Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 18, Issue 1, 2024, Pages 15-37
This paper is available online at http://www.frdint.com/
Published online June 16, 2024

THE $m\Theta$ COMPLETION OF FRACTION FIELD OF $m\Theta$ p-ADIC INTEGERS

PEMHA BINYAM GABRIEL CEDRIC*, UM LAURENCE EMILIE and YVES JONATHAN NDJE

Department of Mathematics and Computer Sciences
Faculty of Sciences
University of Douala
P.O. Box 24157, Douala
Cameroon
e-mail: gpemha@yahoo.fr
laurence.um@gmail.com
nyjo.yves@gmail.com

Abstract

The notion of modal Θ-valent set $(m\Theta s)$ noted $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$, p prime, is defined by F. Ayissi Eteme in [3]. In this note, the purpose is to construct the $m\Theta$ completion of Fraction Field of p-adic numbers on $\mathbb{F}_{p\mathbb{Z}}$; which respects the structure of $m\Theta s$. We think that this approach will bring something of interest to the notion of set of p-adic numbers $\mathbb{Z}_p$ as presented by Alain M. Robert in 2000 [2].

Keywords and phrases: modal Θ-valent converges $(m\Theta cv)$, modal Θ-valent sets $(m\Theta s)$, modal Θ-valent fields $(m\Theta f)$.

2020 Mathematics Subject Classification: 94B60, 94B75.

*Corresponding author

Received April 17, 2024; Accepted May 28, 2024

0.1. Introduction

A $m\Theta$ approach of the notion of set [1] has allowed to bring out the new classes of sets: $m\Theta$ sets. The $m\Theta$ sets present an enrichment from the logical view-point compared with the classical sets. Indeed, with the notion of $m\Theta$ sets, we can mathematically speak of $m\Theta s$ of $m\Theta$ p-adic integers $\mathbb{Z}_{p\mathbb{Z}}$ such that the subset of the $m\Theta$ invariants of $\mathbb{Z}_{p\mathbb{Z}}$ is $\mathbb{Z}_p$ the classical p-adic set,

$$C(\mathbb{Z}_{p\mathbb{Z}}, F'_\alpha) = \mathbb{Z}_p.$$

A $m\Theta$ approach of $m\Theta$ ring $\mathbb{Z}_{p\mathbb{Z}}$ would consist in enriching the alphabet $\mathbb{F}_p$ by taking instead of this one, a richer alphabet as the prime $m\Theta$ field $\mathbb{F}_{p\mathbb{Z}}$ with p^2 elements [3].

The purpose of this paper is to define on the $m\Theta$ set $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$ a notion of completion of Fraction Field of p-adic Integers which respects its structure of $m\Theta$ set in the same manner completion of Fraction Field of p-adic Integers is defined on a finite classical field.

Section 2 recalls the essential notions of $m\Theta$ set for our purpose. Section 3 presents first and briefly $\mathbb{Z}_p$, the classical set of p-adic integers as in [2] and then defines a study of $m\Theta$ Ring $\mathbb{Z}_{p\mathbb{Z}}$. Section 4 is devoted to establish the $m\Theta$ fraction field of $\mathbb{Z}_p$. Section 5 presents the $m\Theta$ completion of $\mathbb{Q}_{p\mathbb{Z}}$.

0.2. The Modal Θ-valent Set Structure and the Algebra of $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$

0.2.1. The modal Θ-valent set structure

$m\Theta$ sets are considered to be non-classical sets which are compatible with a non-classical logic called the chrysippian $m\Theta$ logic.

Definition 0.1. Let E be a non-empty set, I be a chain whose first and last elements are 0 and 1, respectively, $(F_\alpha)_{\alpha\in I_*}$ where $I_* = I \setminus \{0\}$ be a family of applications form E to E.

A $m\Theta$ set is the pair $(E, (F_\alpha)_{\alpha\in I_*})$ simply denoted by (E, F_α) satisfying the following four axioms:

- $\bigcap_{\alpha} F_\alpha(E) = \bigcap_{\alpha\in I_*}\{F_\alpha(x) : x \in \{E\} \neq \varnothing;$

- $\forall\alpha, \beta \in I_*,$ if $\alpha \neq \beta,$ then $F_\alpha \neq F_\beta;$

- $\forall\alpha, \beta \in I_*,\ F_\alpha \circ F_\beta = F_\beta;$

- $\forall x, y \in E,$ if $\forall\alpha \in I_*,\ F_\alpha(x) = F_\alpha(y),$ then $x = y.$

Theorem 0.1. (The theorem of $m\Theta$ determination)

Let (E, F_α) *be a* $m\Theta$ *set.*

$$\forall x, y \in E,\ x =_\Theta y \text{ if and only if } \forall\alpha \in I_*,\ F_\alpha(x) = F_\alpha(y).$$

Proof 0.1. [3].

Definition 0.2. Let $C(E, F_\alpha) = \bigcap_{\alpha\in I_*} F_\alpha(E).$ We call $C(E, F_\alpha)$ the set of $m\Theta$ invariant elements of the $m\Theta$ set $(E, F_\alpha).$

Proposition 0.1. *Let* (E, F_α) *be a* $m\Theta$ *set. The following properties are equivalent*:

1. $x \in \bigcap_{\alpha\in I_*} F_\alpha(E);$

2. $\forall\alpha \in I_*,\ F_\alpha(x) = x;$

3. $\forall \alpha, \beta \in I_*, F_\alpha(x) = F_\beta(x)$;

4. $\exists \mu \in I_*, x = F_\mu(x)$.

Proof 0.2. [3].

Definition 0.3. Let (E, F_α) and (E', F'_α) be two $m\Theta$ sets. Let X be a nonempty set. We shall call

1. (E', F'_α) a modal Θ-valent subset of (E, F_α) if the structure of $m\Theta$ set (E', F'_α) is the restriction to E' of the structure of the $m\Theta$ set (E, F_α), this means:

- $E' \subseteq E$;
- $\forall \alpha : \alpha \in I_*, F'_\alpha = F_{\alpha|E'}$.

2. X a modal Θ-valent subset of (E, F_α) if:

- $X \subseteq E$;
- $(X, F_{\alpha|X})$ is a $m\Theta s$ which is a modal Θ-valent subset of (E, F_α).

In all what follows we shall write $F_\alpha x$ for $F_\alpha(x)$, $F_\alpha E$ for $F_\alpha(E)$, etc.

Example 0.1. For $n \in N^*$, we define the closed chain

$$I = \begin{cases} \{0, 1, 2\} & \text{if} \quad n = 2; \\ \mathbb{N}_{n-1} = \{0, 1, \ldots, n-1\} & \text{if} \quad n \geq 3. \end{cases}$$

The $m\Theta$ set $(\mathbb{Z}_{n\mathbb{Z}}, \mathbb{Z}, F_\alpha)$.

