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COMMON FIXED POINT THEOREMS IN *C*-COMPLETE COMPLEX VALUED METRIC SPACES FOR RATIONAL CONTRACTION

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Abstract

The aim of this manuscript is to establish common fixed point results satisfying rational type contraction conditions in the setting of *C*-complete complex valued metric space. Also we use weakly compatible maps, weakly compatible along with (CLR) and (E.A.) properties that generalizes the existing results. Also the results proved herein are the generalization and extension of some well known results in the existing literature [7, 8, 9].

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1. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric space which is the generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expressions.

Afterwards several authors have dealt and studied with fixed points and common fixed point theory in complex valued metric space (see for instance [2-6, 9]).

Recently, Sintunavarat et al. [7] introduced the concept of a *C*-Cauchy sequence in *C*-complete complex valued metric space and established the existence of common fixed point theorem in *C*-complete complex valued metric spaces.

In 2016, Kumar et al. [8] proved common fixed point theorems for weakly compatible maps, weakly compatible along with (CLR) and (E.A.) properties in *C*-complete complex valued metric spaces.

Very recently, Dubey et al. [9] established common fixed point theorems for a pair of mappings satisfying rational inequality in *C*-complete complex valued metric spaces.

So, in this paper, we shall establish and prove some common fixed point results for mappings satisfying rational type contractive condition in which the constant has been replaced by control function, in the framework of *C*-complete complex valued metric space. Our results unify, generalize and compliment the comparable results from the current literature [7-8].

2. Preliminary

The following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex number and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$, that is, $z_1 \precsim z_2$ if one of the following hold:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we will write $z_1 \prec z_2$ if only (C_4) is satisfied.

Remark 2.1. We denote that following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz, \forall z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Definition 2.2 [1]**.** Let X be a non empty set and let the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(d_1) $0 \precsim d(x, y)$ for all $x, y \in X$;

(d_2) $d(x, y) = 0$ iff $x = y, \forall x, y \in X$;

(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_4) $d(x, y) \precsim d(x, z) + d(z, y)\ \forall x, y, z \in X$.

Then d is called a complex valued metric on X and the pair (X, d) is called a complex valued metric space.

Example 2.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then (X, d) is a complex valued metric space.

Definition 2.4 [1]**.** Let (X, d) be a complex valued metric space. Then

(1) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$\beta(x, r) = \{y \in X : d(x, y) < r\} \subseteq A.$$

(2) A point $x \in X$ is called a limit point of A whenever for all $0 \prec r \in \mathbb{C}$,

$$B(x, r) \bigcap (A - \{x\}) \neq \emptyset.$$

(3) A set $A \subseteq X$ is called an open set whenever each element of A is an interior point of A.

(4) A set $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

(5) A sub-basis for a Hausdorff topology τ on X is the family

$$F\{B(x, r) : x \in X \text{ and } 0 \prec r\}.$$

Definition 2.5 [1]**.** Let (X, d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and let $x \in X$.

(1) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$. Then $\{x_n\}$ is said to be convergent to a point $x \in X$ and x is the limit point of $\{x_n\}$. We denote this by

$$\lim_{n \to \infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

(2) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called Cauchy sequence in X.

(3) If every Cauchy sequence in X is convergent, then (X, d) is said to be complete complex valued metric space.

Lemma 2.6 [1]**.** *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a* *sequence in* X. *Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n, x)| \to 0$ *as* $n \to \infty$.

Lemma 2.7 [1]**.** *Let* (X, d) *be a complex valued metric space and* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$ *where* $m \in \mathbb{N}$.

Further, Sintunavarat et al. [7], introduced the notation of a *C*-Cauchy sequence in *C*-complete complex valued metric space as follows:

Definition 2.8 [7]**.** Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X as $x \in X$.

(i) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exits $k \in \mathbb{N}$ such that for all $m, n > k$, $|d(x_n, x_{n+m})| \prec c$, then $\{x_n\}$ is called a *C*-Cauchy sequence in X.

(ii) If every *C*-Cauchy sequence in X is convergent, then (X, d) is said to be a *C*-complete complex valued metric space.

And he also proved the following fixed point result:

Theorem 2.9 [7]**.** *Let* S *and* T *be self-mappings of a C-complete complex valued metric space* (X, d). *If there exist mappings* $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ *such that for all* $x, y \in X$:

(a) $\alpha(x) + \beta(x) < 1$,

(b) *The mapping* $\gamma : \mathbb{C}_+ \to [0, 1)$ *defined by*

$$\gamma(x) = \frac{\alpha(x)}{1 - \beta(x)} \in \Gamma.$$

(c) $d(Sx, Ty) \precsim \alpha(d(x, y))d(x, y) + \beta(d(x, y))\dfrac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}$.

Then S *and* T *have a unique common fixed point.*

After that, in 2018, Dubey et al. [9], proved the following fixed point results:

Theorem 2.10 [9]**.** *Let* S *and* T *be self-mappings of a C-complete complex valued metric space* (X, d). *If there exist mappings* $\alpha, \beta, \gamma : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *such that for all* $x, y \in X$:

(i) $\alpha(x) + \beta(x) + \gamma(x) < 1$,

(ii) *The mapping* $\mu : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *defined by*

$$\mu(x, y) = \frac{\alpha(x, y)}{1 - \beta(x, y)} \in \Gamma.$$

(iii) $$d(Sx, Ty) \precsim \alpha(x, y) d(x, y) + \beta(x, y) \frac{d(x, Tx)[1 + d(x, Sx)]}{1 + d(x, y)} + \gamma(x, y) \frac{d(y, Sx)[1 + d(x, Ty)]}{1 + d(x, y)}.$$

Then S *and* T *have a unique common fixed point.*

3. Main Results

Throughout this paper, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{C}_+$ denotes a set $\{c \in \mathbb{C}_+ : 0 \precsim c\}$ and Γ denotes the class of functions $\mu : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ which satisfies the condition:

for (x_n, y_n) in $\mathbb{C}_+ \times \mathbb{C}_+$,

$\mu(x_n, y_n) \to 1 \Rightarrow (x_n, y_n) \to 0.$

We extend and generalize of the results [7, 8, 9] and obtain common fixed point theorem for rational expression in *C*-complete complex valued metric space.

