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COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR THREE SELF-MAPPINGS WITH A NEW TYPE OF HARDY-ROGERS CONTRACTIVE CONDITION IN CONE HEPTAGONAL METRIC SPACE

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Abstract

In this paper, we generalize, extend, and unify some results due to Rangamma and Prudhvi [1], Huang and Zhang [2], and Vetro [3] with a new type of contractive condition related to the Hardy-Rogers contraction [4] by using a function which is defined from a Banach space in to another Banach space [5]. Consequently, we also generalize and extend some results due to Malhotra et al. [5].

1. Introduction

Let (X, d) be a metric space. Recall Hardy and Rogers [5] that a map $T : X \mapsto X$ is called a Hardy-Rogers contraction if

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$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty)$$
$$+cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

for all $x, y \in X$, where $a, b, c, e, f \geq 0$ and satisfy $a + b + c + e + f < 1$. Now we introduce the following:

Definition 1.1. Let (X, d) be a metric space, and $T, g, h : X \mapsto X$. We will say the pair (T, g) is a Hardy-Rogers contraction with respect to h if

$$d(Tx, gy) \leq ad(hx, Tx) + bd(hy, gy)$$
$$+cd(hx, gy) + ed(hy, Tx) + fd(hx, hy)$$

for all $x, y \in X$, where $a, b, c, e, f \geq 0$ and satisfy $a + b + c + e + f < 1$.

Remark 1.2. If $g = T$, then we say T is a Hardy-Rogers contraction with respect to h. If $g = T$ and h is the identity, then T is simply a Hardy-Rogers contraction.

This paper is organized as follows. Section 2 gives some preliminary ideas that would be useful in the sequel. Using Definition 2.14, we re-define Definition 1.1 as Definition 2.21. The main result, Theorem 3.1, is presented in Section 3. Theorem 3.1 unifies many similar results in the literature. In particular, Theorem 3.1 generalizes Theorem 2.1 of Malhotra et al. [5] and the results therein.

2. Preliminaries

Definition 2.1. Let E be a real Banach space and P be a subset of E. The set P is called a cone if:

(a) P is closed, nonempty and $P \neq \{0_E\}$, where 0_E is the zero vector of E.

(b) $a, b \in \mathbb{R}, a, b \geq 0; x, y \in P$ implies $ax + by \in P$.

(c) $x \in P$ and $-x \in P$ implies $x = 0_E$.

Remark 2.2. Given a cone $P \subset E$, we define a partial ordering "$\preceq$" with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate $x \preceq y$ but $x \neq y$. While $x \ll y$ iff $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of P.

Definition 2.3. Let P be a cone in a real Banach space E, then P is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$; $0_E \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least number K, for which $\|x\| \leq K\|y\|$ holds is called the normal constant of P.

Definition 2.4. P is called solid if $\text{int}(P) \neq \varnothing$, that is, interior of P is nonempty.

Definition 2.5. Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ iff $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Remark 2.6. If we replace (c) of the previous definition with the following, which we call the heptagonal property, $d(x, y) \preceq d(x, z) + d(z, w) + d(w, u)$ $d(u, v) + d(v, t) + d(t, y)$ for all $x, y, z, w, u, v, t \in X$ and for all distinct points $z, w, u, v, t \in X - \{x, y\}$, then we say d is a cone heptagonal metric on X, and we call (X, d) a cone heptagonal metric space.

Remark 2.7. A metric space is a cone metric space with $E + \mathbb{R}$ and $P = [0, +\infty)$.

Remark 2.8. If (X, d) is a cone metric space (respectively, cone heptagonal metric space) and the underlying cone is normal, then we will say (X, d) is a normal cone metric space (respectively, normal cone heptagonal metric space).

In the sequel, we will need the following from Jungck et al. [6]:

Remark 2.9. Let P be a cone in a real Banach space E with zero vector 0_E and $a, b, c \in P$, then

(a) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(b) If $a \ll b$ and $b \ll c$, then $a \ll c$.

(c) If $0_E \preceq u \ll c$ for each $c \in \text{int}(P)$, then $u = 0_E$.

(d) If $c \in \text{int}(P)$ and $a_n \to 0_E$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll c$.

(e) If $0_E \preceq a_n \preceq b_n$ for each n and $a_n \to a$, $b_n \to b$, then $a \preceq b$.

(f) If $a \preceq \lambda a$ where $0 \leq \lambda < 1$, then $a = 0_E$.

