo, Xu and Qi proved in [18] that the double-side inequalities

$$\sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W(n) \leq \frac{4}{3}\left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \tag{1.2}$$

for $n \geq 2$ is valid with the best possible constants $\sqrt{\frac{e}{\pi}}$ and $\frac{4}{3}$. They also obtained the approximation formula

$$W(n) \sim \chi(n) := \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \qquad n \to \infty. \tag{1.3}$$

Recently, Qi and Mortici [17] improved the approximation formula (1.3) as

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}}, \qquad n \to \infty. \tag{1.4}$$

Motivated by these works, in this paper we will apply the multiple-correction method [19-21] to obtain an improved asymptotic expansion for the Wallis ratio based on continued fraction as follows:

Theorem 1. *For the Wallis ratio* $W(n) = \frac{(2n-1)!!}{(2n)!!}$, *we have*

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}}$$

$$\times \exp\left(\frac{1}{n^2} \frac{a_1}{n + b_1 + \frac{a_2}{n + b_2 + \frac{a_3}{n + b_3}}}\right), \qquad n \to \infty, \tag{1.5}$$

where $a_1 = \frac{1}{144}$, $b_1 = \frac{1}{60}$; $a_2 = \frac{781}{3600}$, $b_2 = -\frac{4309}{109340}$; $a_3 = \frac{51396085}{89664267}$, $b_3 = \frac{25682346121}{449571834712}$.

Using Theorem 1, we provide some double-side inequalities for the Wallis ratio.

Theorem 2. *For every integer* $n > 1$, *it holds*:

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2\left(n+\frac{1}{3}\right)}\right)^{n+\frac{1}{3}}\frac{1}{\sqrt{n}}\exp\left(\frac{1}{n^2}\frac{a_1}{n+b_1+\frac{a_2}{n+b_2+\frac{a_3}{n+b_3}}}\right)$$

$$> W(n) > \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2\left(n+\frac{1}{3}\right)}\right)^{n+\frac{1}{3}}\frac{1}{\sqrt{n}}\exp\left(\frac{1}{n^2}\frac{a_1}{n+b_1+\frac{a_2}{n+b_2}}\right), \tag{1.6}$$

where $a_1 = \frac{1}{144}$, $b_1 = \frac{1}{60}$; $a_2 = \frac{781}{3600}$, $b_2 = -\frac{4309}{109340}$; $a_3 = \frac{51396085}{89664267}$, $b_3 = \frac{25682346121}{449571834712}$.

To prove Theorem 1, we need the following lemma which was used in [22-24] and is very useful for constructing asymptotic formulas.

Lemma 1. *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{1.7}$$

with $s > 1$, *then*

$$\lim_{n\to+\infty} n^{s-1}x^n = \frac{l}{s-1}. \tag{1.8}$$

Lemma 1 was proved by Mortici in [22]. From Lemma 1, we can see that the rate of convergence of the sequences $(x_n)_{n\in\mathbb{N}}$ increases together with the values s satisfying (1.8).

The rest of this paper is arranged as follows. In Section 2, we will apply the *multiple-correction method* to construct a new asymptotic expansion for the Wallis ratio based on continued fraction and prove Theorem 1 by the *multiple-correction method*. In Section 3, we established the double-side inequality for the Wallis ratio. In Section 4, we give some numerical computations which demonstrate the superiority of our new series over some formulas found recently.

2. Proof of Theorem 1

According to the argument of the Theorem 5.1 in [17], we can introduce a sequence $(u(n))_{n\geq 1}$ by the relation

$$W(n) = \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} \exp u(n), \tag{2.1}$$

and say that an approximation $W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}}$ is better if the speed of convergence of $u(n)$ is higher.

(Step 1) The initial-correction. When $n \to \infty$, we define a sequence $(u_0(n))_{n\geq 1}$

$$W(n) = \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} \exp u_0(n). \tag{2.2}$$

From (2.2), we have

$$u_0(n) = \ln W(n) - \ln \sqrt{\frac{e}{\pi}} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right) - \frac{1}{2} \ln \frac{1}{n}. \quad (2.3)$$

Thus,

$$u_0(n) - u_0(n+1) = \ln \frac{2n+2}{2n+1} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right) - \frac{1}{2} \ln \frac{1}{n}$$

$$+ \left(n + \frac{4}{3}\right) \ln \left(1 - \frac{1}{2\left(n + \frac{4}{3}\right)}\right) + \frac{1}{2} \ln \frac{1}{n+1}. \quad (2.4)$$

Developing (2.4) into power series expansion in $1/n$, we have

$$u_0(n) - u_0(n+1) = \frac{1}{48} \frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \quad (2.5)$$

By Lemma 1, we know that the rate of convergence of the sequence $(u_0(n))_{n \geq 1}$ is n^{-3}.

