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CONDITIONS FOR n-th ROOTS OF REAL NUMBERS BEING RATIONAL NUMBERS

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Abstract

This paper provides a survey on the recent results of us which completely answer the challenging question "Given a real number x: for which value of a natural number n $(n \geq 2)$ the n-th root of x is still rational and, if so, what is the precise value of $\sqrt[n]{x}$?". We show, in particular, that it is sufficient to answer the question for the special case that x is a natural number, because - as we prove - this also covers the more general case of x being a real number. Finally, we also offer extremely simple tests which allow one to determine with a high probability, for an arbitrary natural number x, that: $\sqrt[n]{x} \notin \mathbf{Q}$ $\forall n \in \mathbf{N}$, $n \geq 2$.

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1. Introduction

Since ancient times the problem "Is the value $\sqrt{x}$ (for $x \in \mathbf{N}$) rational or irrational?" has been investigated. As an example of those activities, more than 2000 years ago, Euclid has proven that $\sqrt{2} \notin \mathbf{Q}$ [1].

Meanwhile, the mathematical research related to the square roots of natural numbers has led to numerous further results, cf. [2]. Moreover, also investigating the properties of n-th roots of natural numbers has become an important research topic, cf. [3, 4]. To the best of our knowledge, until the very recent past, the mathematical research related to the general question "Given a real number x: for which value of a natural number n $(n \geq 2)$ the n-th root of x is still rational and, if so, what is the precise value of $\sqrt[n]{x}$?" was still in its infancy. This status led to the motivation for us to search for a solution of this question - a solution which should be both, as complete and at the same time as simple as possible. Fortunately, we succeeded in our efforts to significantly advance the state-of-the-art in the domain of n-th roots of real numbers. In particular, during the last year, we discovered and published quite a few innovative results, cf. [5, 6, 7]. As our results related to $\sqrt[n]{x}$ should be of quite strong interest to numerous mathematicians (and non-mathematicians), we decided to publish this survey paper. It summarizes in a compact manner the most fundamental results and insights gained regarding the rationality resp. irrationality of n-th roots of real numbers as well as the (pleasingly simple) algorithm to calculate the value of $\sqrt[n]{x}$ (if $\sqrt[n]{x} \in \mathbf{Q}$).

Let us now introduce several abbreviations (denoting specific sets) which will allow us to simplify our argumentation and the notation throughout this paper:

- $\mathbf{N}_{\geq 2} := \{x \in \mathbf{N} \mid x \geq 2\}$.

- $X_{\text{rat\_roots}} := \{x \in \mathbf{R} \mid \exists n \in \mathbf{N}_{\geq 2}, \sqrt[n]{x} \in \mathbf{Q}\}$.
- $X_{\text{irrat\_roots}} := \{x \in \mathbf{R} \mid \sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}\}$.
- $\mathbf{Q}^- = \{x \in \mathbf{Q} \mid x < 0\}$; $\mathbf{Q}^+ = \{x \in \mathbf{Q} \mid x > 0\}$;

  analogous definitions for: $\mathbf{R}^-$ and $\mathbf{R}^+$.

We want to demonstrate now, that for various sets of numbers (S_N), we obtain: $\sqrt[n]{x} \notin \mathbf{Q}\ \forall x \in S_N,\ n \in \mathbf{N}_{\geq 2}$.

In particular, in [5] we have proven that:

- $\mathbf{R} \setminus \mathbf{Q} \subset X_{\text{irrat\_roots}}$. Example: $\sqrt[n]{\pi} \notin \mathbf{Q}\ \forall n \in \mathbf{N}_{\geq 2}$.
- For $x \in \mathbf{R}^-$:
  - if n is even then $\sqrt[n]{x} \in \mathbf{C}$ ($\mathbf{C}$ denoting the set of complex numbers),
  - if n is odd then $\sqrt[n]{x}$ is considered to be undefined ($\rightarrow$ view V_1) by some mathematicians or others argue $\sqrt[n]{x} = -\sqrt[n]{|x|}$ (for n odd, and $x \in \mathbf{R}^-$), ($\rightarrow$ view V_2, which represents the more conventional view).

So, to summarize, we get two different results for both views:

» view V_1: $x \in \mathbf{R}^-$, then: $x \notin X_{\text{rat\_roots}}$.