Theorem 3.1. *Let* K *and* L *be self-mappings of a C-complete complex valued metric space* (X, d). *If there exist mappings* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *such that*

$$\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) + \lambda_4(x, y) + \lambda_5(x, y) < 1. \tag{3.1}$$

The mapping $\mu : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *defined by*

$$\mu(x, y) = \frac{\lambda_1(x, y) + \lambda_4(x, y)}{1 - \lambda_2(x, y) - \lambda_4(x, y)} \in \Gamma. \tag{3.2}$$

$$d(Kx, Ly) \precsim \lambda_1(x, y)d(x, y) + \lambda_2(x, y)\frac{d(x, Kx)d(y, Ly)}{1 + d(x, y)}$$

$$+ \lambda_3(x, y)\frac{d(x, Lx)d(y, Kx)}{1 + d(x, y)} + \lambda_4(x, y)\frac{d(x, Kx)d(x, Ky)}{1 + d(x, y)}$$

$$+ \lambda_5(x, y)\frac{d(y, Ly)d(y, Kx)}{1 + d(x, y)}. \tag{3.3}$$

Then K *and* L *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X such that

$$x_{2l+1} = Kx_{2l},$$

$$x_{2l+2} = Lx_{2l+1}, \text{ for all } l > 0. \tag{3.4}$$

For $l \geq 0,$ we get

$$d(x_{2l+1}, x_{2l+2}) = d(Kx_{2l}, Lx_{2l+1})$$

$$\precsim \lambda_1(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+1}) + \lambda_2(x_{2l}, x_{2l+1})\frac{d(x_{2l}, Kx_{2l})d(x_{2l+1}, Lx_{2l+1})}{1 + d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_3(x_{2l}, x_{2l+1})\frac{d(x_{2l}, Lx_{2l+1})d(x_{2l+1}, Kx_{2l})}{1 + d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_4(x_{2l}, x_{2l+1})\frac{d(x_{2l}, Kx_{2l})d(x_{2l}, Lx_{2l+1})}{1 + d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_5(x_{2l}, x_{2l+1})\frac{d(x_{2l+1}, Lx_{2l+1})d(x_{2l+1}, Kx_{2l})}{1 + d(x_{2l}, x_{2l+1})}$$

$$= \lambda_1(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+1}) + \lambda_2(x_{2l}, x_{2l+1})\frac{d(x_{2l}, x_{2l+1})d(x_{2l+1}, x_{2l+2})}{1+d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_3(x_{2l}, x_{2l+1})\frac{d(x_{2l}, x_{2l+2})d(x_{2l+1}, x_{2l+1})}{1+d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_4(x_{2l}, x_{2l+1})\frac{d(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+2})}{1+d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_5(x_{2l}, x_{2l+1})\frac{d(x_{2l+1}, x_{2l+2})d(x_{2l+1}, x_{2l+1})}{1+d(x_{2l}, x_{2l+1})}$$

$$= \lambda_1(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+1}) + \lambda_2(x_{2l}, x_{2l+1})\frac{d(x_{2l}, x_{2l+1})d(x_{2l+1}, x_{2l+2})}{1+d(x_{2l}, x_{2l+1})}$$

$$+ \lambda_4(x_{2l}, x_{2l+1})\frac{d(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+2})}{1+d(x_{2l}, x_{2l+1})}$$

$$\precsim \lambda_1(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+1}) + \lambda_2(x_{2l}, x_{2l+1})d(x_{2l+1}, x_{2l+2})$$

$$+\lambda_4(x_{2l}, x_{2l+1})\{d(x_{2l}, x_{2l+1}) + d(x_{2l+1}, x_{2l+2})\},$$

which implies that

$$d(x_{2l+1}, x_{2l+2}) \precsim \mu(x_{2l}, x_{2l+1})d(x_{2l}, x_{2l+1}), \tag{3.5}$$

where $\mu = \dfrac{\lambda_1(x, y) + \lambda_4(x, y)}{1 - \lambda_2(x, y) - \lambda_4(x, y)}$.

Similarly, for $l \geq 0$, we get

$$d(x_{2l+2}, x_{2l+3}) \precsim \mu(x_{2l+1}, x_{2l+2})d(x_{2l+1}, x_{2l+2}). \tag{3.6}$$

From (3.5) and (3.6), we get

$$d(x_{2l}, x_{2l+1}) \precsim \mu(x_{2l-1}, x_{2l})d(x_{2l-1}, x_{2l}), \text{ for all } l \in \mathbb{N}.$$

Therefore, we get

$$|d(x_{2l}, x_{2l+1})| \leq \mu(x_{2l-1}, x_{2l})|d(x_{2l-1}, x_{2l})|$$

$$\leq |\, d(x_{2l-1},\, x_{2l})\,| \text{ for all } l \in \mathbb{N}. \tag{3.7}$$

$\Rightarrow$ The sequence $\{|\, d(x_{2l-1},\, x_{2l})\,|\}_{l\in\mathbb{N}}$ is a monotone non-increasing and bounded below, therefore,

$$|\, d(x_{2l-1},\, x_{2l})\,| \to \delta, \text{ for some } \delta \geq 0.$$

Next we claim that $\delta = 0.$ Assume contrary that $\delta > 0.$

Proceeding limit as $l \to \infty,$ we have from (3.7), $\mu(x_{2l-1},\, x_{2l}) \to 1.$

Since $\mu \in \Gamma,$ so, we get $(x_{2l-1},\, x_{2l}) \to 0,$ that is, $|\, d(x_{2l-1},\, x_{2l})\,| \to 0,$ which is contradiction. Therefore, we have $\delta = 0,$ i.e.,

$$|\, d(x_{2l-1},\, x_{2l})\,| \to 0. \tag{3.8}$$

Next, we show that $\{x_{2l}\}$ is *C*-Cauchy sequence. According to (3.8), it is sufficient to prove that the subsequence $\{x_{2l}\}$ is a *C*-Cauchy sequence. Let, if possible, $\{x_{2l}\}$ is not a *C*-Cauchy sequence. So, there is $c \in \mathbb{C}$ with $0 \prec c,$ for which for all $k \in \mathbb{N},$ there exists $m(k) > l(k) \geq k,$ such that

$$d\big(x_{2l(k)},\, x_{2m(k)}\big) \precsim c. \tag{3.9}$$

Further, corresponding to $l(k),$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > l(k) \geq k$ satisfying (3.9). Then, we have