Definition 2.10. Let (X, d) be a cone heptagonal metric space, and $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We sometimes write $\lim_{n\to\infty} x_n = x$.

Definition 2.11. Let (X, d) be a cone heptagonal metric space, and $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is said to be a Cauchy sequence in X.

Definition 2.12. Let (X, d) be a cone heptagonal metric space. If every Cauchy sequence in X converges to a point in X, then X is called a complete cone heptagonal metric space

Taking inspiration from Huang and Zhang [2], we have the following:

Lemma 2.13. *Let (X, d) be a cone heptagonal metric space, P be a normal* *cone with normal constant K. Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in X*, then

(a) $\{x_n\}$ *is a Cauchy sequence iff* $\lim_{n,m\to\infty} d(x_n, x_m) = 0_E$.

(b) *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *then* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

In the sequel, we will need the following class of functions from Malhotra et al. [5]:

Definition 2.14. Let E and B be two real Banach spaces, P and C be normal cones in E and B, respectively. Let "$\preceq$" and "$\leq$" be partial orderings induced by P and C in E and B, respectively. $\Phi(P, C)$ will denote the set of all functions $\phi : P \mapsto C$ satisfying the following

(a) if $a, b \in P$ with $a \preceq b$, then $\phi[a] \leq k\phi[b]$ for some positive real k.

(b) $\phi[a + b] \leq \phi[a] + \phi[b]$ for all $a, b \in P$.

(c) ϕ is sequentially continuous, that is, if $a_n, a \in P$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} \phi[a_n] = \phi[a]$.

(d) if $\phi[a_n] \to 0_B$, then $a_n \to 0_E$, where 0_E and 0_B are the zero vectors of E and B, respectively.

Remark 2.15. Some elements of $\Phi(P, C)$ are given by Examples 1, 2 and 3, respectively, contained in Malhotra et al. [5].

Remark 2.16. $\phi[a] = 0_B$ iff $a = 0_E$.

Remark 2.17. Let (X, d) be a cone heptagonal metric space with normal cone P and $\phi \in \Phi(P, C)$. Since $d(x, y) \preceq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, t) + d(t, y)$ for all $x, y, z, w, u, v, t \in X$ and for all distinct points $z, w, u, v, t \in X - \{x, y\}$, then $\phi[d(x, y)] \preceq k\phi[d(x, z)] + k\phi[d(z, w)] + k\phi[d(w, u)] + k\phi[d(u, v)] + k\phi[d(v, t)] + k\phi[d(t, y)]$.

Remark 2.18. In the sequel, we will always suppose that E, B are two real Banach spaces, P and C are normal cones in E and B, respectively and "$\preceq$" and "$\leq$" are partial orderings in E and B with respect to P and C, respectively.

Definition 2.19. Let X be a nonempty set and f, g be self-maps on X and $x, z \in X$. Then x is called coincidence point of pair (f, g) if $fx = gx$, and z is called point of coincidence of pair (f, g) if $fx = gx = z$.

Definition 2.20. Let X be a nonempty set and f, g be self-maps on X. Pair (f, g) is called weakly compatible if f and g commute at their coincidence point, that is, $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

Definition 2.21. Let (X, d) be a cone metric space; $T, g, h : X \mapsto X$, and P be a normal cone with normal constant K. We will say the pair (T, g) is a ϕ-Hardy-Rogers contraction with respect to h if

$$\phi[d(Tx, gy)] \leq a\phi[d(hx, Tx)] + b\phi[d(hy, gy)] + c\phi[d(hx, gy)]$$

$$+e\phi[d(hy, Tx)] + f\phi[d(hx, hy)]$$

for all $x, y \in X$; $a, b, c, e, f \geq 0$; $a + b + c + e + f < 1$, and $\phi \in \Phi(P, C)$.

Remark 2.22. In the definition immediately above if we take $E = B$, $P = C$, and defining $\phi : P \mapsto P$ by $\phi[a] = a$ for all $a \in P$, then we recover Definition 1.1.