» view V_2: $x \in \mathbf{R}^-$ and n odd, then: $\sqrt[n]{x} \in \mathbf{Q} \Leftrightarrow \sqrt[n]{|x|} \in \mathbf{Q}$

which means $x \in X_{\text{rat\_roots}} \Leftrightarrow |x| \in X_{\text{rat\_roots}}$.

Example. $x = -8$, then according to view V_2, $\sqrt[3]{-8} = -2 \in \mathbf{Q}$ and therefore $-8 \in X_{\text{rat\_roots}}$.

• For $x \in \mathbf{Q}^+$:

x can be represented by $x = p/q$, where $p, q \in \mathbf{N} \setminus \{0\}$, p and q being coprime.

Then, $x \in X_{\text{rat\_roots}} \Leftrightarrow \exists n \in \mathbf{N}_{\geq 2},\ \sqrt[n]{p} \in \mathbf{Q}$ and $\sqrt[n]{q} \in \mathbf{Q}$.

Example. $\frac{8}{27} \in X_{\text{rat\_roots}}$, because for $n = 3$, we get $\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$ $= \frac{2}{3} \in \mathbf{Q}$, but $\frac{8}{9} \notin X_{\text{rat\_roots}}$, though $\sqrt[3]{8} \in \mathbf{Q}$ and $\sqrt[2]{9} \in \mathbf{Q}$ (because: 3 and 2 are coprime).

So, in summary, we have proven that the question "$\sqrt[n]{x} \in \mathbf{Q}$ (for $x \in \mathbf{R}$)?" can be simplified by investigating the question "$\sqrt[n]{x} \in \mathbf{Q}$ (for $x \in \mathbf{N}_{\geq 2}$)?". We will make use of this fact in the rest of this paper.

The paper is structured as follows: In Section 2, we tackle the challenging problem to find out whether a natural number $n \in \mathbf{N}_{\geq 2}$ exists such that $\sqrt[n]{x} \in \mathbf{Q}$ (for $x \in \mathbf{N}_{\geq 2}$) and - if so - to precisely determine the value of $\sqrt[n]{x}$. Section 3 introduces various elementary tests which allow one to determine whether $\sqrt[n]{x}$ (for $x \in \mathbf{N}_{\geq 2}$) is rational or irrational by only looking at the prime factorization of x. In Section 4, we no longer assume the availability of the prime factorization of x. Anyway - for a large percentage of all $x \in \mathbf{N}_{\geq 2}$ - we still are able to apply a set of (rather trivial) tests to decide with certainty that $\sqrt[n]{x}$ is an irrational number. Section 5 shortly concludes this survey paper.

2. Criteria for the n-th Root of a Natural Number x being Rational and an Algorithm to Determine the Corresponding Value of $\sqrt[n]{x}$

In this section, we want to answer the rather general and challenging question of $\sqrt[n]{x}$ being rational or irrational (for $n, x \in \mathbf{N}_{\geq 2}$). The answer to this question becomes astonishingly simple if we make use of the (unique) prime factorization of x, cf. [8], as it has been suggested in [5].

So, let us assume that the prime factorization of x, $x \in \mathbf{N}_{\geq 2}$, is given by:

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_i^{k\_i} \bullet\bullet\bullet\, p_m^{k\_m}, \tag{1}$$

where p_i are prime numbers $\forall i \in \{1, 2, \ldots, m\}$ and $p_i \neq p_j$ $\forall i \neq j$, $m \geq 1$, $k_i \in \mathbf{N}$ $\forall i \in \{1, 2, \ldots, m\}$.

Note. k_i to be read as k_i.

The result derived in [5] was:

For any $n, x \in \mathbf{N}$, $n, x \geq 2$:

$$\sqrt[n]{x} \in \mathbf{Q} \Leftrightarrow n \in \mathrm{CD}(\{k_1, k_2, \ldots, k_m\}), \tag{2}$$

where $\mathrm{CD}(M)$, for M being a subset of $\mathbf{N}$, denotes the set of common divisors c, $c \geq 2$, of all elements of the set M.

Therefore, if $m \geq 2$:

$$\mathrm{CD}(\{k_1, k_2, \ldots, k_m\}) := \{c \in \mathbf{N}, c \geq 2 \mid \forall k_i\ \exists \nu_i = \nu_i(k_i) \in \mathbf{N} : c \bullet \nu_i = k_i\},$$

and, if $m = 1$:

$$\mathrm{CD}(\{k_1\}) := \{c \in \mathbf{N}, c \geq 2 \mid \exists \nu \in \mathbf{N} : c \bullet \nu = k_1\}.$$

It should be noted that for answering our question "$\sqrt[n]{x} \in \mathbf{Q}$?" by applying eq. (2) the values of the p_i appearing in eq. (1) are not of interest but only the values of the exponents k_i (resp. k_i in eq. (1)) are relevant.

