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COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE INTERPOLATIVE BERINDE WEAK MAPPING OF INTEGRAL TYPE IN METRIC SPACES USING (CLR) PROPERTY

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Abstract

In this paper, we introduce the notion of weakly compatible interpolative Berinde weak mapping of integral type in metric spaces, and obtain common fixed point theorem for two pair of self-mappings using the (CLR) property. Some consequences of the main result are stated.

1. Introduction and Preliminaries

Definition 1.1. [1] A coincidence point of a pair of self-mappings $A, B : X \mapsto X$ is a point $x \in X$ for which $Ax = Bx$.

Definition 1.2. [1] A common fixed point of a pair of self-mappings $A, B : X \mapsto X$ is a point $x \in X$ for which $Ax = Bx = x$.

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Definition 1.3. [2] A pair of self-mappings $A, B : X \mapsto X$ is weakly compatible if they commute at their coincidence points, that is, if there exists a point $x \in X$ such that $ABx = BAx$ whenever $Ax = Bx$.

Definition 1.4. [1] Let (X, d) be a metric space and $A, B, P, Q : X \mapsto X$ be four self-maps. The pairs (A, Q) and (B, P) satisfy the common limit range property with respect to mappings Q and P, denoted by (CLR_{PQ}) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in QX \cap PX.$$

Notation 1.5. [1] Φ will denote the family of all functions $\phi : [0, \infty) \mapsto [0, \infty)$ satisfying the following:

(1) ϕ is lower semi-continuous.

(2) $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

(3) ϕ is discontinuous at $t = 0$.

Notation 1.6. [1] Ψ will denote the family of all functions $\psi : [0, \infty) \mapsto [0, \infty)$ satisfying the following:

(1) ψ is monotonically increasing and continuous.

(2) $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

Lemma 1.7. [3] *Let* $\varphi \in \Phi$ *and* $\{r_n\}_{n\in\mathbb{N}}$ *be a nonnegative sequence with* $\lim_{n\to\infty} r_n = a$. *Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^{a} \varphi(t)dt.$$

Theorem 1.8. [1] *Let* (X, d) *be a metric space and* A, B, P, Q be *four self-maps on* X *satisfying the following*:

(1) *The pairs* (A, P) *and* (B, Q) *share the* (CLR_{PQ}) *property.*

(2) $$\psi\left(\int_0^{d(Ax, By)} \varphi(t)dt\right) \leq \psi\left(\int_0^{d(Ax, Px)+d(By, Qy)} \varphi(t)dt\right) - \phi\left(\int_0^{d(Ax, Px)+d(By, Qy)} \varphi(t)dt\right).$$

If the pairs (A, P) *and* (B, Q) *are weakly compatible, then* A, B, P, Q *have a unique common fixed point in* X.

Definition 1.9. [4] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows:

Definition 1.10. [4] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in [0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

2. Main Result

Theorem 2.1. *Let* (X, d) *be a metric space and* A, B, P, Q *be four self-maps on* X *satisfying the following*:

(1) *The pairs* (A, P) *and* (B, Q) *share* (CLR_{PQ}) *property.*

(2) *For all* $x, y \in X$, $x, y \notin \{Fix(A), Fix(P), Fix(B), Fix(Q)\}$, *and* $\alpha \in (0, 1)$, *we have*

$$\psi\left(\int_0^{d(Ax, By)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(Px, Qy)^\alpha d(Px, Ax)^{1-\alpha}} \varphi(t)dt\right)$$

$$- \phi\left(\int_0^{\frac{1}{2^\alpha} d(Px, Qy)^\alpha d(Px, Ax)^{1-\alpha}} \varphi(t)dt\right).$$

If the pairs (A, P) *and* (B, Q) *are weakly compatible, then* A, B, P, *and* Q *have a unique common fixed point in* X.

Proof. Suppose that the pairs (A, P) and (B, Q) share the (CLR_{PQ}) property, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z$$

$$\text{for some } z \in PX \cap QX. \tag{2.1}$$

Since $z \in PX$, there exists a point $s \in X$ such that $Ps = z$. From (2.1), we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Ps. \tag{2.2}$$

Now we claim that $As = Ps$. Let if possible $As \neq Ps$. Putting $x = s$ and $y = y_n$ in (2) of the theorem we get

$$\psi\left(\int_0^{d(As, By_n)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(Ps, Qy_n)^\alpha d(Ps, As)^{1-\alpha}} \varphi(t)dt\right). \tag{2.3}$$

Passing to the limit as $n \to \infty$ in (2.3) and using (2.2) we obtain

$$\psi\left(\int_0^{d(As, z)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(z, z)^\alpha d(z, As)^{1-\alpha}} \varphi(t)dt\right) = \psi(0) = 0.$$