Let us set $x_{n\mathbb{Z}} = (p + \alpha r)_{\alpha \in I_*}$ where $x \in \mathbb{Z} \setminus n\mathbb{Z}$ $(x = pn + r;\ p, r$

$\in \mathbb{Z}; 1 \leq r \leq n-1)$.

$$x_{n\mathbb{Z}} \in \begin{cases} \mathbb{Z}^2 & \text{if} \quad n = 2; \\ \mathbb{Z}^{n-1} & \text{if} \quad n \geq 3. \end{cases}$$

Let us set

$$\mathbb{Z}_{n\mathbb{Z}} = \mathbb{Z} \cup \{x_{n\mathbb{Z}} : \neg\,(x \equiv 0 \pmod n)\}.$$

We define for all $\alpha \in I_*$;

$F_\alpha : \mathbb{Z}_{n\mathbb{Z}} \to \mathbb{Z}_{n\mathbb{Z}}$

$$a \mapsto \begin{cases} F_\alpha a = a \quad \mathit{if} \quad a \in \mathbb{Z}, \\ F_\alpha a = b_1 + \alpha b_2 \quad \mathit{if} \quad a = b_{n\mathbb{Z}},\, b \in \mathbb{Z} \setminus n\mathbb{Z} \\ \qquad (b = b_1 n + b_2 : b_2, b_1 \in \mathbb{Z}; 1 \leq b_2 \leq n-1). \end{cases}$$

$(\mathbb{Z}_{n\mathbb{Z}}, F_\alpha)$ is a $m\Theta$ set such that $C(\mathbb{Z}_{n\mathbb{Z}}, F_\alpha) = \mathbb{Z}$.

Consider $(\mathbb{Z}_{2\mathbb{Z}}, F_\alpha)$

$$\mathbb{Z}_{2\mathbb{Z}} = \mathbb{Z} \cup \{1_{2\mathbb{Z}}, 3_{2\mathbb{Z}}, 5_{2\mathbb{Z}}, 7_{2\mathbb{Z}}, \ldots\};$$

$$1_{2\mathbb{Z}} = (0 + \alpha \cdot 1)_{\alpha \in \{1, 2\}} = \{1, 2\} \in \mathbb{Z}^2;$$

$$3_{2\mathbb{Z}} = (1 + \alpha \cdot 1)_{\alpha \in \{1, 2\}} = \{2, 3\} \in \mathbb{Z}^2;$$

$$5_{2\mathbb{Z}} = (2 + \alpha \cdot 1)_{\alpha \in \{1, 2\}} = \{3, 4\} \in \mathbb{Z}^2;$$

$$7_{2\mathbb{Z}} = (3 + \alpha \cdot 1)_{\alpha \in \{1, 2\}} = \{4, 5\} \in \mathbb{Z}^2;$$

$$\vdots$$

$$F_1\mathbb{Z} = F_2\mathbb{Z} = \mathbb{Z};$$

$$F_1 1_{2\mathbb{Z}} = 0 + 1 \cdot 1 = 1; \quad F_2 1_{2\mathbb{Z}} = 0 + 2 \cdot 1 = 2;$$

$$F_1 3_{2\mathbb{Z}} = 2; \ F_2 3_{2\mathbb{Z}} = 3.$$

0.2.2. The Algebra of $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$

Let $p \in \mathbb{N}$, a prime number. Let us recall that if $a \in \mathbb{F}_{p\mathbb{Z}}$, then

$$\mathbb{F}_{p\mathbb{Z}} = \mathbb{F}_p \cup \{x_{p\mathbb{Z}} : \neg (x \equiv 0 \pmod{p})\}; \ \mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}.$$

We define the $m\Theta$ support of a denoted $s(a)$ as follows:

$$s(a) = \begin{cases} a & \text{if} \quad a \in \mathbb{F}_p; \\ x & \text{if} \quad a = x_{p\mathbb{Z}} \quad \text{with } \rceil (x \equiv 0 \pmod{p}). \end{cases}$$

Thus $s(a) \in \mathbb{F}_p$.

Definition 0.4. Let $\perp$ be a binary operation on $\mathbb{F}_p$. So, $\forall a, b \in \mathbb{F}_p$, $a \perp b \in \mathbb{F}_p$. Let $x, y \in \mathbb{F}_{p\mathbb{Z}}$. We define a binary operation $\perp^*$ on $\mathbb{F}_{p\mathbb{Z}}$ as follows:

$$x \perp^* y = \begin{cases} s(x) \perp s(y) & \text{if} \quad \begin{cases} x, y \in \mathbb{F}_p, \\ (s(x) \perp s(y)) \equiv 0 \pmod{p} & \text{otherwise}, \end{cases} \\ (s(x) \perp s(y))_{p\mathbb{Z}} & \text{otherwise}. \end{cases}$$

$\perp^*$ as defined above on $\mathbb{F}_{p\mathbb{Z}}$ will be called a $m\Theta$ law on $\mathbb{F}_{p\mathbb{Z}}$ for $x, y \in \mathbb{F}_{p\mathbb{Z}}$. Thus we can define $x + y \in \mathbb{F}_{p\mathbb{Z}}$ and $x \times y \in \mathbb{F}_{p\mathbb{Z}}$ for every $x, y \in \mathbb{F}_{p\mathbb{Z}}$, where $+$ and $\times$ are $m\Theta$ addition and $m\Theta$ multiplication, respectively.

Theorem 0.2. $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha, +, \times)$ *is a* $m\Theta$ *ring of unity* 1 *and of* $m\Theta$ *unity* $\frac{1}{p\mathbb{Z}}$.

Proof 0.3. [1].

Remark 0.1. Since p is prime, $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$ is a $m\Theta$ field.

Definition 0.5. x is a divisor of zero in $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$ if there exists

$y \in \mathbb{F}_{p\mathbb{Z}}$ such that $x \times y = 0$.

Example 0.2. 1. $p = 2$, we have $\mathbb{F}_{2\mathbb{Z}} = \{0, 1, 1_{2\mathbb{Z}}, 3_{2\mathbb{Z}}\}$.

The table of $m\Theta$ determination and tables laws of $\mathbb{F}_{2\mathbb{Z}}$

$\mathbb{F}_{2\mathbb{Z}}$	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
F_1	0	1	1	0
F_2	0	1	0	1

$+^{\Theta}$	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
0	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
1	1	0	0	0
$1_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$	0	0	0
$3_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$	0	0	0

$\times^{\Theta}$	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
0	0	0	0	0
1	0	1	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
$1_{2\mathbb{Z}}$	0	$1_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$
$3_{2\mathbb{Z}}$	0	$3_{2\mathbb{Z}}$	$3_{2\mathbb{Z}}$	$1_{2\mathbb{Z}}$

Observation.