$$d\big(x_{2l(k)},\, x_{2m(k)}\big) \precsim c, \tag{3.10}$$

and

$$d\big(x_{2l(k)},\, x_{2m(k)-2}\big) \prec c. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned} c &\precsim d\big(x_{2l(k)},\, x_{2m(k)}\big) \\ &\precsim d\big(x_{2l(k)},\, x_{2m(k)-2}\big) + d\big(x_{2m(k)-2},\, x_{2m(k)-1}\big) + d\big(x_{2m(k)-1},\, x_{2m(k)}\big) \end{aligned}$$

$$\prec c + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).$$

This implies that

$$|c| \leq |d(x_{2l(k)}, x_{2m(k)})|$$

$$\leq |c| + d|(x_{2m(k)-2}, x_{2m(k)-1})| + d|(x_{2m(k)-1}, x_{2m(k)})|.$$

On taking limit $k \to \infty$, we get

$$|d(x_{2l(k)}, x_{2m(k)})| \to |c|. \tag{3.12}$$

Further, we have

$$d(x_{2l(k)}, x_{2m(k)}) \precsim d(x_{2l(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)})$$

$$\precsim d(x_{2l(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1})$$

$$+ d(x_{2m(k)+1}, x_{2m(k)})$$

$$\Rightarrow |d(x_{2l(k)}, x_{2m(k)})| \leq |d(x_{2l(k)}, x_{2m(k)})|$$

$$+ |d(x_{2m(k)}, x_{2m(k)+1})| + |d(x_{2m(k)+1}, x_{2m(k)})|.$$

Letting, $k \to \infty$ and using (3.8) and (3.12), we get

$$|d(x_{2l(k)}, x_{2m(k)+1})| \to |c|. \tag{3.13}$$

Now

$$d(x_{2l(k)}, x_{2m(k)+1})$$

$$\precsim d(x_{2l(k)}, x_{2l(k)+1}) + d(x_{2l(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2m(k)+1})$$

$$= d(x_{2l(k)}, x_{2l(k)+1}) + d(Kx_{2l(k)}, Lx_{2m(k)+1}) + d(x_{2m(k)+2}, x_{2m(k)+1})$$

$$\precsim d(x_{2l(k)}, x_{2l(k)+1}) + \lambda_1(x_{2l(k)}, x_{2m(k)+1})d(x_{2l(k)}, x_{2m(k)+1})$$

$$+ \lambda_2(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, Kx_{2l(k)})d(x_{2m(k)+1}, Lx_{2m(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_3(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, Lx_{2m(k)+1})d(x_{2m(k)+1}, Kx_{2l(k)})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_4(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, Kx_{2l(k)})d(x_{2l(k)}, Lx_{2m(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_5(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2m(k)+1}, Lx_{2m(k)+1})d(x_{2m(k)+1}, Kx_{2l(k)})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ d(x_{2m(k)+2}, x_{2m(k)+1})$$

$$= d(x_{2l(k)}, x_{2l(k)+1}) + \lambda_1(x_{2l(k)}, x_{2m(k)+1})d(x_{2l(k)}, x_{2m(k)+1})$$

$$+ \lambda_2(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_3(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, x_{2m(k)+2})d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_4(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2l(k)}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ \lambda_5(x_{2l(k)}, x_{2m(k)+1})\frac{d(x_{2m(k)+1}, x_{2m(k)+2})d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}$$

$$+ d(x_{2m(k)+2}, x_{2m(k)+1}).$$

Implies that

$$|d(x_{2l(k)}, x_{2m(k)+1})|$$

$$\le |d(x_{2l(k)}, x_{2l(k)+1})| + \lambda_1(x_{2l(k)}, x_{2m(k)+1})|d(x_{2l(k)}, x_{2m(k)+1})|$$

$$+ \lambda_2(x_{2l(k)}, x_{2m(k)+1})\left|\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+ \lambda_3(x_{2l(k)}, x_{2m(k)+1}) \left| \frac{d(x_{2l(k)}, x_{2m(k)+2}) d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \lambda_4(x_{2l(k)}, x_{2m(k)+1}) \left| \frac{d(x_{2l(k)}, x_{2l(k)+1}) d(x_{2l(k)}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \lambda_5(x_{2l(k)}, x_{2m(k)+1}) \left| \frac{d(x_{2m(k)+1}, x_{2m(k)+2}) d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| d(x_{2m(k)+2}, x_{2m(k)+1}) \right|$$

$$\leq \left| d(x_{2l(k)}, x_{2l(k)+1}) \right| + \lambda_1(x_{2l(k)}, x_{2m(k)+1}) \left| d(x_{2l(k)}, x_{2m(k)+1}) \right|$$

$$+ \left| \frac{d(x_{2l(k)}, x_{2l(k)+1}) d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| \frac{d(x_{2l(k)}, x_{2m(k)+2}) d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| \frac{d(x_{2l(k)}, x_{2l(k)+1}) d(x_{2l(k)}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| \frac{d(x_{2m(k)+1}, x_{2m(k)+2}) d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| d(x_{2m(k)+2}, x_{2m(k)+1}) \right|$$

$$\leq \left| d(x_{2l(k)}, x_{2l(k)+1}) \right|$$

$$+ \frac{(\lambda_1 + \lambda_2)(x_{2l(k)}, x_{2m(k)+1})}{1 - (\lambda_2 + \lambda_4)(x_{2l(k)}, x_{2m(k)+1})} \left| d(x_{2l(k)}, x_{2m(k)+1}) \right|$$

$$+ \left| \frac{d(x_{2l(k)}, x_{2l(k)+1}) d(x_{2m(k)+1}, x_{2m(k)+2})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+ \left| \frac{d(x_{2l(k)}, x_{2m(k)+2}) d(x_{2m(k)+1}, x_{2l(k)+1})}{1 + d(x_{2l(k)}, x_{2m(k)+1})} \right|$$

$$+\left|\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2l(k)}, x_{2m(k)+2})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+\left|\frac{d(x_{2m(k)+1}, x_{2m(k)+2})d(x_{2m(k)+1}, x_{2l(k)+1})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|+\left|d(x_{2m(k)+2}, x_{2m(k)+1})\right|$$