Thus, from the above equation we have $d(As, z) = 0$, that is $As = z$. Thus, $As = Ps = z$. Similarly, since $z \in Qx$, there exist a point $v \in X$ such that $Qv = z$. Thus, (2.1) becomes

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Qv.$$

Now we claim that $Bv = Qv$. To support this claim, let $Bv \neq Qv$. Then on putting $x = x_n$ and $y = v$ in (2) of the theorem, we conclude that

$$Bv = Qv = z.$$

Thus, we have

$$As = Ps = Bv = Qv = z.$$

Next we show that z is a common fixed point of A, B, P, and Q. To this aim, since the pairs (A, P) and (B, Q) are weakly compatible, then using the above equation, we have

$$As = Ps \Rightarrow PAs = APS \Rightarrow Az = Pz$$

and

$$Bv = Qv \Rightarrow QBv = BQv \Rightarrow Bz = Qz.$$

We will show next that $Az = z$. Otherwise if $Az \neq z$, then using (2) of the theorem with $x = z$ and $y = v$, we have

$$\psi\left(\int_0^{d(Az, Bv)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(Pz, Qv)^\alpha d(Pz, Az)^{1-\alpha}} \varphi(t)dt\right) = \psi(0) = 0.$$

Hence it follows that $Az = Bv = z$. Thus

$$Az = Pz = z.$$

Similarly, setting $x = u$ and $y = z$ in (2) of the theorem and using

$As = Ps = Bv = Qv = z$ and $Bv = Qv \Rightarrow QBv = BQv \Rightarrow Bz = Qz$, we conclude that

$$Bz = Qz = z.$$

Therefore, it follows that

$$Az = Bz = Qz = Pz = z,$$

that is, z is a common fixed point of A, B, Q and P. Finally, we prove the uniqueness of the common fixed point of A, B, Q and P. Assume that z_1 and z_2 are two distinct common fixed points of A, B, Q and P. Then replacing x by z_1 and y by z_2 in (2) of the theorem, we have

$$\psi\left(\int_0^{d(Az_1, Bz_2)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(Pz_1, Qz_2)^\alpha d(Pz_1, Az_1)^{1-\alpha}} \varphi(t)dt\right) = \psi(0) = 0.$$

Thus, $d(Az_1, Bz_2) = d(z_1, z_2) = 0$. Hence $z_1 = z_2$. Therefore, A, B, Q and P have a unique common fixed point in X and the proof is complete.

Corollary 2.2. *Let (X, d) be a metric space and let A, P, Q be three self-maps on X satisfying the following*:

(1) *The pairs (A, P) and (A, Q) share the (CLR_{PQ}) property.*

(2) *For all $x, y \in X$, $x, y \notin \{Fix(A), Fix(P), Fix(Q)\}$ and $\alpha \in (0, 1)$, we have*

$$\psi\left(\int_0^{d(Ax, Ay)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha} d(Px, Qy)^\alpha d(Px, Ax)^{1-\alpha}} \varphi(t)dt\right)$$

$$- \phi\left(\int_0^{\frac{1}{2^\alpha} d(Px, Qy)^\alpha d(Px, Az)^{1-\alpha}} \varphi(t)dt\right).$$

If the pairs (A, Q) *and* (A, P) *are weakly compatible, then* A, P, Q *have a unique common fixed point in* X.

Corollary 2.3. *Let* (X, d) *be a metric space and* A, Q *be two self-maps on* X *satisfying the following*:

(1) *The pair* (A, Q) *share* (CLR_Q) *property.*

(2) *For all* $x, y \in X$, $x, y \notin \{Fix(A), Fix(Q)\}$, *and* $\alpha \in (0, 1)$, *we have*

$$\psi\left(\int_0^{d(Ax, Ay)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha}d(Qx, Qy)^\alpha d(Qx, Ax)^{1-\alpha}} \varphi(t)dt\right)$$

$$- \phi\left(\int_0^{\frac{1}{2^\alpha}d(Qx, Qy)^\alpha d(Qx, Ax)^{1-\alpha}} \varphi(t)dt\right).$$

If the pair (A, Q) *is weakly compatible, then* A *and* Q *have a unique common fixed point in* X.

Corollary 2.4. *Let* (X, d) *be a metric space and* A *be a self-map on* X *satisfying the following condition*:

$$\psi\left(\int_0^{d(Ax, Ay)} \varphi(t)dt\right) \leq \psi\left(\int_0^{\frac{1}{2^\alpha}d(x, y)^\alpha d(x, Ax)^{1-\alpha}} \varphi(t)dt\right)$$

$$- \phi\left(\int_0^{\frac{1}{2^\alpha}d(x, y)^\alpha d(x, Ax)^{1-\alpha}} \varphi(t)dt\right)$$

for all $x, y \in X$, $x, y \notin Fix(A)$, *and* $\alpha \in (0, 1)$. *Then* A *has a unique fixed point in* X.

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