3. Main Results

Theorem 3.1. *Let (X, d) be a cone heptagonal metric space and P be a normal cone with normal constant K. Suppose $T, g, h : X \mapsto X$ and the pair (T, g) is a ϕ-Hardy-Rogers contraction with respect to h. If $T(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps T, g, h have a unique point of coincidence in X. Moreover, if (T, h) and (g, h) are weakly compatible pairs, then T, g, h have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of *X*. Since $T(X) \cup g(X) \subset h(X)$ starting with x_0 we define a sequence $\{y_n\}$ such that $y_{2n} = Tx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$ for all $n \geq 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence in *X*. If $y_n = y_{n+1}$ for some *n*, say $y_{2n} = y_{2n+1}$, then since the pair (T, g) is a ϕ-Hardy-Rogers contraction with respect to *h* one obtains

$$\phi[d(y_{2n+2}, y_{2n+1})] = \phi[d(Tx_{2n+2}, gx_{2n+1})]$$

$$\leq a\phi[d(hx_{2n+2}, Tx_{2n+2})] + b\phi[d(hx_{2n+1}, gx_{2n+1})] + c\phi[d(hx_{2n+2}, gx_{2n+1})]$$

$$+e\phi[d(hx_{2n+1}, Tx_{2n+2})] + f\phi[d(hx_{2n+2}, hx_{2n+1})]$$

$$= a\phi[d(y_{2n+1}, y_{2n+2})] + b\phi[d(y_{2n}, y_{2n+1})] + c\phi[d(y_{2n+1}, y_{2n+1})]$$

$$+e\phi[d(y_{2n}, y_{2n+2})] + f\phi[d(y_{2n+1}, y_{2n})].$$

Since $y_{2n} = y_{2n+1}$, from the above we deduce that

$$\phi[d(y_{2n+2}, y_{2n+1})] \leq a\phi[d(y_{2n+1}, y_{2n+2})] + e\phi[d(y_{2n}, y_{2n+2})].$$

By Definition 2.14 and since $y_{2n} = y_{2n+1}$, one deduces from the above that

$$\phi[d(y_{2n+2}, y_{2n+1})] \leq (a + e)\phi[d(y_{2n+1}, y_{2n+2})].$$

Since $a + e < 1$, it follows from Remark 2.9 that one has $\phi[d(y_{2n+2}, y_{2n+1})] = 0_B$, and since $\phi \in \Phi(P, C)$, we have $[d(y_{2n+2}, y_{2n+1})] = 0_E$, that is, $y_{2n+2} = y_{2n+1}$. Similalry, one obtains

$$y_{2n} = y_{2n+1} = \cdots = v \text{ (say)}.$$

It follows that $\{y_n\}$ is a Cauchy sequence. Suppose $y_n \neq y_{n+1}$ for all *n*

$$\phi[d(y_{2n}, y_{2n+1})] = \phi[d(Tx_{2n}, gx_{2n+1})]$$

$$\leq a\phi[d(hx_{2n}, Tx_{2n})] + b\phi[d(hx_{2n+1}, gx_{2n+1})] + c\phi[d(hx_{2n}, gx_{2n+1})]$$

$$+e\phi[d(hx_{2n+1}, Tx_{2n})] + f\phi[d(hx_{2n}, hx_{2n+1})]$$

$$= a\phi[d(y_{2n-1}, y_{2n})] + b\phi[d(y_{2n}, y_{2n+1})] + c\phi[d(y_{2n-1}, y_{2n+1})]$$

$$+e\phi[d(y_{2n}, y_{2n})] + f\phi[d(y_{2n-1}, y_{2n})].$$

Using Definition 2.14 in the above, one deduces that

$$(1 - b - c)\phi[d(y_{2n}, y_{2n+1})] \leq (a + f + c)\phi[d(y_{2n-1}, y_{2n})]$$

from which it follows that $\phi[d(y_{2n}, y_{2n+1})] \leq h\phi[d(y_{2n-1}, y_{2n})]$, where $h := \frac{a + f + c}{1 - b - c} < 1$. Now put $d_n = \phi[d(y_n, y_{n+1})]$, then $d_{2n} \leq hd_{2n-1}$. Now observe that

$$\phi[d(y_{2n+2}, y_{2n+1})] = \phi[d(Tx_{2n+2}, gx_{2n+1})]$$

$$\leq a\phi[d(hx_{2n+2}, Tx_{2n+2})] + b\phi[d(hx_{2n+1}, gx_{2n+1})] + c\phi[d(hx_{2n+2}, gx_{2n+1})]$$

$$+e\phi[d(hx_{2n+1}, Tx_{2n+2})] + f\phi[d(hx_{2n+2}, hx_{2n+1})]$$

$$= a\phi[d(y_{2n+1}, y_{2n+2})] + b\phi[d(y_{2n}, y_{2n+1})] + c\phi[d(y_{2n+1}, y_{2n+1})]$$