$\mathbb{F}_{2\mathbb{Z}}$ has no divisor of zero, is a $m\Theta$ ring from four elements, that is a

$m\Theta$ field of four elements.

2. $p = 3$, we have $\mathbb{F}_{3\mathbb{Z}} = \{0, 1, 2, 1_{3\mathbb{Z}}, 2_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 5_{3\mathbb{Z}}, 7_{3\mathbb{Z}}, 8_{3\mathbb{Z}}\}$.

The table of $m\Theta$ determination and tables laws of $\mathbb{F}_{3\mathbb{Z}}$

$\mathbb{F}_{3\mathbb{Z}}$	0	1	2	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
F_1	0	1	2	1	2	2	0	0	1
F_2	0	1	2	2	1	0	2	1	0

+	0	1	2	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
0	0	1	2	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
1	1	2	0	$2_{3\mathbb{Z}}$	0	$5_{3\mathbb{Z}}$	0	$8_{3\mathbb{Z}}$	0
2	2	0	1	0	$4_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$
$1_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	0	$2_{3\mathbb{Z}}$	0	$5_{3\mathbb{Z}}$	0	$8_{3\mathbb{Z}}$	0
$2_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	0	$4_{3\mathbb{Z}}$	0	$4_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$
$4_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	0	$5_{3\mathbb{Z}}$	0	$8_{3\mathbb{Z}}$	0	$2_{3\mathbb{Z}}$	0
$5_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$	0	$4_{3\mathbb{Z}}$
$7_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	0	$8_{3\mathbb{Z}}$	0	$2_{3\mathbb{Z}}$	0	$5_{3\mathbb{Z}}$	0
$8_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$	0	$4_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$

$\times$	0	1	2	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
0	0	0	0	0	0	0	0	0	0

1	0	1	2	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
2	0	2	1	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$
$1_{3\mathbb{Z}}$	0	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$
$2_{3\mathbb{Z}}$	0	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$
$4_{3\mathbb{Z}}$	0	$4_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$
$5_{3\mathbb{Z}}$	0	$5_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$
$7_{3\mathbb{Z}}$	0	$7_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$
$8_{3\mathbb{Z}}$	0	$8_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$8_{3\mathbb{Z}}$	$7_{3\mathbb{Z}}$	$5_{3\mathbb{Z}}$	$4_{3\mathbb{Z}}$	$2_{3\mathbb{Z}}$	$1_{3\mathbb{Z}}$

0.3. The $m\Theta$ Ring $\mathbb{Z}_{p\mathbb{Z}}$ of p-adic Integers

0.3.1. $\mathbb{Z}_p$ the set of p-adic integers [2]

Definition 0.6. A p-adic integer is a formal series $\sum_{i\geq 0} a_i p^i$ with integral coefficients a_i satisfying

$$0 \leq a_i \leq p-1.$$

In particular, if $a = \sum_{i\geq 0} a_i p^i$, $b = \sum_{i\geq 0} b_i p^i$ (with $a_i, b_i \in \mathbb{F}_p$), we have

$$a = b \Leftrightarrow a_i = b_i \quad \text{for all} \quad i \geq 0.$$

Remark 0.2. From the definition, we immediately infer that the set of p-adic integers is not countable.

0.3.2. The $m\Theta$ ring $\mathbb{Z}_{p\mathbb{Z}}$

$\mathbb{Z}_{p\mathbb{Z}}$ is $m\Theta s$ of p-adic integers which has $C(\mathbb{Z}_{p\mathbb{Z}}, F'_\alpha) = \mathbb{Z}_p$ as the subset of modal Θ-valent invariants.

Definition 0.7. A $m\Theta$ p-adic integer is a formal series $\sum_{i\geq 0} a_i p^i$ with integral coefficients $a_i \in \mathbb{F}_{p\mathbb{Z}}$ satisfying

$$F_\alpha(a_i) \in \mathbb{F}_p; \quad \forall \alpha \in I_*.$$

With this definition, a $m\Theta$ p-adic integer $a = \sum_{i\geq 0} a_i p^i$ can be identified with the sequence $(a_i)_{i\geq 0}$ of its coefficients, and the set of $m\Theta$ p-adic integers coincides with the Cartesian product

$$X_{p\mathbb{Z}} = \prod_{i\geq 0} \mathbb{F}_{p\mathbb{Z}} = \mathbb{F}_{p\mathbb{Z}}^{\mathbb{N}}.$$

The usefulness of the series representation will be revealed when we introduce algebraic operations on these $m\Theta$ p-adic integers. Let us already observe that the expansions in base p of natural integers produce $m\Theta$ p-adic integers, and we obtain a canonical embedding of the set of natural integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ into $X_{p\mathbb{Z}}$.

1. Addition of $m\Theta$ p-adic integers

Let us define the sum of two $m\Theta$ p-adic integers a and b by the following procedure. The first component of the sum is $F_\alpha(a_0) + F_\alpha(b_0)$, $\forall \alpha \in I_*$ if this is less than or equal to $p - 1$, or $F_\alpha(a_0) + F_\alpha(b_0) - p$ otherwise. In the second case, we add a carry to the $m\Theta$ component of p and proceed by addition of the next $m\Theta$ components. In this way, we obtain a series for the sum that has $m\Theta$ components in the desired range. More succinctly, we can say that addition is defined $m\Theta$ componentwise, using the system of carries to keep them in $\mathbb{F}_p$.

Example 0.3. Let $p = 3$ and

$$a = 1 \times 3^0 + 1_{3\mathbb{Z}} \times 3^1 + 8_{3\mathbb{Z}} \times 3^2 + 2_{3\mathbb{Z}} \times 3^3 + 0 \times 3^4 + \dots ,$$

$$b = 5_{3\mathbb{Z}} \times 3^0 + 4_{3\mathbb{Z}} \times 3^1 + 2 \times 3^2 + 7_{3\mathbb{Z}} \times 3^3 + 0 \times 3^4 + \dots$$

with an infinity of zero coefficients from $i = 4$.