$$\leq \left|d(x_{2l(k)}, x_{2l(k)+1})\right|+\mu(x_{2l(k)}, x_{2m(k)+1})\left|d(x_{2l(k)}, x_{2m(k)+1})\right|$$

$$+\left|\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2m(k)+1}, x_{2m(k)+2})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+\left|\frac{d(x_{2l(k)}, x_{2m(k)+2})d(x_{2m(k)+1}, x_{2l(k)+1})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+\left|\frac{d(x_{2l(k)}, x_{2l(k)+1})d(x_{2l(k)}, x_{2m(k)+2})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+\left|\frac{d(x_{2m(k)+1}, x_{2m(k)+2})d(x_{2m(k)+1}, x_{2l(k)+1})}{1+d(x_{2l(k)}, x_{2m(k)+1})}\right|$$

$$+\left|d(x_{2m(k)+2}, x_{2m(k)+1})\right|.$$

Taking limit as $k \to \infty$, we get

$$|c| \leq \lim_{k\to\infty} \mu(x_{2l(k)}, x_{2m(k)+1})|c| \leq |c|$$

$$\Rightarrow \lim_{k\to\infty} \mu(x_{2l(k)}, x_{2m(k)+1}) = 1.$$

Since $\mu \in \Gamma$, we get $(x_{2l(k)}, x_{2m(k)+1}) \to 0$, i.e., $d(x_{2l(k)}, x_{2m(k)+1}) \to 0$ which contradicts $0 \prec c$. Therefore we can conclude that $\{x_{2l}\}$ is C-Cauchy and hence $\{x_l\}$ is C-Cauchy sequence in X and X is complete, so there exists a point u in K such that $x_l \to u$ as $l \to \infty$.

Next we prove that $Ku = u$. If $Ku \neq u$, then $d(Ku, u) > 0$.

Now

$$d(Ku, u) \precsim d(u, x_{2l+2}) + d(x_{2l+2}, Ku)$$

$$= d(u, x_{2l+2}) + d(Lx_{2l+1}, Ku)$$

$$= d(u, x_{2l+2}) + d(Su, Lx_{2l+1})$$

$$\precsim d(x_{2l+2}, u) + \lambda_1(u, x_{2l+1})d(u, x_{2l+1})$$

$$+ \lambda_2(u, x_{2l+1})\frac{d(u, Ku)d(x_{2l+1}, Lx_{2l+1})}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_3(u, x_{2l+1})\frac{d(u, Lx_{2l+1})d(x_{2l+1}, Ku)}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_4(u, x_{2l+1})\frac{d(u, Ku)d(u, Lx_{2l+1})}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_5(u, x_{2l+1})\frac{d(x_{2l+1}, Lx_{2l+1})d(x_{2l+1}, Ku)}{1 + d(u, x_{2l+1})}$$

$$= d(x_{2l+2}, u) + \lambda_1(u, x_{2l+1})d(u, x_{2l+1})$$

$$+ \lambda_2(u, x_{2l+1})\frac{d(u, Ku)d(x_{2l+1}, x_{2l+2})}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_3(u, x_{2l+1})\frac{d(u, x_{2l+2})d(x_{2l+1}, Ku)}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_4(u, x_{2l+1})\frac{d(u, Ku)d(u, x_{2l+2})}{1 + d(u, x_{2l+1})}$$

$$+ \lambda_5(u, x_{2l+1})\frac{d(x_{2l+1}, x_{2l+2})d(x_{2l+1}, Ku)}{1 + d(u, x_{2l+1})}.$$

Letting $l \to \infty$, we get

$$d(u, Ku) \precsim d(u, u) + \lambda_1(u, u)d(u, u) + \lambda_2 d(u, u)\frac{d(u, Ku)d(u, u)}{1 + d(u, u)}$$

$$+ \lambda_3 d(u, u)\frac{d(u, u)d(u, Ku)}{1 + d(u, u)} + \lambda_4 d(u, u)\frac{d(u, Ku)d(u, u)}{1 + d(u, u)}$$

$$+ \lambda_5 d(u, u)\frac{d(u, u)d(u, Ku)}{1 + d(u, u)}$$

$\Rightarrow |\, d(u, Ku)\,| = 0,$ which is a contradiction.

Thus, we get $Ku = u$.

Similarly, we get $Lu = u$.

Therefore, $u = Ku = Lu$, that is, u is common fixed point of K and L.

Finally, we show that, u is the unique common fixed point of K and L. Assume that there exists another point v such that $v = Kv = Lv$.

From (3.1), we have

$$d(u, v) = d(Ku, Lv)$$

$$\precsim \lambda_1(u, v)d(u, v) + \lambda_2 d(u, v)\frac{d(u, Ku)d(v, Lv)}{1 + d(u, v)}$$

$$+ \lambda_3 d(u, v)\frac{d(u, Lv)d(v, Ku)}{1 + d(u, v)} + \lambda_4 d(u, v)\frac{d(u, Ku)d(v, Ku)}{1 + d(u, v)}$$

$$+ \lambda_5 d(u, v)\frac{d(v, Lv)d(v, Ku)}{1 + d(u, v)}$$

$$= \lambda_1(u, v)d(u, v) + \lambda_3 d(u, v)\frac{d(u, Lv)d(v, Ku)}{1 + d(u, v)}$$

$$\precsim [\lambda_1(u, v) + \lambda_3 d(u, v)]d(u, v)$$

$$\Rightarrow |\, d(u, v)\,| \le [\lambda_1(u, v) + \lambda_3 d(u, v)]\, |\, d(u, v)\,|$$

$$\Rightarrow \lambda_1(u, v) + \lambda_3 d(u, v) \ge 1,$$

which is contradiction and hence $u = v$.

Therefore, u is a unique common fixed point of K and L.

Corollary 3.2. *Let S and T be self-mappings of a C-complete complex valued metric space (X, d) satisfying the following*:

$$d(Kx, Ly) \precsim \alpha d(x, y) + \beta \frac{d(x, Kx)d(y, Ly)}{1 + d(x, y)} + \gamma \frac{d(x, Ly)d(y, Kx)}{1 + d(x, y)}$$

$$+ \mu \frac{d(x, Kx)d(x, Ly)}{1 + d(x, y)} + \delta \frac{d(y, Ly)d(y, Kx)}{1 + d(x, y)}. \tag{3.14}$$

Then S and T have a unique common fixed point.