$$+e\phi[d(y_{2n}, y_{2n+2})] + f\phi[d(y_{2n+1}, y_{2n})].$$

Using Definition 2.14 in the above, one deduces that

$$(1 - a - e)\phi[d(y_{2n+2}, y_{2n+1})] \leq (b + e + f)\phi[d(y_{2n+1}, y_{2n})]$$

from which it follows that $\phi[d(y_{2n+2}, y_{2n+1})] \leq h_1\phi[d(y_{2n+1}, y_{2n})]$, where $h_1 := \frac{b + f + e}{1 - a - e} < 1$. Now put $d_n = \phi[d(y_n, y_{n+1})]$, then $d_{2n+1} \leq h_1 d_{2n}$. Now observe that

$$\phi[d(y_{2n+2}, y_{2n+3})] = \phi[d(Tx_{2n+2}, gx_{2n+3})]$$

$$\leq a\phi[d(hx_{2n+2}, Tx_{2n+2})] + b\phi[d(hx_{2n+3}, gx_{2n+3})] + c\phi[d(hx_{2n+2}, gx_{2n+3})]$$

$$+e\phi[d(hx_{2n+3}, Tx_{2n+2})] + f\phi[d(hx_{2n+2}, hx_{2n+3})]$$

$$= a\phi[d(y_{2n+1}, y_{2n+2})] + b\phi[d(y_{2n+2}, y_{2n+3})] + c\phi[d(y_{2n+1}, y_{2n+3})]$$

$$+e\phi[d(y_{2n+2}, y_{2n+2})] + f\phi[d(y_{2n+1}, y_{2n+3})].$$

From the above one deduces that

$$(1-b)\phi[d(y_{2n+2}, y_{2n+3})] \le a\phi[d(y_{2n+1}, y_{2n+2})]$$

$$+(c+f)\phi[d(y_{2n+1}, y_{2n+3})].$$

Using Definition 2.14 in the above one deduces that

$$(1-b-c-f)\phi[d(y_{2n+2}, y_{2n+3})]$$

$$\le (a+c+f)\phi[d(y_{2n+1}, y_{2n+2})].$$

Now putting $d_n = \phi[d(y_n, y_{n+1})]$ in the above, we get $d_{2n+2} \le h_2 d_{2n+1}$, where $h_2 := \frac{a+f+c}{1-b-c-f} < 1.$ Now observe that

$$\phi[d(y_{2n+4}, y_{2n+3})] = \phi[d(Tx_{2n+4}, gx_{2n+3})]$$

$$\le a\phi[d(hx_{2n+4}, Tx_{2n+4})] + b\phi[d(hx_{2n+3}, gx_{2n+3})] + c\phi[d(hx_{2n+4}, gx_{2n+3})]$$

$$+e\phi[d(hx_{2n+3}, Tx_{2n+4})] + f\phi[d(hx_{2n+4}, hx_{2n+3})]$$

$$= a\phi[d(y_{2n+3}, y_{2n+4})] + b\phi[d(y_{2n+2}, y_{2n+3})] + c\phi[d(y_{2n+3}, y_{2n+3})]$$

$$+e\phi[d(y_{2n+2}, y_{2n+4})] + f\phi[d(y_{2n+3}, y_{2n+2})].$$

From the above one deduces that

$$(1-a)\phi[d(y_{2n+4}, y_{2n+3})] \le e\phi[d(y_{2n+2}, y_{2n+4})]$$

$$+(b+f)\phi[d(y_{2n+3}, y_{2n+2})].$$

Now using Definition 2.14 in the above, we observe that

$$(1-a-e)\phi[d(y_{2n+4}, y_{2n+3})] \le (b+f+e)\phi[d(y_{2n+3}, y_{2n+2})].$$

Now putting $d_n = \phi[d(y_n, y_{n+1})]$ in the above, we get $d_{2n+3} \le h_3 d_{2n+2}$, where $h_3 := \frac{b + f + e}{1 - a - e} < 1$. Now observe that

$$\phi[d(y_{2n+4}, y_{2n+5})] = \phi[d(Tx_{2n+4}, gx_{2n+5})]$$

$$\le a\phi[d(hx_{2n+4}, Tx_{2n+4})] + b\phi[d(hx_{2n+5}, gx_{2n+5})] + c\phi[d(hx_{2n+4}, gx_{2n+5})]$$