To simplify the notation, we will simply note

$$a = (1, 1_{3\mathbb{Z}}, 8_{3\mathbb{Z}}, 2_{3\mathbb{Z}}, 0, \dots) \quad \text{and} \quad b = (5_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 2, 7_{3\mathbb{Z}}, 0, \dots).$$

We calculate the different $m\Theta$ values of a and b.

$$F_1(a) = (1, 1, 1, 2, 0, \dots) \text{ and } F_2(a) = (1, 2, 0, 1, 0, \dots),$$

$$F_1(b) = (0, 2, 2, 0, 0, \dots) \text{ and } F_2(b) = (2, 0, 2, 1, 0, \dots).$$

Thus

$$\begin{aligned} a + b &= (F_1(a) + F_1(b),\ F_2(a) + F_2(b)) \\ &= ((1, 0, 1, 0, 1, 0, \dots), (0, 0, 0, 0, 1, 0, \dots)) \\ &= (8_{3\mathbb{Z}}, 0, 8_{3\mathbb{Z}}, 0, 1, 0, \dots). \end{aligned}$$

So

$$a + b = (8_{3\mathbb{Z}}, 0, 8_{3\mathbb{Z}}, 0, 1, 0, \dots).$$

2. Product of two $m\Theta$ p-adic integers

Let us define the product of two $m\Theta$ p-adic integers by multiplying their expansions $m\Theta$ componentwise, using the system of carries to keep these $m\Theta$ components in the desired range $\mathbb{F}_p$.

This multiplication is defined in such a way that it extends the usual multiplication of elements of $\mathbb{F}_{p\mathbb{Z}}$, written in base p.

Example 0.4. Let $p = 3$ and

$$a = (7_{3\mathbb{Z}}, 1, 2, 4_{3\mathbb{Z}}, 5_{3\mathbb{Z}}, 0, \ldots); \quad b = (1_{3\mathbb{Z}}, 0, 2, 8_{3\mathbb{Z}}, 2_{3\mathbb{Z}}, 0, \ldots)$$

with an infinity of zero coefficients from $i = 6$. We calculate the different $m\Theta$ values of a and b.

$$F_1(a) = (0, 1, 2, 2, 0, 0, \ldots) \quad \text{and} \quad F_2(a) = (1, 1, 1, 2, 0, 0, 0, \ldots),$$

$$F_1(b) = (1, 0, 2, 1, 2, 0, \ldots) \quad \text{and} \quad F_2(b) = (2, 0, 2, 0, 1, 0, \ldots).$$

Thus

$$\begin{aligned} a \times b &= (F_1^2(a \times b),\ F_2^2(a \times b)) \\ &= (F_1(a) \times F_1(b),\ F_2(a) \times F_2(b)) \\ &= ((0, 2, 1, 0, 2, 1, 1, 1, 0, \ldots), (1, 0, 1, 1, 0, 2, 1, 2, 0, \ldots)) \\ &= (7_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 1, 7_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 1_{3\mathbb{Z}}, 1, 1_{3\mathbb{Z}}, 0, \ldots). \end{aligned}$$

So

$$a \times b = (7_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 1, 7_{3\mathbb{Z}}, 4_{3\mathbb{Z}}, 1_{3\mathbb{Z}}, 1, 1_{3\mathbb{Z}}, 0, \ldots).$$

Definition 0.8. Let $a = \sum_{i \geq 0} a_i p^i$ be a $m\Theta$ p-adic integer. If $a \neq 0$, that is, $\forall \alpha \in I_*$, $F_\alpha(a) \neq 0$, there is a first index $v = v^\Theta(a) \geq 0$ such that $F_\alpha(a_v) \neq 0$. This index is the p-adic order $v = v^\Theta(a) = ord_{p\mathbb{Z}}(a)$, and we get a $m\Theta$ map

$$v^\Theta = ord_{p\mathbb{Z}} : \mathbb{Z}_{p\mathbb{Z}} - \{0\} \to \mathbb{N}.$$

Proposition 0.2. *The ring $\mathbb{Z}_{p\mathbb{Z}}$ of $m\Theta$ p-adic integers is an $m\Theta$ integral domain.*

Proof 0.4. The commutative $m\Theta$ ring $\mathbb{Z}_{p\mathbb{Z}}$ is not $\{0\}$, and we have to show that it has no zero divisor. Let therefore $a = \sum_{i\geq 0} a_i p^i \neq 0$, $b = \sum_{i\geq 0} b_i p^i \neq 0$, and define $v = v^{\Theta}(a)$, $w = v^{\Theta}(b)$. Then a_v is the first nonzero $m\Theta$ coefficient of a, $\forall \alpha \in I_*$, $0 < F_\alpha(a_v) < p$, and similarly b_w is the first nonzero $m\Theta$ coefficient of b. In particular, p divides neither a_v nor b_w and consequently does not divide their product $a_v b_w$ either. By definition of multiplication, the first nonzero $m\Theta$ coefficient of the product ab is the $m\Theta$ coefficient c_{v+w} of p^{v+w}, and this $m\Theta$ coefficient is defined by

$$\forall \alpha \in I_*, \quad 0 < F_\alpha(c_{v+w}) < p, \quad F_\alpha(c_{v+w}) \equiv F_\alpha^2(a_v b_w) \, (\mathrm{mod}\ p).$$

0.4. The Fraction Field of $\mathbb{Z}_{p\mathbb{Z}}$

Let $(\mathbb{F}_{p\mathbb{Z}}, F_\alpha)$ be the finite $m\Theta$ field with p^2 elements.

Definition 0.9. The $m\Theta$ mapping

$$a = \sum_{i\geq 0} a_i p^i \mapsto a_0 \bmod p\mathbb{Z}$$

defines a ring $m\Theta$ homomorphism $\varepsilon^{\Theta} : \mathbb{Z}_{p\mathbb{Z}} \to \mathbb{F}_{p\mathbb{Z}}$ called reduction mod $p\mathbb{Z}$.

This reduction $m\Theta$ homomorphism is obviously surjective, with kernel

$$\{a \in \mathbb{Z}_{p\mathbb{Z}} : \forall \alpha \in I_*,\ F_\alpha(a_0) = 0\} = \left\{ \sum_{i\geq 1} a_i p^i = p \sum_{i\geq 0} a_{i+1} p^i \right\} = p\mathbb{Z}_{p\mathbb{Z}}.$$

Since the quotient is a $m\Theta$ field, the kernel $p\mathbb{Z}_{p\mathbb{Z}}$ of ε^{Θ} is a maximal ideal of the $m\Theta$ ring $\mathbb{Z}_{p\mathbb{Z}}$.