Proof. By putting $\lambda_1(x, y) = \alpha$, $\lambda_2(x, y) = \beta$, $\lambda_3(x, y) = \gamma$, $\lambda_4(x, y) = \mu$, $\lambda_5(x, y) = \delta$ in Theorem 3.1, we get the required result.

Corollary 3.3. *Let K be self-mapping of a C-complete complex valued metric space (X, d), if there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ satisfying* (3.1), (3.2) *and the following*:

$$d(Kx, Ky) \precsim \lambda_1(x, y)d(x, y) + \lambda_2 \frac{d(x, Kx)d(y, Ky)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ky)d(y, Kx)}{1 + d(x, y)}$$

$$+ \lambda_4 \frac{d(x, Kx)d(x, Ky)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ky)d(y, Kx)}{1 + d(x, y)}. \tag{3.15}$$

Then K has a unique fixed point in X.

Proof. By putting $K = L$ in Theorem 3.1, we get the required result.

Corollary 3.4. *Let K be self-mapping of a C-complete complex valued metric space (X, d) satisfying the following*:

$$d(Kx, Ky) \precsim \alpha d(x, y) + \beta \frac{d(x, Kx)d(y, Ky)}{1 + d(x, y)} + \gamma \frac{d(x, Ky)d(y, Kx)}{1 + d(x, y)}$$

$$+ \mu \frac{d(x, Kx)d(x, Ky)}{1 + d(x, y)} + \delta \frac{d(y, Ky)d(y, Kx)}{1 + d(x, y)} \tag{3.16}$$

for all $x, y \in X$. Then K has a unique fixed point in X.

Proof. By putting $\lambda_1(x, y) = \alpha$, $\lambda_2(x, y) = \beta$, $\lambda_3(x, y) = \lambda$, $\lambda_4(x, y) = \mu$, $\lambda_5(x, y) = \delta$ in Corollary 3.3, we get the required result.

Theorem 3.5. *Let K be a self-map of a C-complete complex valued metric space (X, d), if there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ satisfying* (3.1), (3.2) *and the following*:

$$d(K^n x, K^n y) \precsim \lambda_1(x, y)d(x, y) + \lambda_2 \frac{d(x, K^n x)d(y, K^n y)}{1 + d(x, y)} + \lambda_3 \frac{d(x, K^n y)d(y, K^n x)}{1 + d(x, y)} + \lambda_4 \frac{d(x, K^n x)d(x, K^n y)}{1 + d(x, y)} + \lambda_5 \frac{d(y, K^n y)d(y, K^n x)}{1 + d(x, y)}, \tag{3.17}$$

for all $x, y \in X$ and some $n \in \mathbb{N}$. Then K has a unique fixed point in X.

Proof. From Corollary 3.3, K^n has a fixed point u. But K^n has a fixed point Ku, since $K^n(Ku) = K(K^n u) = Ku$. Therefore, $Ku = u$ by the uniqueness of a fixed point K^n. Therefore, u is also a fixed point of K. Since the fixed point of K is also a fixed point of K^n, the fixed point of K is also unique.

Corollary 3.6. *Let K be self-mapping of a C-complete complex valued metric space (X, d) satisfying the following*:

$$d(K^n x, K^n y) \precsim \alpha d(x, y) + \beta \frac{d(x, K^n x)d(y, K^n y)}{1 + d(x, y)} + \gamma \frac{d(x, K^n y)d(y, K^n x)}{1 + d(x, y)} + \mu \frac{d(x, K^n x)d(x, K^n y)}{1 + d(x, y)} + \delta \frac{d(y, K^n y)d(y, K^n x)}{1 + d(x, y)}, \tag{3.18}$$

where $\alpha, \beta, \gamma, \mu, \delta$ are non-negative reals with $\alpha + \beta + \gamma + \mu + \delta < 1$. Then K has a unique fixed in X.

Proof. By putting $\lambda_1(x, y) = \alpha$, $\lambda_2(x, y) = \beta$, $\lambda_3(x, y) = \gamma$, $\lambda_4(x, y) = \mu$,

$\lambda_5(x, y) = \delta$ in Theorem 3.5, we get the required result.

Now, we deduce the main results of [7] as follows.

Theorem 3.7 [7, Theorem 3.2]**.** *Let (X, d) be a C-complete complex valued metric space. If there exist two mappings $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X$*

(a) $\alpha(x) + \beta(x) < 1$;

(b) *The mapping $\gamma : \mathbb{C}_+ \to [0, 1)$ defined by*

$$\gamma(x) = \frac{\alpha(x)}{1 - \beta(x)} \in \Gamma;$$

(c)

$$d(Sx, Ty) \precsim \alpha(d(x, y))d(x, y) + \beta \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}. \tag{3.19}$$

Then S and T have a unique common fixed point in X.

Proof. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ by $\lambda_1(x, y) = \alpha(d(x, y))$, $\lambda_2(x, y) = \beta(d(x, y))$, $\lambda_3(x, y) = 0$, $\lambda_4(x, y) = 0$, $\lambda_5(x, y) = 0$ and $K = S$, $L = T$ in Theorem 3.1, we get the required result.

Corollary 3.8 [7, Corollary 3.3]**.** *Let S and T be self-mappings of a C-complete complex valued metric space (X, d) satisfying the following*:

$$d(Sx, Ty) \precsim \lambda d(x, y) + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}, \tag{3.20}$$

for all $x, y \in X$, where λ, μ are non-negative reals with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof. By putting $\alpha(d(x, y)) = \lambda$ and $\beta(d(x, y)) = \mu$ in Theorem 3.7, we get the required result.

Corollary 3.9 [7, Corollary 3.4]**.** *Let* (X, d) *be a C-complete complex valued metric space and* $T : X \to X$ *be a mapping. If there exist two mappings* $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$, *such that for all* $x, y \in X$

(a) $\alpha(x) + \beta(x) < 1;$

(b) *The mapping* $\gamma : \mathbb{C}_+ \to [0, 1)$ *defined by*

$$\gamma(x) = \frac{\alpha(x)}{1 - \beta(x)} \in \Gamma,$$

(c)

$$d(Tx, Ty) \precsim \alpha(d(x, y))d(x, y) + \beta \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}. \tag{3.21}$$

Then T *has a unique common fixed point in* X.