$$+e\phi[d(hx_{2n+5}, Tx_{2n+4})] + f\phi[d(hx_{2n+4}, hx_{2n+5})]$$

$$= a\phi[d(y_{2n+3}, y_{2n+4})] + b\phi[d(y_{2n+4}, y_{2n+5})] + c\phi[d(y_{2n+3}, y_{2n+5})]$$

$$+e\phi[d(y_{2n+4}, y_{2n+4})] + f\phi[d(y_{2n+3}, y_{2n+5})].$$

From the above one deduces that

$$(1 - b)\phi[d(y_{2n+4}, y_{2n+5})] \le a\phi[d(y_{2n+3}, y_{2n+4})]$$

$$+(c + f)\phi[d(y_{2n+3}, y_{2n+5})].$$

Now using Definition 2.14 in the above, and putting $d_n = \phi[d(y_n, y_{n+1})]$, we get $d_{2n+4} \le h_4 d_{2n+3}$, where $h_4 := \frac{a + c + f}{1 - b - c - f} < 1$. Now observe that

$$\phi[d(y_{2n+6}, y_{2n+5})] = \phi[d(Tx_{2n+6}, gx_{2n+5})]$$

$$\le a\phi[d(hx_{2n+6}, Tx_{2n+6})] + b\phi[d(hx_{2n+5}, gx_{2n+5})] + c\phi[d(hx_{2n+6}, gx_{2n+5})]$$

$$+e\phi[d(hx_{2n+5}, Tx_{2n+6})] + f\phi[d(hx_{2n+6}, hx_{2n+5})]$$

$$= a\phi[d(y_{2n+5}, y_{2n+6})] + b\phi[d(y_{2n+4}, y_{2n+5})] + c\phi[d(y_{2n+5}, y_{2n+5})]$$

$$+e\phi[d(y_{2n+4}, y_{2n+6})] + f\phi[d(y_{2n+5}, y_{2n+5})].$$

From the above one deduces that

$$(1 - a)\phi[d(y_{2n+6}, y_{2n+5})] \le b\phi[d(y_{2n+4}, y_{2n+5})]$$

$$+e\phi[d(y_{2n+4}, y_{2n+6})].$$

Now using Definition 2.14 in the above, and putting $d_n = \phi[d(y_n, y_{n+1})]$, we get $d_{2n+5} \leq h_5 d_{2n+4}$, where $h_5 := \dfrac{b+e}{1-a-e} < 1$. Now observe that

$$\begin{aligned} d_{2n} &\leq h d_{2n-1} \\ &\leq h_1 h d_{2n-2} \\ &\leq h_2 h_1 h d_{2n-3} \\ &\leq h_3 h_2 h_1 h d_{2n-4} \\ &\leq h_4 h_3 h_2 h_1 h d_{2n-5} \\ &\leq h_5 h_4 h_3 h_2 h_1 h d_{2n-6} \\ &\leq \\ &\vdots \\ &\leq (h_5 h_4 h_3 h_2 h_1 h)^n d_0. \end{aligned}$$

From the above the following is immediate

$$d_{2n+1} \leq h_1^{n+1}(h_5 h_4 h_3 h_2 h)^n d_0$$

$$d_{2n+2} \leq h_1^{n+1} h_2^{n+1}(h_5 h_4 h_3 h)^n d_0,$$

$$d_{2n+3} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1}(h_5 h_4 h)^n d_0,$$

$$d_{2n+4} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1} h_4^{n+1}(h_5 h)^n d_0,$$

$$d_{2n+5} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1} h_4^{n+1} h_5^{n+1} h^n d_0.$$

Now observe from above we have the following

$$d_{2n} + d_{2n+1} \leq h_1^n [h_1 + 1](h_5 h_4 h_3 h_2 h)^n d_0,$$

$$d_{2n+1} + d_{2n+2} \leq h_1^{n+1} h_2^n [1 + h_2](h_5 h_4 h_3 h)^n d_0,$$

$$d_{2n+2} + d_{2n+3} \leq h_1^{n+1} h_2^{n+1} h_3^n [1 + h_3](h_5 h_4 h)^n d_0,$$

$$d_{2n+3} + d_{2n+4} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1} h_4^n [1 + h_4](h_5 h)^n d_0,$$

$$d_{2n+4} + d_{2n+5} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1} h_4^{n+1} h_5^n [1 + h_5] h^n d_0,$$

$$d_{2n+5} + d_{2n+6} \leq h_1^{n+1} h_2^{n+1} h_3^{n+1} h_4^{n+1} h_5^n h^n d_0 [1 + h_5 + h_1 h_2 h_3 h_4 h_5 h].$$