Theorem 0.3. *The* $m\Theta$ *group* $\mathbb{Z}_{p\mathbb{Z}}^{\times}$ *of invertible elements in the* $m\Theta$ *ring* $\mathbb{Z}_{p\mathbb{Z}}$ *consists of the* $m\Theta$ p*-adic integers of order zero, namely*

$$\mathbb{Z}_{p\mathbb{Z}}^{\times} = \left\{ \sum_{i \geq 0} a_i p^i \ : \ a_0 \neq 0 \right\}.$$

Proof 0.5. If a $m\Theta$ p-adic integer a is invertible, so must be its reduction $\varepsilon^{\Theta}(a)$ in $\mathbb{F}_{p\mathbb{Z}}$. This proves the inclusion $\mathbb{Z}_{p\mathbb{Z}}^{\times} \subset \left\{ \sum_{i \geq 0} a_i p^i \ : a_0 \neq 0 \right\}$. Conversely, we have to show that any $m\Theta$ p-adic integer a of order $v^{\Theta}(a) = 0$ is invertible. In this case the reduction $\varepsilon^{\Theta}(a_0) \in \mathbb{F}_{p\mathbb{Z}}$ is not zero, and hence is invertible in this field. Choose $\forall \alpha \in I_*$, $0 < F_{\alpha}(b_0) < p$ with $a_0 b_0 \equiv 1_{p\mathbb{Z}} (\mathrm{mod}\ p\mathbb{Z})$ and write $F_{\alpha}^2(a_0 b_0) = 1 + k \times p$. Hence, if we write $F_{\alpha}(a) = a_0 + p \times k'$, then

$$F_{\alpha}^2(a \cdot b_0) = 1 + k \times p + pk'b_0 = 1 + p \times k''$$

for some $m\Theta$ p-adic integer k''. It suffices to show that the classical p-adic integer $1 + k'' \times p$ is invertible, since we can then write

$$F_{\alpha}^2(a \cdot b_0)(1 + k'' \times p)^{-1} = 1, \quad (F_{\alpha}(a))^{-1} = F_{\alpha}(b_0)(1 + k'' \times p)^{-1}.$$

In other words, it is enough to treat the case $F_{\alpha}(a_0) = 1$, $F_{\alpha}(a) = 1 + k'' \times p$. Let us observe that we can take

$$(1 + k'' \times p)^{-1} = 1 - k'' \times p + (k'' \times p)^2 - \ldots$$

$$= 1 + c_1 \times p + c_1 \times p^2 + \ldots ,$$

with integers $c_i \in \mathbb{F}_p$. This possibility is assured if we apply the rules for carries suitably.

Remark 0.3. The ring $\mathbb{Z}_{p\mathbb{Z}}$ of $m\Theta$ p-adic integers has a unique maximal ideal, namely

$$p\mathbb{Z}_{p\mathbb{Z}} = \mathbb{Z}_{p\mathbb{Z}} - \mathbb{Z}_{p\mathbb{Z}}^{\times}.$$

The statement of the preceding remark corresponds to a partition $\mathbb{Z}_{p\mathbb{Z}} = \mathbb{Z}_{p\mathbb{Z}}^{\times} \coprod p\mathbb{Z}_{p\mathbb{Z}}$ (a disjoint union). In fact, one has a partition

$$\mathbb{Z}_{p\mathbb{Z}} - \{0\} = \coprod_{k \geq 0} p^k \mathbb{Z}_{p\mathbb{Z}}^{\times}.$$

Definition 0.10. Every nonzero $m\Theta$ p-adic integer $a \in \mathbb{Z}_{p\mathbb{Z}}$ has a canonical representation $a = p^v u$, where $v = v^{\Theta}(a)$ is the p-adic order of a and $u \in \mathbb{Z}_{p\mathbb{Z}}^{\times}$ is a $m\Theta$ p-adic unit.

The principal $m\Theta$ ideals of the ring $\mathbb{Z}_{p\mathbb{Z}}$,

$$(p^k) = p^k \mathbb{Z}_{p\mathbb{Z}} = \{x \in \mathbb{Z}_{p\mathbb{Z}} : ord_{p\mathbb{Z}}(x) \geq k\},$$

have an intersection equal to $\{0\}$:

$$\mathbb{Z}_{p\mathbb{Z}} \supset p\mathbb{Z}_{p\mathbb{Z}} \supset \dots \supset p^k \mathbb{Z}_{p\mathbb{Z}} \supset \dots \supset \bigcap_{k \geq 0} p^k \mathbb{Z}_{p\mathbb{Z}} = \{0\}.$$

Indeed, any element $a \neq 0$ has an order $v^{\Theta}(a) = k$, hence $a \notin (p^{k+1})$. In fact, these principal $m\Theta$ ideals are the only nonzero ideals of the ring of $m\Theta$ p-adic integers.

Proposition 0.3. *The ring* $\mathbb{Z}_{p\mathbb{Z}}$ *is a principal* $m\Theta$ *ideal domain. More precisely, its ideals are the principal ideals* $\{0\}$ *and* $p^k \mathbb{Z}_{p\mathbb{Z}}$, $k \in \mathbb{N}$.

Proof 0.6. Let $I \neq \{0\}$ be a nonzero $m\Theta$ ideal of $\mathbb{Z}_{p\mathbb{Z}}$ and $0 \neq a \in I$ an element of minimal order, say $k = v^{\Theta}(a) < \infty$. Write $a = p^k u$ with a

$m\Theta$ p-adic unit u. Hence $p^k = u^{-1}a \in I$ and $(p^k) = p^k\mathbb{Z}_{p\mathbb{Z}} \subset I$. Conversely, for any $b \in I$, let $w = v^{\Theta}(b) \geq k$ and write

$$b = p^w u' = p^k \times p^{w-k}u' \in p^k\mathbb{Z}_{p\mathbb{Z}}.$$

The ring of $m\Theta$ p-adic integers is an $m\Theta$ integral domain (Proposition 0.2). Hence we can define the field of $m\Theta$ p-adic numbers as the fraction field of $\mathbb{Z}_{p\mathbb{Z}}$

$$\mathbb{Q}_{p\mathbb{Z}} = Frac\,(\mathbb{Z}_{p\mathbb{Z}}).$$

We have seen that any nonzero $m\Theta$ p-adic integer $x \in \mathbb{Z}_{p\mathbb{Z}}$ can be written in the form $x = p^m u$ with a $m\Theta$ unit u of $\mathbb{Z}_{p\mathbb{Z}}$ and $m \in \mathbb{N}$ the order of x. The inverse of x in the $m\Theta$ fraction field will thus be $\frac{1}{x} = p^{-m}u^{-1.}$. This shows that this $m\Theta$ fraction field is generated-multiplicatively, and a fortiori as a $m\Theta$ ring by $\mathbb{Z}_{p\mathbb{Z}}$ and the negative powers of p. We can write

$$\mathbb{Q}_{p\mathbb{Z}} = \mathbb{Z}_{p\mathbb{Z}}\left[\frac{1}{p}\right].$$

The representation $\frac{1}{x} = p^{-m}u^{-1.}$ also shows that $\frac{1}{x} \in p^{-m}\mathbb{Z}_{p\mathbb{Z}}$ and

$$\mathbb{Q}_{p\mathbb{Z}} = \coprod_{m \geq 0} p^{-m}\mathbb{Z}_{p\mathbb{Z}}$$

is a union over the positive integers m. These considerations also show that a nonzero $m\Theta$ p-adic number $x \in \mathbb{Q}_{p\mathbb{Z}}$ can be uniquely written as $x = p^m u$ with $m \in \mathbb{Z}$ and a unit $u \in \mathbb{Z}^{\times}_{p\mathbb{Z}}$; hence

$$\mathbb{Q}^{\times}_{p\mathbb{Z}} = \coprod_{m \geq 0} p^{-m}\mathbb{Z}^{\times}_{p\mathbb{Z}}$$

is a disjoint union over the rational integers $m \in \mathbb{Z}$.