In [5] it has been proven that (for $n, x \in \mathbf{N}_{\geq 2}$): $\sqrt[n]{x} \in \mathbf{Q} \Rightarrow \sqrt[n]{x} \in \mathbf{N}_{\geq 2}$.

Examples 2.1.

- $x = p_1^{12} \bullet p_2^{36} \bullet p_3^{6}$, p_i arbitrary (different) prime numbers, $i \in \{1, 2, 3\}$

  $\Rightarrow$ of interest: $\mathrm{CD}(\{12, 36, 6\}) = \{2, 3, 6\}$

  and therefore $\sqrt[2]{x} \in \mathbf{N}$, $\sqrt[3]{x} \in \mathbf{N}$ and $\sqrt[6]{x} \in \mathbf{N}$.

- $x = p_1^{9} \Rightarrow$ of interest: $\mathrm{CD}(\{9\}) = \{3, 9\}$

  and therefore $\sqrt[3]{x} \in \mathbf{N}$, $\sqrt[9]{x} \in \mathbf{N}$.

- $x = p_1^{12} \bullet p_2^{36} \bullet p_3^{5} \Rightarrow$ of interest: $\mathrm{CD}(\{12, 36, 5\}) = \varnothing$

  and therefore $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}_{\geq 2}$.

What remains is the task to calculate the exact value of $\sqrt[n]{x}$, if $\sqrt[n]{x} \in \mathbf{Q}$ (resp. $\sqrt[n]{x} \in \mathbf{N}$) for $n, x \in \mathbf{N}_{\geq 2}$.

So, we can give the complete algorithm (coarse specification using pseudo-code) to determine whether $\sqrt[n]{x}$ is rational or irrational and, in the former case, to calculate the value of $\sqrt[n]{x}$:

Algorithm. $\sqrt[n]{x}$ being rational or not? Value of $\sqrt[n]{x}$? (if $\sqrt[n]{x}$ is rational)

Given: $n, x \in \mathbf{N}_{\geq 2}$.

Step 1: Determine prime factorization of x (cf. eq. (1)).

Step 2: Consider the exponents $k_1, k_2, ..., k_m$ of the prime factorization of x.

Step 3: Evaluate $\text{CD}(\{k_1, k_2, ..., k_m\}) \Rightarrow$ resulting set denoted by N_x: $N_x = \{n_1, n_2, ..., n_r\}$, where $r \geq 1$ or $r = 0$ (i.e., $N_x = \varnothing$).

Step 4: If $N_x = \varnothing$, then $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}_{\geq 2}$ and STOP else go to Step 5.

Step 5: For all $n_j \in N_x$, we know that $\sqrt[n\_j]{x} \in \mathbf{N}$ and $\sqrt[n\_j]{x} = p_1^{k\_1/n\_j} \bullet p_2^{k\_2/n\_j} \bullet\bullet\bullet\, p_m^{k\_m/n\_j}$ is the exact value of $\sqrt[n\_j]{x}$. ■

Examples 2.2.

- $x = p_1^{12} \bullet p_2^{36} \bullet p_3^{6} \Rightarrow \sqrt[2]{x} \in \mathbf{N}$, $\sqrt[3]{x} \in \mathbf{N}$ and $\sqrt[6]{x} \in \mathbf{N}$ yield to the corresponding values of $\sqrt[n\_j]{x}$ (for $n_j \in \{2, 3, 6\}$): $\sqrt[2]{x} = p_1^{6} \bullet p_2^{18} \bullet p_3^{3}$; $\sqrt[3]{x} = p_1^{4} \bullet p_2^{12} \bullet p_3^{2}$; $\sqrt[6]{x} = p_1^{2} \bullet p_2^{6} \bullet p_3$;
- $x = p_1^{9} \Rightarrow \sqrt[3]{x} \in \mathbf{N}$ and $\sqrt[9]{x} \in N$ yield to the corresponding values of $\sqrt[n\_j]{x}$ (for $n_j \in \{3, 9\}$): $\sqrt[3]{x} = p_1^{3}$; $\sqrt[9]{x} = p_1$.