(Step 2) The first-correction. We define the sequence $(u_1(n))_{n \geq 1}$ by the relation

$$W(n) = \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n + \frac{1}{3}} \frac{1}{\sqrt{n}} \exp\left(\frac{1}{n^2} \frac{a_1}{n + b_1}\right) \exp u_1(n). \quad (2.6)$$

From (2.6), we have

$$u_1(n) - u_1(n+1) = \ln \frac{2n+2}{2n+1} - \left(n + \frac{1}{3}\right) \ln \left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)$$

$$-\frac{1}{2}\ln\frac{1}{n}+\frac{1}{2}\ln\frac{1}{n+1}+\left(n+\frac{4}{3}\right)\ln\left(1-\frac{1}{2\left(n+\frac{4}{3}\right)}\right)$$

$$-\frac{1}{n^2}\frac{a_1}{n+b_1}+\frac{1}{(n+1)^2}\frac{a_1}{n+1+b_1}. \qquad (2.7)$$

Developing (2.7) into power series expansion in $1/n$, we have

$$u_1(n)-u_1(n+1)=\left(\frac{1}{48}-3a_1\right)\frac{1}{n^4}$$

$$+\left(-\frac{91}{2160}+a_1(6+4b_1)\right)\frac{1}{n^5}+O\left(\frac{1}{n^6}\right). \qquad (2.8)$$

By Lemma 1, we know that the fastest possible sequence $(u_1(n))_{n\geq 1}$ is obtained as the first item on the right of (2.8) vanishes. So taking $a_1=\frac{1}{144}$, $b_1=\frac{1}{60}$, we can get the rate of convergence of the sequence $(u_1(n))_{n\geq 1}$ is at least n^{-5}.

(Step 3) The second-correction. We define the sequence $(u_2(n))_{n\geq 1}$ by the relation

$$W(n)=\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2\left(n+\frac{1}{3}\right)}\right)^{n+\frac{1}{3}}\frac{1}{\sqrt{n}}$$

$$\times\exp\left(\frac{1}{n^2}\frac{a_1}{n+b_1+\frac{a_2}{n+b_2}}\right)\exp u_2(n). \qquad (2.9)$$

Using the same method as above, we obtain that the sequence $(u_2(n))_{n\geq 1}$

converges fastest only if $a_2 = \frac{781}{3600}$, $b_2 = -\frac{4309}{109340}$.

(Step 4) The third-correction. Similarly, define the sequence $(u_3(n))_{n\geq 1}$ by the relation

$$W(n) = \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}}$$

$$\times \exp\left(\frac{1}{n^2} \frac{a_1}{n + b_1 + \frac{a_2}{n + b_2 + \frac{a_3}{n + b_3}}}\right) \exp u_3(n). \tag{2.10}$$

Using the same method as above, we obtain that the sequence $(u_3(n))_{n\geq 1}$ converges fastest only if $a_3 = \frac{51396085}{89664267}$, $b_3 = \frac{25682346121}{449571834712}$.

The new asymptotic (1.5) is obtained.

3. Proof of Theorem 2

The double-side inequality (1.6) may be written as follows:

$$f(n) = \ln W(n) - \frac{1}{2} + \frac{1}{2}\ln \pi - \left(n + \frac{1}{3}\right)\ln\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)$$

$$- \frac{1}{2}\ln\frac{1}{n} - \frac{1}{n^2} \frac{\frac{1}{144}}{n + \frac{1}{60} + \frac{\frac{781}{3600}}{n - \frac{4309}{109340}}}$$

and

$$g(n) = \ln W(n) - \frac{1}{2} + \frac{1}{2}\ln \pi - \left(n + \frac{1}{3}\right)\ln\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)$$

$$- \frac{1}{2}\ln\frac{1}{n} - \frac{1}{n^2}\cfrac{\frac{1}{144}}{n + \frac{1}{60} + \cfrac{\frac{781}{3600}}{n - \frac{4309}{109340} + \cfrac{\frac{51396085}{89664267}}{n + \frac{256823461 21}{449571834712}}}}.$$