Proof. By putting $S = T$ in Theorem 3.7, we get the required result.

Corollary 3.10 [7, Corollary 3.5]**.** *Let* (X, d) *be a C-complete complex valued metric space and* $T : X \to X$ *be a mapping. If* T *satisfies*

$$d(Tx, Ty) \precsim \lambda d(x, y) + \mu \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \tag{3.22}$$

for all $x, y \in X$, *where* λ, μ *are non-negative reals with* $\lambda + \mu < 1$, *then* T *has a unique fixed point in* X.

Proof. By putting $\alpha d(x, y) = \lambda$ and $\beta d(x, y) = \mu$ in Corollary 3.9, we get the required result.

Theorem 3.11 [7, Corollary 3.6]**.** *Let* (X, d) *be a C-complete complex valued metric space and* $T : X \to X$ *be a mapping. If there exist two mappings* $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$, *such that, for all* $x, y \in X$

(a) $\alpha(x) + \beta(x) < 1;$

(b) *The mappings* $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ *defined by*

$$\gamma(x) = \frac{\alpha(x)}{1 - \beta(x)} \in \Gamma;$$

(c)

$$d(T^n x, T^n y) \precsim \alpha(d(x, y))d(x, y) + \beta \frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)} \tag{3.23}$$

for some $n \in \mathbb{N}$. *Then* T *has a unique common fixed point in* X.

Proof. From Corollary 3.9, we get T^n has a fixed point u. Since $T^n(Tu) = T(T^n u) = Tu$, we get Tu is a fixed point of T^n. Therefore, $Tu = u$ by the uniqueness of a fixed point T^n. Since the fixed point of T is also fixed point of T^n, then we get that fixed point of T is also unique.

Corollary 3.12 [7, Corollary 3.7]**.** *Let* T *be a self-mapping of a C-complete complex valued metric space* (X, d) *satisfying the following*:

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \mu \frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)}, \tag{3.24}$$

for all $x, y \in X$ *and* $n \in \mathbb{N}$ *where* λ, μ *are non-negative reals with* $\lambda + \mu < 1$, *then* T *has a unique fixed point in* X.

Proof. By putting $\alpha d(x, y) = \lambda$ and $\beta d(x, y) = \mu$ in Theorem (3.11), we get the required result.

4. Weakly Compatible Maps

In 1996, Jungck [10] introduced the concept of weakly compatible maps as follows:

Definition 4.1. Two self-maps f and g are said to be weakly compatible if they commute at coincidence points.

Lemma 4.2 [11]**.** *Let* $T : X \to X$ *be a function, then there exists a subset* $E \subseteq X$ *such that* $T(E) = T(X)$ *and* $T : E \to X$ *is one to one.*

Theorem 4.3. *Let* K *and* L *be self-mappings of a complex valued metric space* (X, d) *such that* $L(\alpha) \subseteq K(X)$ *and* $K(X)$ *is C-complete. If there exit mappings* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *satisfying* (3.1), (3.2) *and the following*:

$$d(Lx, Ly) \precsim \lambda_1(Kx, Ky)d(Kx, Ky) + \lambda_2(Kx, Ky)\frac{d(Kx, Lx)d(Ky, Ly)}{1 + d(Kx, Ky)}$$

$$+ \lambda_3(Kx, Ky)\frac{d(Kx, Ly)d(Ky, Lx)}{1 + d(Kx, Ky)}$$

$$+ \lambda_4(Kx, Ky)\frac{d(Kx, Lx)d(Kx, Ly)}{1 + d(Kx, Ky)}$$

$$+ \lambda_5(Kx, Ky)\frac{d(Ky, Ly)d(Ky, Lx)}{1 + d(Kx, Ky)}, \tag{4.1}$$

for all $x, y \in X$. *Then* K *and* L *have a unique fixed point of coincidence in* X. *Moreover, if* K *and* L *are compatible, then* K *and* L *have a unique common fixed point.*

Proof. Consider the mapping $K : X \to X$. By Lemma 4.2, there exists $E \subset X$ such that $K(E) = K(X)$ and $K : E \to X$ is one to one.

Next we define a mapping $\Theta : K(E) \to K(E)$ by $\Theta(Kx) = Lx$, for all $Kx \in K(E)$. Therefore, Θ is well defined, since K is one to one on L. Since $\Theta \circ K = L$, using (4.3) we get

$$d(\Theta Kx, \Theta Ky) \precsim \lambda_1(Kx, Ky)d(Kx, Ky) + \lambda_2(Kx, Ky)\frac{d(Kx, \Theta Kx)d(Ky, \Theta Ky)}{1 + d(Kx, Ky)}$$

$$+ \lambda_3(Kx, Ky)\frac{d(Kx, \Theta Ky)d(Ky, \Theta Kx)}{1 + d(Kx, Ky)}$$

$$+ \lambda_4(Kx, Ky)\frac{d(Kx, \Theta Kx)d(Kx, \Theta Ky)}{1 + d(Kx, Ky)}$$

$$+\lambda_5(Kx, Ky)\frac{d(Ky, \Theta Ky)d(Ky, \Theta Kx)}{1+d(Kx, Ky)} \tag{4.2}$$

for all $Kx, Ky \in K(E)$. Since $K(E) = K(\alpha)$ is C-complete and (4.2) holds, we can apply Corollary 3.3, with a mapping Θ. Therefore, there exists a unique fixed point u in $K(X)$ such that $\Theta u = u$.

Next since $u \in K(X)$, there exists a point v in X such that $u = Kv$, $\Theta(Kv) = Kv$, that is $Lv = Kv$.

Therefore, K and L have a unique point of coincidence.

Next, we show that K and L have a common fixed point. Now, we have $u = Lv = Kv$. Since K and L are weakly compatible, we get $Ku = KLv = Lv$. This implies that, u is a point of coincidence of K and L.

Finally, we prove the uniqueness of common fixed point of K and L. Assume that, w is another common fixed point of K and L. So, $w = Kw = Lw$ and w is also a point of coincidence of K and L.

However, we know that, u is a unique point of coincidence of K and L. Therefore, we get $w = u$, that is, u is a unique common fixed point of K and L.