3. Is $\sqrt[n]{x}$ Rational or Irrational? - Elementary Tests if the Prime Factorization of x is Available

The aim of this section is to provide simple test methods to decide whether $\sqrt[n]{x}$ is rational or irrational $(n, x \in \mathbf{N}_{\geq 2})$. We assume that the notation introduced in the previous section continues to hold and also that the prime factorization of x is known and still given by eq. (1).

$$\text{Thus: } x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_i^{k\_i} \bullet\bullet\bullet\, p_m^{k\_m}.$$

Furthermore, we denote the set of exponents k_i (resp. k_i) appearing in the prime factorization by $E_x = \{k_1, k_2, ..., k_m\}$ and the set of common divisors of E_x is denoted again by $\mathrm{CD}(E_x)$.

Now, $\mathrm{CD}(E_x)$ can be used (cf. Section 2 and [5]) to determine in a simple manner whether $x \in X_{\text{rat\_roots}}$ or $x \in X_{\text{irrat\_roots}}$.

In particular,

- $\mathrm{CD}(E_x) \neq \varnothing \Leftrightarrow x \in X_{\text{rat\_roots}}$, (3)

- $\mathrm{CD}(E_x) = \varnothing \Leftrightarrow x \in X_{\text{irrat\_roots}}$. (4)

It is evident that a sufficient condition for $\mathrm{CD}(E_x) \neq \varnothing$ is that at least one value of n exists such that $\sqrt[n]{x} \in \mathbf{Q}$ (also implying $\sqrt[n]{x} \in \mathbf{N}$).

And it is also evident that a sufficient condition for $\mathrm{CD}(E_x) = \varnothing$ is that at least for a subset of E_x, denoted by E_x^*, we get $\mathrm{CD}(E_x^*) = \varnothing$. This holds because: $\mathrm{CD}(E_x) \subseteq \mathrm{CD}(E_x^*)$ if $E_x^* \subseteq E_x$.

We now consider the special case that - in the prime factorization of x - there exists at least one i that the value of exponent k_i (resp. k_i) of p_i is $k\_i = 1$. Then it is sufficient to choose the subset E_x^* as $E_x^* = \{k_i\}$ which implies $\mathrm{CD}(E_x^*) = \mathrm{CD}(\{1\}) = \varnothing$ and thus $\mathrm{CD}(E_x) = \varnothing$, implying $x \in X_{\text{irrat\_roots}}$.

To summarize, we have presented rather simple tests which allow us to recognize whether $\sqrt[n]{x}$ is rational or irrational (for $n, x \in \mathbf{N}_{\geq 2}$) if the prime factorization of x is available.

Examples 3.1. $\sqrt[n]{x}$ being rational

- $x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_m^{k\_m}$ and all k_i are even numbers

  $\Rightarrow \mathrm{CD}(E_x) \supseteq \{2\} \Rightarrow \mathrm{CD}(E_x) \neq \emptyset \Rightarrow x \in X_{\text{rat\_roots}}$.

- $x = p_1^{k\_1}$ and the prime factorization of x does not include other primes than p_1, i.e., $m = 1$, and furthermore we assume $k\_1 \geq 2 \Rightarrow \mathrm{CD}(E_x)$ is equal to the set of all divisors of k_1 and as $k\_1 \geq 2$: $\mathrm{CD}(E_x) \supseteq \{k\_1\} \Rightarrow \mathrm{CD}(E_x) \neq \emptyset \Rightarrow x \in X_{\text{rat\_roots}}$.

Examples 3.2. $\sqrt[n]{x}$ being irrational

- $x = p_1 \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_m^{k\_m} \Rightarrow \mathrm{CD}(E_x) = \emptyset \Rightarrow x \in X_{\text{irrat\_roots}}$.

- $x = p$; p being a prime number. This is a special case of the precedent example and it implies that, for all prime numbers, we recognize that $p \in X_{\text{irrat\_roots}}$ (besides, this can also be considered as a rather simple proof of Euclid's theorem, i.e., $\sqrt{2} \notin \mathbf{Q}$).