Let $n, m \in \mathbb{N}$, for the sequence $\{y_n\}$, we consider $\phi[d(y_n, y_m)]$ in two cases. First observe that

$$\begin{aligned} \phi[d(y_n, y_m)] &\leq k\phi[d(y_n, y_{n+1})] + k\phi[d(y_{n+1}, y_{n+2})] \\ &\quad + k\phi[d(y_{n+2}, y_{n+3})] + k\phi[d(y_{n+3}, y_{n+4})] + k\phi[d(y_{n+4}, y_{n+5})] \\ &\quad + k\phi[d(y_{n+5}, y_{n+6})] + \cdots + k\phi[d(y_{m-1}, y_m)] \\ &\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + d_{n+4} + d_{n+5} + \cdots + d_{m-1}]. \end{aligned}$$

If n is even and $m > n$, then we consider the following cases:

Subcase I. Replacing n with $\frac{n}{2}$, then

$$\begin{aligned} \phi[d(y_n, y_m)] &\leq k[h_1^{\frac{n}{2}} [h_1 + 1](h_5 h_4 h_3 h_2 h)^{\frac{n}{2}} d_0 \\ &\quad + h_1^{\frac{n+2}{2}} [h_1 + 1](h_5 h_4 h_3 h_2 h)^{\frac{n+2}{2}} d_0 + \cdots] \\ &\leq k h_1^{\frac{n}{2}} [h_1 + 1](h_5 h_4 h_3 h_2 h)^{\frac{n}{2}} d_0 [1 + h_1 h_5 h_4 h_3 h_2 h + \cdots] \\ &\leq \frac{k h_1^{\frac{n}{2}} [h_1 + 1](h_5 h_4 h_3 h_2 h)^{\frac{n}{2}} d_0}{1 - h_1 h_5 h_4 h_3 h_2 h}. \end{aligned}$$

Subcase II. Replacing n with $\frac{n-2}{2}$, then

$$\phi[d(y_n, y_m)] \leq k[h_1^{\frac{n-2}{2}+1} h_2^{\frac{n-2}{2}+1} h_3^{\frac{n-2}{2}} [1 + h_3](h_5 h_4 h)^{\frac{n-2}{2}} d_0$$

$$+ h_1^{\frac{n}{2}+1} h_2^{\frac{n}{2}+1} h_3^{\frac{n}{2}} [1 + h_3](h_5 h_4 h)^{\frac{n}{2}} d_0 + \cdots]$$

$$\leq kh_1^{\frac{n-2}{2}+1} h_2^{\frac{n-2}{2}+1} h_3^{\frac{n-2}{2}} [1 + h_3](h_5 h_4 h)^{\frac{n-2}{2}} d_0 [1 + h_1 h_5 h_4 h_3 h_2 h + \cdots]$$

$$\leq \frac{kh_1^{\frac{n-2}{2}+1} h_2^{\frac{n-2}{2}+1} h_3^{\frac{n-2}{2}} [1 + h_3](h_5 h_4 h)^{\frac{n-2}{2}} d_0}{1 - h_1 h_5 h_4 h_3 h_2 h}.$$

Subcase III. Replacing n with $\frac{n-4}{2}$, then,

$$\phi[d(y_n, y_m)]$$

$$\leq k[h_1^{\frac{n-4}{2}+1} h_2^{\frac{n-4}{2}+1} h_3^{\frac{n-4}{2}+1} h_4^{\frac{n-4}{2}+1} h_5^{\frac{n-4}{2}} [1 + h_5] h^{\frac{n-4}{2}} d_0$$

$$+ h_1^{\frac{n-2}{2}+1} h_2^{\frac{n-2}{2}+1} h_3^{\frac{n-2}{2}+1} h_4^{\frac{n-2}{2}+1} h_5^{\frac{n-2}{2}} [1 + h_5] h^{\frac{n-2}{2}} d_0 + \cdots]$$

$$\leq kh_1^{\frac{n-4}{2}+1} h_2^{\frac{n-4}{2}+1} h_3^{\frac{n-4}{2}+1} h_4^{\frac{n-4}{2}+1} h_5^{\frac{n-4}{2}} [1 + h_5] h^{\frac{n-4}{2}} d_0 [1 + h_1 h_5 h_4 h_3 h_2 h + \cdots]$$

$$\leq \frac{kh_1^{\frac{n-4}{2}+1} h_2^{\frac{n-4}{2}+1} h_3^{\frac{n-4}{2}} h_4^{\frac{n-4}{2}} h_5^{\frac{n-4}{2}} [1 + h_5] h^{\frac{n-4}{2}} d_0}{1 - h_1 h_5 h_4 h_3 h_2 h}.$$