5. Weakly Compatible and (CLR) Property

In 2011, Sintunavarat et al. [12], introduced the notion of (CLR) property as follows:

Definition 5.1. Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_f) property if there exists a sequence $\{\alpha_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx,$$

for some $x \in X$.

In a similar mode, we use this property in complex valued metric space.

Theorem 5.2. *Let* K *and* L *be self-mappings of a complex valued metric space* (X, d) *such that*

(a) K *and* L *are weakly compatible*;

(b) K *and* L *satisfy* (CLR_f) *property*,

$$d(Lx, Ly) \precsim \lambda_1(Kx, Ky)d(Kx, Ky) + \lambda_2(Kx, Ky)\frac{d(Kx, Lx)d(Ky, Ly)}{1 + d(Kx, Ky)}$$

$$+ \lambda_3(Kx, Ky)\frac{d(Kx, Ly)d(Ky, Lx)}{1 + d(Kx, Ky)}$$

$$+ \lambda_4(Kx, Ky)\frac{d(Kx, Ly)d(Ky, Lx)}{1 + d(Kx, Ky)}$$

$$+ \lambda_5(Kx, Ky)\frac{d(Ky, Ly)d(Ky, Lx)}{1 + d(Kx, Ky)}, \tag{5.1}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : \mathbb{C}_+ \times \mathbb{C}_+ \to [0, 1)$ *be the mappings satisfying* (3.1). *Then* K *and* L *have a unique common fixed point*.

Proof. Since K and L satisfy the (CLR_f) property, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = Kx,$$

for $x \in X$.

We claim that

$$Kx = Lx.$$

From (5.1), we have

$$d(Lx_n, Lx) \precsim \lambda_1(Kx_n, Kx)d(Kx_n, Kx) + \lambda_2(Kx_n, Kx)\frac{d(Kx_n, Lx_n)d(Kx, Lx)}{1 + d(Kx_n, Kx)}$$

$$+ \lambda_3(Kx_n, Kx)\frac{d(Kx_n, Lx)d(Kx, Lx_n)}{1 + d(Kx_n, Kx)}$$

$$+ \lambda_4(Kx, Kx) \frac{d(Kx_n, Lx) d(Kx_n, Lx)}{1 + d(Kx_n, Kx)}$$

$$+ \lambda_5(Kx, Kx) \frac{d(Kx, Lx) d(Kx, Lx_n)}{1 + d(Kx_n, Kx)}. \tag{5.2}$$

Letting $L \to \infty$, we get $d(Kx, Lx) \precsim 0$, which implies that $|d(Kx, Lx)| \leq 0$, that is $Kx = Lx$.

Let $u = Kx = Lx$. Since K and L are weakly compatible mappings, $KLx = LKx$, implies that

$$Ku = KLu = LKu = Lu.$$

Now, we claim that, $Lu = u$. Let, if possible, $Lu \neq u$. From (5.1), we have

$$d(Lu, u) = d(Lu, Lx)$$

$$\precsim \lambda_1(Ku, Kx) d(Ku, Kx) + \lambda_2(Ku, Kx) \frac{d(Ku, Lu) d(Kx, Lx)}{1 + d(Ku, Kx)}$$

$$+ \lambda_3(Ku, Kx) \frac{d(Ku, Lx) d(Kx, Lu)}{1 + d(Ku, Kx)} + \lambda_4(Ku, Kx) \frac{d(Ku, Lx) d(Ku, Lx)}{1 + d(Ku, Kx)}$$

$$+ \lambda_5(Ku, Kx) \frac{d(Kx, Lx) d(Kx, Lu)}{1 + d(Ku, Kx)}$$

$$= \lambda_1(Lu, u) d(Lu, u) + \lambda_3(Lu, u) \frac{d(Lu, u) d(u, Lu)}{1 + d(Lu, u)}$$

$$\precsim \lambda_1(Lu, u) d(Lu, u) + \lambda_3(Lu, u) d(Lu, u)$$

$\Rightarrow |d(Lu, u)| \leq [\lambda_1(Lu, u) + \lambda_3(Lu, u)] |d(Lu, u)|$, a contradiction.

Therefore $Lu = u = Ku$. So, u is the common fixed point of K and L.

For the uniqueness, let v be another common fixed point of K and L such that $v \neq u$.

From (5.1)

$$d(u, v) = d(Lu, Lv)$$

$$\precsim \lambda_1(Ku, Kv)d(Ku, Kv) + \lambda_2(Ku, Kv)\frac{d(Ku, Lu)d(Kv, Lv)}{1 + d(Ku, Kv)}$$

$$+ \lambda_3(Ku, Kv)\frac{d(Ku, Lv)d(Kv, Lu)}{1 + d(Ku, Kv)} + \lambda_4(Ku, Kv)\frac{d(Ku, Lu)d(Kv, Lu)}{1 + d(Ku, Kv)}$$

$$+ \lambda_5(Ku, Kv)\frac{d(Kv, Lv)d(Kv, Lu)}{1 + d(Ku, Kv)}$$

$$= \lambda_1(Lu, u)d(u, v) + \lambda_3(u, v)\frac{d(u, v)d(v, u)}{1 + d(u, v)}$$

$$\precsim \lambda_1(u, v)d(u, v) + \lambda_3(u, v)d(u, v)$$

$\Rightarrow | d(u, v) | \le [\lambda_1(u, v) + \lambda_3(u, v)] | d(u, v) |,$ a contradiction.

Hence $u = v$. Therefore, K and L have a unique common fixed point.

6. Weakly Compatible and E.A. Property

In 2002, Aamri et al. [13], introduced the notion of weakly compatible maps as follows:

Definition 6.1. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Theorem 6.2. *Let K and L be such mappings of a complex valued metric space (X, d), satisfying condition* (b) *of Theorem* 5.2 *and* (5.1) *and the following K and L satisfy the E.A. property*

$$L(\alpha) \subseteq K(X). \tag{6.1}$$

In the range of K or L is a C-complete subspace of X, then K and L have a *unique common fixed point in* X.

Proof. Since K and L satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = u,$$

for some $u \in X$.

Since $L(X) \subseteq K(X)$, there exists a sequence $\{y_n\}$ in X such that

$$Kx_n = Ky_n.$$

Hence $\lim_{n\to\infty} Ky_n = u$.

Now, we shall show that

$$Ly_n = u.$$

Now, let $\lim_{n\to\infty} Ly_n = t$.