4. Elementary Tests to Determine the Value of $\sqrt[n]{x}$ if $\sqrt[n]{x}$ is Irrational without given Prime Factorization

In Sections 2 and 3, the question "whether $\sqrt[n]{x} \in \mathbf{Q}$ or $\sqrt[n]{x} \notin \mathbf{Q}$?" has been answered completely (i.e., for any $n, x \in \mathbf{N}_{\geq 2}$). Nevertheless, we were not fully satisfied with our results, in particular, for cases when the value of x is extremely large. It is well-known that in cases of very large numbers x even very powerful supercomputers are reaching practical limits when they try to determine the corresponding prime factorization of x. So, we looked for solutions (or at least for partial solutions) to answer the question "Is $\sqrt[n]{x}$ irrational for all $n \in \mathbf{N}_{\geq 2}$?" assuming now that the prime factorization of x is not available.

Indeed, we succeeded - at least for a high percentage of all natural numbers - to determine with certainty whether $\sqrt[n]{x} \notin \mathbf{Q}\ \forall n \in \mathbf{N}_{\geq 2}$, even for the case that the prime factorization of x is unavailable. These results have been published in [7] and they are summarized in this section.

We discovered that for a lot of $x \in \mathbf{N}_{\geq 2}$, it can be proven in an extremely simple manner that: $\sqrt[n]{x} \notin \mathbf{Q}\ \forall n \in \mathbf{N}_{\geq 2}$.

To come up with this result, we made use of the fact that if x is an integer multiple of a prime number p but no integer multiple of p^2, then $\sqrt[n]{x} \notin \mathbf{Q}\ \forall n \in \mathbf{N}_{\geq 2}$, i.e., $x \in X_{\text{irrat\_roots}}$. Though we do not have to know the exact structure of the prime factorization of x, we see that, here, it exists $z \in \mathbf{N}$ such that: $x = p \bullet z$, p and z being coprime. So, eq. (4) can be applied and yield: $x \in X_{\text{irrat\_roots}}$.

Therefore - for an arbitrary $x \in \mathbf{N}_{\geq 2}$ given - to determine whether $x \in X_{\text{irrat\_roots}}$, we just can apply the following:

• **Test $\mathbf{T}_p$.** We choose a prime number p. Then we can conclude:

if x is an integer multiple of p and no integer multiple of p^2, then $x \in X_{\text{irrat\_roots}}$. ■

Evidently, it is possible to repeat test T_p by means of using the following:

Algorithm. Test, whether $x \in X_{\text{irrat\_roots}}$.

Given: $x \in \mathbf{N}_{\geq 2}$; p denotes a prime number.

Step 1: $p := 2$.

Step 2: if $p \neq 2$, then replace p by the next higher prime number.

Step 3: Apply Test T_p to x based on the prime number p currently considered. If the test yields $x \in X_{\text{irrat\_roots}}$ then STOP.

Step 4: If test for an additional prime number is desirable, then go to Step 2 else STOP. ■

Remarks regarding this algorithm:

- The coarse specification should be sufficiently elementary to be understandable even to non-Computer Scientists;

- Test T_p is applied choosing $p = 2; 3; 5; 7; 11; 13; \ldots$ in this sequence and the algorithm can be terminated in Step 4, e.g., because of the steadily increasing expenditure (in particular, executing Steps 2 and 3) or because we are satisfied with the result obtained up to this point;

- Tests T_2, T_3 and T_5 can be even executed by hand (even for large numbers of x), cf. Examples 4.1.

Examples 4.1.

• **Test $\mathbf{T_2}$.** $p = 2$: here, the following two conditions have to be tested:

- is x an integer multiple of 2? $\rightarrow$ i.e., is x an even number?
- is x NOT an integer multiple of $2^2 = 4$? $\rightarrow$ i.e., are the last two digits of x NOT an integer multiple of 4?

Result of Test T_2: sufficient condition for $x \in X_{\text{irrat\_roots}}$: the last two digits of x are 02, 06, 10, 14, 18, 22, ..., 90, 94, 98 $\rightarrow$ corresponds to 25% of all natural numbers.

• **Test $\mathbf{T_3}$.** $p = 3$: here, the following two conditions have to be tested:

- is x an integer multiple of 3? $\rightarrow$ i.e., is the sum of all digits of x an integer multiple of 3?

- is x NOT an integer multiple of $3^2 = 9$? $\rightarrow$ i.e., is the sum of all digits of x NOT an integer multiple of 9?