0.5. $m\Theta$ Completion of $\mathbb{Q}_{p\mathbb{Z}}$

0.5.1. The $m\Theta s$ of $\mathbb{Q}_{p\mathbb{Z}}$ - sequences

Definition 0.11. Intrinsic metric of $\mathbb{Q}_{p\mathbb{Z}}$ or $m\Theta$ metric space on $\mathbb{Q}_{p\mathbb{Z}}$ is any list defined as follows:

$$(\mathbb{Q}_{p\mathbb{Z}}, F_\alpha, d_{p\mathbb{Z}}), \text{ or simply } (\mathbb{Q}_{p\mathbb{Z}}, d_{p\mathbb{Z}})$$

with $d_{p\mathbb{Z}}$ a $m\Theta$ p-adic metric such that:

$$\forall x = (x_i)_{i\geq 0}, \quad y = (y_i)_{i\geq 0} \in \mathbb{Q}_{p\mathbb{Z}}; \quad d_{p\mathbb{Z}}(x, y) = \sup_{i\geq 0} \frac{\delta(x_i, y_i)}{p^i} = \frac{1}{p^{\upsilon^{\Theta}(x-y)}},$$

where $\delta(x_i, y_i) = \begin{cases} 1 & \text{if} \quad x_i \neq y_i \\ 0 & \text{if} \quad x_i = y_i \end{cases}$ is the discrete topology.

Definition 0.12. (a_m) is a $m\Theta$ Cauchy sequence of $\mathbb{Q}_{p\mathbb{Z}}$, $a_m = \sum_{i\geq 0} a_{m,\, i} p^i$, if and only if (a_m) verifies the following statement:

$$\forall \epsilon \in \mathbb{Q}_+^*, \quad \exists m_0 \in \mathbb{N}^* : m', m'' \geq m_0 \Rightarrow d_{p\mathbb{Z}}(a_{m'}, a_{m''}) \leq \epsilon.$$

$\mho = \mho(\mathbb{Q}_p)$ is the set of sequences of elements of $\mathbb{Q}_p$ noted $(a_m) : \forall m;\ a_m \in \mathbb{Q}_p$.

$\Omega = \Omega(\mathbb{Q}_p)$ is the set of Cauchy sequences of $\mathbb{Q}_p$.

Let $\mho_{p\mathbb{Z}}$ be the set of sequences of elements of $\mathbb{Q}_{p\mathbb{Z}} : \forall (a_m)$; $\forall \alpha \in I_*,\ F_\alpha(a_m) = (F_\alpha a_m) \in \mho(\mathbb{Q}_p)$.

Remark 0.4. 1. $(\mho_{p\mathbb{Z}}, F_\alpha)$ is a $m\Theta s$ whose the subset of $m\Theta$ invariants, $C(\mho_{p\mathbb{Z}})$ is $\mho(\mathbb{Q}_p)$; so

$$C(\mho_{p\mathbb{Z}}, F_\alpha) = \mho(\mathbb{Q}_p).$$

2. $(\Omega_{p\mathbb{Z}}, F_\alpha)$ is a $m\Theta s$ whose the subset of $m\Theta$ invariants, $C(\Omega_{p\mathbb{Z}})$ is $\Omega(\mathbb{Q}_p)$; so

$$C(\Omega_{p\mathbb{Z}}, F_\alpha) = \Omega(\mathbb{Q}_p).$$

Definition 0.13. Let $\Omega_{p\mathbb{Z}} = \Omega_{m\Theta}(\mathbb{Q}_{p\mathbb{Z}})$ be the set of $m\Theta$ Cauchy sequences of $(\mathbb{Q}_{p\mathbb{Z}})$.

0.5.2. $m\Theta$ construction of $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$

Definition 0.14. We define $\mathcal{R}_{p\mathbb{Z}}$ in $\Omega_{p\mathbb{Z}}$ as follows:

$$\forall (a_m), (b_m) \in \Omega_{p\mathbb{Z}}, (a_m)\mathcal{R}_{p\mathbb{Z}}(b_m)$$

$$\Leftrightarrow (\forall \epsilon \in \mathbb{Q}_+^*, \exists m_0 \in \mathbb{N}^* : m \geq m_0 \Rightarrow d_{p\mathbb{Z}}(a_m, b_m) \leq \epsilon).$$

Consequence 0.1. 1. Let $\mathcal{R} = \mathcal{R}_{p\mathbb{Z}}/\Omega$. By definition,

$$\forall (a_m), (b_m) \in \Omega, (a_m)\mathcal{R}(b_m)$$

$$\Leftrightarrow (\forall \epsilon \in \mathbb{Q}_+^*, \exists m_0 \in \mathbb{N}^* : m \geq m_0 \Rightarrow d(a_m, b_m) \leq \epsilon),$$

where $d(a_m, b_m) = \dfrac{1}{p^{\upsilon(a_m - b_m)}}$.

2. $\exists (a_m), (b_m)$; $(a_m), (b_m) \in \Omega_{p\mathbb{Z}}$ such that $(a_m) \in \Omega_{p\mathbb{Z}} - \Omega$, $b_m \in \Omega$ but $(a_m)\mathcal{R}_{p\mathbb{Z}}(b_m)$.

Therefore $(a_m)\mathcal{R}_{p\mathbb{Z}}(b_m)$; either $(a_m), (b_m) \in \Omega_{p\mathbb{Z}} - \Omega$ or $(a_m), (b_m) \in \Omega$.