From (5.1), we have

$$\begin{aligned} d(Lx_n, Ly_n) \precsim\ & \lambda_1(Kx_n, Ky_n)d(Kx_n, Ky_n) \\ & + \lambda_2(Kx_n, Ky_n)\frac{d(Kx_n, Lx_n)d(Ky_n, Ly_n)}{1 + d(Kx_n, Ky_n)} \\ & + \lambda_3(Kx_n, Ky_n)\frac{d(Kx_n, Ly_n)d(Ky_n, Lx_n)}{1 + d(Kx_n, Ky_n)} \\ & + \lambda_4(Kx_n, Ky_n)\frac{d(Kx_n, Ly_n)d(Ky_n, Lx_n)}{1 + d(Kx_n, Ky_n)} \\ & + \lambda_5(Kx_n, Ky_n)\frac{d(Ky_n, Ly_n)d(Ky_n, Lx_n)}{1 + d(Kx_n, Ky_n)}. \end{aligned}$$

Letting $n \to \infty$, we get

$$
\begin{aligned}
d(u, t) &\precsim \lambda_1(u, u)d(u, u) + \lambda_2(u, u)\frac{d(u, u)d(u, t)}{1 + d(u, u)} \\
&\quad + \lambda_3(u, u)\frac{d(u, t)d(u, u)}{1 + d(u, u)} + \lambda_4(u, u)\frac{d(u, u)d(u, t)}{1 + d(u, u)} \\
&\quad + \lambda_5(u, u)\frac{d(u, t)d(u, u)}{1 + d(u, u)} \\
&= 0 \\
&\Rightarrow |d(u, t)| \le 0 \\
&\Rightarrow u = t.
\end{aligned}
$$

Hence $\lim_{n\to\infty} Ly_n = u$.

Now, suppose that $K(X)$ is C-complete subspace of X. Then, there exists $u \in X$ such that

$$u = Ku.$$

Subsequently, we have

$$
\begin{aligned}
\lim_{n\to\infty} Kx_n &= \lim_{n\to\infty} Lx_n \\
&= \lim_{n\to\infty} Ky_n \\
&= \lim_{n\to\infty} Ly_n \\
&= u \\
&= Ku.
\end{aligned}
$$

Now, we will show that

$$Ku = Lu.$$

From (5.1), we have

$$d(Lx_n, Lu) \precsim \lambda_1(Kx_n, Ku)d(Kx_n, Ku)$$

$$+ \lambda_2(Kx_n, Ku)\frac{d(Kx_n, Lx_n)d(Ku, Lu)}{1 + d(Kx_n, Ku)}$$

$$+ \lambda_3(Kx_n, Ku)\frac{d(Kx_n, Lu)d(Ku, Lx_n)}{1 + d(Kx_n, Ku)}$$

$$+ \lambda_4(Kx_n, Ku)\frac{d(Kx_n, Lu)d(Ku, Lx_n)}{1 + d(Kx_n, Ku)}$$

$$+ \lambda_5(Kx_n, Ku)\frac{d(Ku, Lu)d(Ku, Lx_n)}{1 + d(Kx_n, Ku)}.$$

Letting $n \to \infty$, we get

$$d(Ku, Lu) = 0$$

$$\Rightarrow d(Ku, Lu) \leq 0$$

$$\Rightarrow Ku = Lu = u.$$

Since, K and L be weakly compatible so, $KLu = LKu$

$$\Rightarrow KLu = LKu$$

$$= LLu$$

$$= LKu.$$

Now, we claim that Lu is the common fixed point of K and L.

Let, if possible, $Lu \neq LLu$.

From (5.1) we have

$$d(Lu, LLu) \precsim \lambda_1(Ku, KLu)d(Ku, KLu)$$

$$+ \lambda_2(Ku, KLu)\frac{d(Ku, Lu)d(KLu, LLu)}{1 + d(Ku, KLu)}$$

$$+\lambda_3(Ku, KLu)\frac{d(Ku, KLu)d(KLu, Lu)}{1+d(Ku, KLu)}$$

$$+\lambda_4(Ku, KLu)\frac{d(Ku, Lu)d(Ku, LLu)}{1+d(Ku, KLu)}$$

$$+\lambda_5(Ku, KLu)\frac{d(KLu, LLu)d(KLu, Lu)}{1+d(Ku, KLu)}$$

$$=\lambda_1(Ku, KLu)d(Ku, KLu)+\lambda_3(Lu, LLu)\frac{d(Lu, LLu)d(LLu, Lu)}{1+d(Lu, LLu)}$$

$$\precsim |\lambda_1(Ku, KLu)+\lambda_3(Lu, LLu)|d(Ku, KLu)$$

$$\Rightarrow |d(Lu, LLu)| \leq [\lambda_1(Ku, KLu)+\lambda_3(Lu, LLu)]| d(Lu, LLu)|,$$

which is a contradiction. Hence $Lu = LLu = KLu$.

Therefore, Lu is the common fixed point of K and L.

For the uniqueness, Let u and v be any two common fixed point of K and L such that $u \neq v$.

$$d(u, v) = d(Lu, Lv)$$

$$\precsim \lambda_1(Ku, Kv)d(Ku, Kv)+\lambda_2(Ku, Kv)\frac{d(Ku, Lv)d(Kv, Lv)}{1+d(Ku, Kv)}$$

$$+\lambda_3(Ku, Kv)\frac{d(Ku, Kv)d(Kv, Lu)}{1+d(Ku, Kv)}$$

$$+\lambda_4(Ku, Kv)\frac{d(Ku, Lu)d(Ku, Lv)}{1+d(Ku, Kv)}$$

$$+\lambda_5(Ku, Kv)\frac{d(Kv, Lv)d(Kv, Lu)}{1+d(Ku, Kv)}$$

$$=\lambda_1(u, v)d(u, v)+\lambda_3(u, v)\frac{d(u, v)d(v, u)}{1+d(u, v)}$$

$$\precsim \lambda_1(u, v)d(u, v)+\lambda_3(u, v)d(u, v)$$

$$\Rightarrow |d(u, v)| \leq [\lambda_1(u, v) + \lambda_3(u, v)] |d(u, v)|,$$

which is a contradiction. Hence $u = v$.

Therefore, K and L have a unique common fixed point.

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