If n is odd and $m > n$, then we consider the following cases:

Subcase I. Replacing n with $\frac{n-1}{2}$, then

$$\phi[d(y_n, y_m)]$$

$$\leq k[h_1^{\frac{n-1}{2}+1} h_2^{\frac{n-1}{2}} [1+h_2](h_5h_4h_3h)^{\frac{n-1}{2}} d_0 + h_1^{\frac{n+1}{2}+1} h_2^{\frac{n+1}{2}}$$

$$\times [1+h_2](h_5h_4h_3h)^{\frac{n+1}{2}} d_0 + \cdots]$$

$$\leq kh_1^{\frac{n-1}{2}+1} h_2^{\frac{n-1}{2}} [1+h_2](h_5h_4h_3h)^{\frac{n-1}{2}} d_0 [1 + h_1h_5h_4h_3h_2h + \cdots]$$

$$\leq \frac{kh_1^{\frac{n-1}{2}+1} h_2^{\frac{n-1}{2}} [1+h_2](h_5h_4h_3h)^{\frac{n-1}{2}} d_0}{1 - h_1h_5h_4h_3h_2h}.$$

Subcase II. Replacing n with $\frac{n-3}{2}$, then

$$\phi[d(y_n, y_m)]$$

$$\leq k[h_1^{\frac{n-3}{2}+1} h_2^{\frac{n-3}{2}+1} h_3^{\frac{n-3}{2}+1} h_4^{\frac{n-3}{2}} [1+h_4](h_5h)^{\frac{n-3}{2}} d_0$$

$$+ h_1^{\frac{n-1}{2}+1} h_2^{\frac{n-1}{2}+1} h_3^{\frac{n-1}{2}+1} h_4^{\frac{n-1}{2}} [1+h_4](h_5h)^{\frac{n-1}{2}} d_0 + \cdots]$$

$$\leq kh_1^{\frac{n-3}{2}+1} h_2^{\frac{n-3}{2}+1} h_3^{\frac{n-3}{2}+1} h_4^{\frac{n-3}{2}} [1+h_4](h_5h)^{\frac{n-3}{2}} d_0 [1 + h_1h_5h_4h_3h_2h + \cdots]$$

$$\leq \frac{kh_1^{\frac{n-3}{2}+1} h_2^{\frac{n-3}{2}+1} h_3^{\frac{n-3}{2}+1} h_4^{\frac{n-3}{2}} [1+h_4](h_5h)^{\frac{n-3}{2}} d_0}{1 - h_1h_5h_4h_3h_2h}.$$

Subcase III. Replacing n with $\frac{n-5}{2}$, then

$$\phi[d(y_n, y_m)]$$

$$\leq k[h_1^{\frac{n-5}{2}+1} h_2^{\frac{n-5}{2}+1} h_3^{\frac{n-5}{2}+1} h_4^{\frac{n-5}{2}+1} h_5^{\frac{n-5}{2}} h^{\frac{n-5}{2}} d_0 [1 + h_5 + h_1h_2h_3h_4h_5h]$$

$$+ h_1^{\frac{n-3}{2}+1} h_2^{\frac{n-3}{2}+1} h_3^{\frac{n-3}{2}+1} h_4^{\frac{n-3}{2}+1} h_5^{\frac{n-3}{2}} h^{\frac{n-3}{2}} d_0 [1 + h_5 + h_1h_2h_3h_4h_5h + \cdots]$$

$$\le kh_1^{\frac{n-5}{2}+1}h_2^{\frac{n-5}{2}+1}h_3^{\frac{n-5}{2}+1}h_4^{\frac{n-5}{2}+1}h_5^{\frac{n-5}{2}}h^{\frac{n-5}{2}}d_0[1+h_5+h_1h_2h_3h_4h_5h]$$

$$\times[1+h_1h_5h_4h_3h_2h+\cdots]$$

$$\le \frac{kh_1^{\frac{n-5}{2}+1}h_2^{\frac{n-5}{2}+1}h_3^{\frac{n-5}{2}+1}h_4^{\frac{n-5}{2}+1}h_5^{\frac{n-5}{2}}h^{\frac{n-5}{2}}d_0[1+h_5+h_1h_2h_3h_4h_5h]}{1-h_1h_5h_4h_3h_2h}.$$