Observation 0.1. 1. Since $\mathcal{R}_{p\mathbb{Z}}$ is respectful with the $m\Theta$ structure of $\Omega_{p\mathbb{Z}}$

$$(a_m)\mathcal{R}(b_m) \Rightarrow \forall\alpha \in I_*, \quad F_\alpha(a_m)\mathcal{R}F_\alpha(b_m).$$

Therefore let $\widehat{\Omega_{p\mathbb{Z}}} = \dfrac{\Omega_{p\mathbb{Z}}}{\mathcal{R}_{p\mathbb{Z}}}$ be the $m\Theta$ quotient $m\Theta s$ modulo $\mathcal{R}_{p\mathbb{Z}}$ of $\Omega_{p\mathbb{Z}}$.

2. Let $\mathcal{N}$ be defined as follows:

$$\mathcal{N} = \{(a_m);\ (a_m) \in \Omega : (a_m)\mathcal{R}(0)\}.$$

It is known that $\mathcal{N}$ is a maximal ideal of Ω. $\mathcal{N}_{p\mathbb{Z}}$ defined as follows:

$$\mathcal{N}_{p\mathbb{Z}} = \{(a_m);\ (a_m) \in \Omega_{p\mathbb{Z}} : \forall\alpha \in I_*,\ F_\alpha(a_m) \in \mathcal{N}\}$$

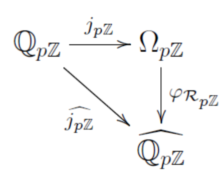
respects all the $m\Theta$ structures of $\Omega_{p\mathbb{Z}}$ and even is a $m\Theta$ maximal ideal of $\Omega_{p\mathbb{Z}}$ for all its $m\Theta$ algebraic structures. Obviously, $\dfrac{\Omega_{p\mathbb{Z}}}{\mathcal{N}_{p\mathbb{Z}}} = \dfrac{\Omega_{p\mathbb{Z}}}{\mathcal{R}_{p\mathbb{Z}}}$.

Let $\widehat{\mathbb{Q}_{p\mathbb{Z}}} = \dfrac{\Omega_{p\mathbb{Z}}}{\mathcal{N}_{p\mathbb{Z}}}$.

The following remark results from the preceding observation.

Remark 0.5. $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ is a $m\Theta$ quasifield whose the subfield of the $m\Theta$ invariants, $C(\widehat{\mathbb{Q}_{p\mathbb{Z}}})$ is $\widehat{\mathbb{Q}_p} : C(\widehat{\mathbb{Q}_{p\mathbb{Z}}}) = \widehat{\mathbb{Q}_p}$.

Consider the following commutative diagram:



With the following definitions $\forall a \in \mathbb{Q}_{p\mathbb{Z}} : j_{p\mathbb{Z}} : a \mapsto j_{p\mathbb{Z}}(a) = (a_m)$.

$$\varphi_{\mathcal{R}_{p\mathbb{Z}}} : (a_m) \mapsto \varphi_{\mathcal{R}_{p\mathbb{Z}}}((a_m)) = \frac{(a_m)}{\mathcal{R}_{p\mathbb{Z}}}; \qquad \widehat{j_{p\mathbb{Z}}} : x \mapsto \widehat{j_{p\mathbb{Z}}}(a) = \frac{(a_m)}{\mathcal{R}_{p\mathbb{Z}}}.$$

1. By construction, $\widehat{j_{p\mathbb{Z}}} = \varphi_{\mathcal{R}_{p\mathbb{Z}}} \circ j_{p\mathbb{Z}}$ is a $m\Theta$ isomorphism of $m\Theta$ ring from the $m\Theta f\ \mathbb{Q}_{p\mathbb{Z}}$ over the sub $m\Theta f\ \hat{j}\,(\mathbb{Q}_{p\mathbb{Z}})$ of $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$.

2. $\hat{j} = \widehat{j_{p\mathbb{Z}}}\big|_{\mathbb{Q}_p}$ is a field isomorphism of $\mathbb{Q}_p$ over the subfield $\hat{j}(\mathbb{Q}_p)$ of $\widehat{\mathbb{Q}_p}$.

Definition 0.15. 1. Let $a_0 \in \mathbb{Q}_{p\mathbb{Z}}$, (a_m) a sequence of elements of $\mathbb{Q}_{p\mathbb{Z}}$; (a_m) $m\Theta$ converges $(m\Theta cv)$ to a_0; notation $(a_m)\ m\Theta cv \to a_0$ if and only if:

$$\forall \epsilon \in \mathbb{Q}_+^*, \exists m_0 \in \mathbb{N}^* : m \geq m_0 \to d_{p\mathbb{Z}}(a_m, a_0) \leq \epsilon.$$

2. (a_m) $m\Theta$ converges to a in $\mathbb{Q}_{p\mathbb{Z}}$ is equivalent to the following statement:

$$\forall \alpha \in I_*,\ F_\alpha(a_m) \text{ converges to } F_\alpha(a) \text{ in } \mathbb{Q}_p.$$

Proposition 0.4. $\forall a \in \mathbb{Q}_{p\mathbb{Z}}$, $\forall J : \emptyset \neq J \subset I_*$, $\forall (a_m) \in \mho_{p\mathbb{Z}}$. *If* $\forall \alpha \in J$, $F_\alpha(a_m)$ *converges to* $F_\alpha(a) \in \mathbb{Q}_p$, *then* (a_m) J*-converges to* $a \in \mathbb{Q}_{p\mathbb{Z}}$.

Proof 0.7. Since $\forall \alpha$, $\alpha \in J$, $\exists m_{0\alpha} \in \mathbb{N}$ such that $m \geq m_{0\alpha} \to d(F_\alpha a_m, F_\alpha a) \leq \frac{\epsilon}{Card\ J}\ \forall \epsilon \in \mathbb{Q}_+^*$, $\forall \alpha \in J$.

Let $m_0 = \max\{m_{0\alpha}\}$; then $m \geq m_0 \to d(F_\alpha a_m, F_\alpha a) \leq \frac{\epsilon}{Card\ J}$ $\forall \epsilon \in \mathbb{Q}_+^*$, $\forall \alpha \in J$. Therefore $\sum_{\alpha \in J} d(F_\alpha(a_m), F_\alpha(a)) \leq \epsilon$ what is $d_J(a_m, a) \leq \epsilon$ and (a_m) J-converges to a in $\mathbb{Q}_{p\mathbb{Z}}$.

Lemma 0.1. *If* $\varrho \in \widehat{\mathbb{Q}_{p\mathbb{Z}}}$, *then*

$$\forall (a_m) \in \varrho;\ \hat{j}(a_m)\ m\Theta \text{ converges to } \varrho \text{ in } \widehat{\mathbb{Q}_{p\mathbb{Z}}}.$$

Proof 0.8. It is obvious that in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ as well as in $\mathbb{Q}_{p\mathbb{Z}}$ the following statements are equivalent:

1. (a_m) $m\Theta$ converges to ϱ in $\mathbb{Q}_{p\mathbb{Z}}$ and

2. $\forall \alpha \in I_*$, $F_\alpha \circ \widehat{j_{p\mathbb{Z}}}(a_m)$ converges to $F_\alpha \varrho$ in $\widehat{\mathbb{Q}_p}$.