Result of Test T_3: sufficient condition for $x \in X_{\text{irrat\_roots}}$: $x \in \{3, 6, 12, 15, 21, 24, 30, 33, 39, 42, 48, \ldots\}$ $\rightarrow$ corresponds to about 22% of all natural numbers.

• **Test $\mathbf{T_5}$.** $p = 5$: here, the following two conditions have to be tested:

- is x an integer multiple of 5? $\rightarrow$ i.e., is the last digit of x "0" or "5"?

- is x NOT an integer multiple of $5^2 = 25$? $\rightarrow$ i.e., are the last two digits of x different from "00", "25", "50", "75"?

Result of Test T_5: sufficient condition for $x \in X_{\text{irrat\_roots}}$: the last two digits of x are 05, 10, 15, 20, 30, 35, 40, 45, 55, 60, 65, 70, 80, 85, 90, 95 $\rightarrow$ corresponds to 16% of all natural numbers.

Remarks. The tests T_2 and T_5 are indeed trivial to be carried out and even for numbers of x possessing millions of digits their application is just a matter of seconds. Nevertheless, they cover a pleasingly high percentage of natural numbers x given $(x \in \mathbf{N}_{\geq 2})$ for which the question

"$x \in X_{\text{irrat\_roots}}$?" can be answered with complete certainty. For example, in [7] it has been shown that even the simple tests introduced in this section allow one to prove that $x \in X_{\text{irrat\_roots}}$ for about 85% of all $x \in \mathbf{N}$, $x \leq 100$.

5. Conclusions

This paper summarizes the major results published most recently which represent a significant progress in the mathematical state-of-the-art regarding the understanding of n-th roots of real numbers. In particular, it is possible now to determine in a straightforward manner whether $\sqrt[n]{x} \in \mathbf{Q}$ or $\sqrt[n]{x} \notin \mathbf{Q}$ (for $n \in \mathbf{N}_{\geq 2}$, $x \in \mathbf{R}$). We showed how understanding the properties of $\sqrt[n]{x}$ (for $n, x \in \mathbf{N}_{\geq 2}$) helps to understand the more general case "n-th root of a real number x".

Most of the research results presented here have been obtained by relying on the prime factorization of x, if we investigate $\sqrt[n]{x}$ $(n, x \in \mathbf{N}_{\geq 2})$. The availability of the prime factorization is fundamental, e.g., because it allows one to easily determine whether $\sqrt[n]{x}$ is rational or irrational. Moreover, the value of $\sqrt[n]{x}$ can also be calculated in a rather trivial manner based on basic properties of the prime factorization of x. Anyway, for numerous $x \in \mathbf{N}_{\geq 2}$, even without prime factorization of x, extremely simple tests allow us to recognize that $\sqrt[n]{x} \notin \mathbf{Q}$ (cf. Section 4).

In this paper, we have primarily considered the n-th root of x (for $n \in \mathbf{N}_{\geq 2}$, $x \in \mathbf{R}^+$). Anyway, our results for roots with natural root index, i.e., for $\sqrt[n]{x}$, also cover the more general case of roots with (positive) rational root index $\sqrt[r]{x}$ (e.g., for $r \in \mathbf{Q}^+ \setminus \mathbf{N}$, $x \in \mathbf{R}^+$). Evidently, if $r \in \mathbf{Q}^+ \setminus \mathbf{N}$, this implies that $\exists p, q : r = \frac{p}{q}$, $p \in \mathbf{N}$,

$q \in \mathbf{N}_{\geq 2} \Rightarrow \sqrt[r]{x} = \sqrt[p/q]{x} = \sqrt[q]{x^p}$. If we set $q = n$ and $y = (x)^p$, we obtain $\sqrt[q]{x^p} = \sqrt[n]{y}$ (and $n \in \mathbf{N}_{\geq 2}$). Moreover, if $x \in \mathbf{N}_{\geq 2} \Rightarrow y \in \mathbf{N}_{\geq 2}$, and, if $x \in \mathbf{Q}^+ \Rightarrow y \in \mathbf{Q}^+$, and, if $x \in \mathbf{R}^+ \Rightarrow y \in \mathbf{R}^+$. So, it becomes evident that our solutions presented do not only cover the roots of type $\sqrt[n]{x}$ (for $n \in \mathbf{N}_{\geq 2}$, $x \in \mathbf{R}^+$) but also the more general roots $\sqrt[r]{x}$ (for $r \in \mathbf{Q}^+$, $x \in \mathbf{R}^+$).

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