Suppose $F(n) = f(n+1) - f(n)$ and $G(n) = g(n+1) - g(n)$. For every $x > 1$, we can get

$$F''(x) = \frac{A(x-1)}{\left(\begin{array}{l}160x^4(1+x)^4(1+2x)^2(1+3x)(4+3x) \\ \times (6x-1)^2(5+6x)^2\psi_1^3(x;\ 2)\psi_2^3(x;\ 2)\end{array}\right)} < 0 \tag{3.1}$$

and

$$G''(x) = \frac{B(x)}{\left(\begin{array}{l}32x^4(1+x)^4(1+2x)^2(1+3x)(4+3x) \\ \times (-1+6x)^2(5+6x)^2\psi_1^3(x;\ 3)\psi_2^3(x;\ 3)\end{array}\right)} > 0, \tag{3.2}$$

where

$$\psi_1(x;\ 2) = 10642 - 1119x + 49203x^2,$$

$$\psi_2(x;\ 2) = 58726 + 97287x + 49203x^2,$$

$$\psi_1(x;\ 3) = 113505180 + 4083418255x + 178132605x^2$$

$$+ 5180725368x^3,$$

$$\psi_2(x;\ 3) = 9555781408 + 19981859569x + 15720308709x^2$$
$$+ 5180725368x^3.$$

$A(x) = -461003, \ldots, 512n^{18} - \ldots$ is a polynomial of 18th degree with all negative coefficients and $B(x) = 212876, \ldots, 696x^{22} + \ldots$ is a polynomial of 22th degree with all positive coefficients.

It shows that $F(x)$ is strictly concave and $G(x)$ is strictly convex on $(0, \infty)$. According to Theorem 1, when $n \to \infty$ it holds that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$; thus $\lim_{n\to\infty} F(n) = \lim_{n\to\infty} G(n) = 0$. As a result, we can make sure that $F(n) < 0$ and $G(n) > 0$ on $(0, \infty)$. Consequently, the sequence $f(n)$ is strictly increasing and $g(n)$ is strictly decreasing while they both converge to 0. As a result, we conclude that $f(n) > 0$, and $g(n) < 0$ for every integer $n > 1$.

The proof of Theorem 2 is complete.

4. Numerical Computations

In this section, we give Table 1 to demonstrate the superiority of our new series respectively. From what has been discussed above, we found out the new asymptotic function as follows:

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}}$$

$$\times \exp\left(\frac{1}{n^2}\frac{\frac{1}{144}}{n + \frac{1}{60} + \frac{\frac{781}{3600}}{n - \frac{4300}{109340} + \frac{\frac{51396085}{89664267}}{n + \frac{256823461 21}{449571834712}}}}\right) = \sigma(n). \quad (4.1)$$

Chen and Qi [10] gave:

$$W(n) \sim \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}} = \alpha(n). \quad (4.2)$$

Qi and Mortici [17] gave the improved formula:

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2n}\right)^{n}\sqrt{\frac{1}{n}}$$

$$\times \exp\left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5}\right) = \beta(n) \quad (4.3)$$

and

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1 - \frac{1}{2\left(n + \frac{1}{3}\right)}\right)^{n+\frac{1}{3}} \frac{1}{\sqrt{n}} = \gamma(n). \quad (4.4)$$

We can easily observe that the new formula converges fastest of the other three formulas.

Table 1. Simulations for $\alpha(n)$, $\beta(n)$, $\gamma(n)$ and $\sigma(n)$

n	$\frac{W(n)-\alpha(n)}{W(n)}$	$\frac{W(n)-\beta(n)}{W(n)}$	$\frac{W(n)-\gamma(n)}{W(n)}$	$\frac{W(n)-\sigma(n)}{W(n)}$
50	-6.1876×10^{-6}	7.3576×10^{-14}	5.5532×10^{-8}	-3.8082×10^{-19}
500	-6.2438×10^{-8}	7.1643×10^{-20}	5.5554×10^{-11}	-3.8138×10^{-28}
1000	-1.5617×10^{-8}	1.1177×10^{-21}	6.9443×10^{-12}	-7.4489×10^{-31}
2000	-3.9053×10^{-9}	1.7452×10^{-23}	8.6805×10^{-13}	-1.4549×10^{-33}

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