So that it then remains the same to state that $\forall \alpha \in I_*$, $\hat{j}(F_\alpha a_m)$ converges to $F_\alpha \varrho$ in $\widehat{\mathbb{Q}_p}$. That is

$$\forall \alpha \in I_*, \forall \epsilon \in \mathbb{Q}_+^*, \exists m_{0\alpha} \in \mathbb{N}^* : m \geq m_{0\alpha}$$

$$\Rightarrow d(j(F_\alpha a_m), F_\alpha \varrho) \leq \frac{\epsilon}{Card\ I_*}.$$

Let $m_0 = \max_{\alpha \in I_*} \{m_{0\alpha}\}$, if $\forall m \geq m_0$, then $\forall \alpha \in I_*$, $d(j(F_\alpha a_m), F_\alpha \varrho) \leq \frac{\epsilon}{Card\ I_*}$. Therefore $d(j(F_\alpha a_m), F_\alpha \varrho) \leq \epsilon$.

This means that $\forall \epsilon \in \mathbb{Q}_+^*$, $m \geq m_0 \Rightarrow d_{p\mathbb{Z}}(\widehat{j_{p\mathbb{Z}}}(a_m), \varrho) \leq \epsilon$. So that $\hat{j}(a_m)$ $m\Theta$ converges to ϱ in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$.

Remark 0.6. It is obvious that $\forall J : \varnothing \neq J \subset I_*$ if $\varrho \in \widehat{\mathbb{Q}_{p\mathbb{Z}}}$, $\forall (a_m) \in \varrho \cap \Omega_J(\mathbb{Q}_{p\mathbb{Z}})$, then (a_m) J cv to ϱ in $\mathbb{Q}_{p\mathbb{Z}}$. One would then say that $\mathbb{Q}_{p\mathbb{Z}}$ is J-dense in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$.

$\mathbb{Q}_{p\mathbb{Z}}$ is $m\Theta$ dense in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$.

Theorem 0.4. *Let* $(\mathbb{K}_{p\mathbb{Z}}, F'_\alpha)$ *be a* $m\Theta f$ *such as* $\mathbb{Q}_{p\mathbb{Z}}$ *is* $m\Theta$ *dense in* $\mathbb{K}_{p\mathbb{Z}}$ *and any* $m\Theta$ *Cauchy sequence of* $\mathbb{K}_{p\mathbb{Z}}$ $m\Theta cv$ *in* $\mathbb{K}_{p\mathbb{Z}}$, *then* $\exists\, \widehat{f_{p\mathbb{Z}}} : \widehat{\mathbb{Q}_{p\mathbb{Z}}} \to \mathbb{K}_{p\mathbb{Z}}$ *such that* $\widehat{f_{p\mathbb{Z}}}$ *is a* $m\Theta$ *isomorphism of* $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ *over* $\mathbb{K}_{p\mathbb{Z}}$, $s(f_{p\mathbb{Z}})$ *is unique and* $\widehat{f_{p\mathbb{Z}}}_{|\mathbb{Q}_{p\mathbb{Z}}} = id_{\mathbb{Q}_{p\mathbb{Z}}}$.

Proof 0.9. Any $m\Theta$ Cauchy sequence of $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ is a $m\Theta$ Cauchy sequence of $\mathbb{K}_{p\mathbb{Z}}$. Let then $\hat{a} \in \widehat{\mathbb{Q}_{p\mathbb{Z}}}$ and $(a_m) \in \Omega_{p\mathbb{Z}}$ such that $\hat{a} = \frac{(a_m)}{\mathcal{R}_{p\mathbb{Z}}}$. It is known that (a_m) $m\Theta cv$ say to $\hat{a}$ in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$.

As a Cauchy sequence of $\mathbb{K}_{p\mathbb{Z}}$, let a' be the $m\Theta$ limit of (a_m) in $\mathbb{K}_{p\mathbb{Z}}$. Define now f as follows:

$$\widehat{f_{p\mathbb{Z}}} : \widehat{\mathbb{Q}_{p\mathbb{Z}}} \to \mathbb{K}_{p\mathbb{Z}} : \hat{a} \mapsto a'.$$

By definition of $\widehat{f_{p\mathbb{Z}}}$ and the respective laws of $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ and $\mathbb{K}_{p\mathbb{Z}}$, $\widehat{f_{p\mathbb{Z}}}_{|\mathbb{Q}_{p\mathbb{Z}}} = id_{\mathbb{Q}_{p\mathbb{Z}}}$.

0.6. Conclusion

This note shows that $\mathbb{Q}_{p\mathbb{Z}}$ is a $m\Theta$ Fraction Field of $\mathbb{Z}_{p\mathbb{Z}}$, the ring of $m\Theta$ p-adic integers compatible with the $m\Theta$ structure as presented in [4].

$\mathbb{Q}_{p\mathbb{Z}}$ is $m\Theta$ dense in $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$. $\widehat{\mathbb{Q}_{p\mathbb{Z}}}$ is the unique $m\Theta$ completion of $\mathbb{Q}_{p\mathbb{Z}}$ and

$$C(\widehat{\mathbb{Q}_{p\mathbb{Z}}}, F'_\alpha) = \widehat{\mathbb{Q}_p}.$$

The results obtained in this paper can be used in the construction of the compact topological $m\Theta$ ring $\mathbb{Z}_{p\mathbb{Z}}$ of $m\Theta$ p-adic integers and of its quotient $m\Theta$ field $\mathbb{Q}_{p\mathbb{Z}}$ the locally compact $m\Theta$ field of p-adic numbers.

References

[1] F. Ayissi Eteme, *chrm*Θ Introducing Pure and Applied Mathematics, Lambert Academic Publishing Saarbruken, Germany, 2015.

[2] Alain M. Robert, A Course in p-adic Analysis, Springer-Verlag New York, 2000.

[3] F. Ayissi Eteme, Logique et Algèbre de structure mathématiques modales Θ- valentes chrysippiennes, Edition Hermann, Paris, 2009.

[4] F. Ayissi Eteme, Complétion chrysippienne d'une algèbre de Lukasiewicz Θ- valent CRAS Paris, 299, série 1(3), 1984, pp. 69-72.

[5] Alain Escassut, Analytic Elements in p- adic Analysis, World Scientific, 1995.

[6] F. Ayissi Eteme, Anneau chrysippien Θ- valent, CRAS, Paris 298, série 1, 1984, pp. 1-4.