Since $h, h_1, h_2, h_3, h_4, h_5 < 1$, it follows that $hh_1h_2h_3h_4h_5 < 1$, so in all sub-cases when n is even or odd, we have $\phi[d(y_n, y_m)] \to 0_B$ as $n \to \infty$, and since $\phi \in \Phi(P, C)$, we have $d(y_n, y_m) \to 0_E$. Thus by Lemma 2.13, $\{y_n\} = \{hx_{n-1}\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists $v \in h(X)$ and $u \in X$ such that $\lim_{n\to\infty} y_n = v$ and $v = hu$. Now, we show that u is coincidence point of the pairs (T, h) and (g, h), that is, $Tu = gu = hu$. If $Tu \neq hu$ then $0_E \prec d(Tu, hu)$. Now observe that

$$\phi[d(Tu, y_{2n+1})] = \phi[d(Tu, gx_{2n+1})]$$

$$\le a\phi[d(hu, Tu)] + b\phi[d(hx_{2n+1}, gx_{2n+1})] + c\phi[d(hu, gx_{2n+1})]$$

$$+e\phi[d(hx_{2n+1}, Tu)] + f\phi[d(hu, hx_{2n+1})]$$

$$= a\phi[d(hu, Tu)] + b\phi[d(y_{2n}, y_{2n+1})] + c\phi[d(hu, y_{2n+1})]$$

$$+e\phi[d(y_{2n}, Tu)] + f\phi[d(hu, y_{2n})]$$

$$= a\phi[d(hu, Tu)] + bd_{2n} + c\phi[d(hu, y_{2n+1})] + e\phi[d(y_{2n}, Tu)]$$

$$+f\phi[d(hu, y_{2n})].$$

Since $y_{2n} \to hu$, $d_{2n} \to 0_B$, $d(Tu, y_{2n+1}) \to d(Tu, hu)$ as $n \to \infty$ and $\phi \in \Phi(P, C)$, therefore taking limits in the above and using Remark 2.9, we deduce, since $a + e < 1$, that

$$\phi[d(Tu, hu)] \le (a+e)\phi[d(hu, Tu)] < \phi[d(hu, Tu)]$$

which is a contradiction, thus $hu = Tu$. Similarly, we can show $hu = gu$, thus $Tu = gu = hu = v$. Thus, v is a point of coincidence of the pairs (T, h) and (g, h). Finally, we show uniqueness of the point of coincidence. Suppose w is another point of coincidence of these pairs, that is, $Tz = gz = hz = w$ for some $z \in X$. Now observe that

$$\begin{aligned}
&\phi[d(w, v)] = \phi[d(Tz, gu)] \\
&\leq a\phi[d(hz, Tz)] + b\phi[d(hu, gu)] + c\phi[d(hz, gu)] \\
&\quad +e\phi[d(hu, Tz)] + f\phi[d(hz, hu)] \\
&\leq a\phi[d(w, w)] + b\phi[d(v, v)] + c\phi[d(w, v)] + e\phi[d(v, w)] + f\phi[d(w, v)] \\
&\leq (c + e + f)\phi[d(w, v)].
\end{aligned}$$

Since $c + e + f < 1$ and using Remark 2.9, we deduce that $\phi[d(w, v)] = 0_B$, that is $w = v$. Thus, point of coincidence is unique. If the pairs (T, h) and (g, h) are weakly compatible, then since $Tu = gu = hu = v$, we have, $Tv = Thu = hTu = hv$ and $gv = ghu = hgu = hv$, therefore $Tv = gv = hv = p$ (say). This shows that p is another point of coincidence, therefore by uniqueness we must have $p = v$, that is, $Tv = gv = hv = v$. Thus, v is unique common fixed point of the self-maps T, g, h.

Taking $B = \mathbb{R}$, $C = [0, \infty)$ and $\phi[a] = \|a\|$, then C is a normal cone with normal constant 1, $\phi \in \Phi(P, C)$ with $k = K$ (normal constant of P). Thus from the above, we have the following:

Corollary 3.2. *Let (X, d) be a cone heptagonal metric space and P be a normal cone with normal constant K. Suppose $T, g, h : X \mapsto X$ satisfy the condition*

$$\begin{aligned}
\|d(Tx, gy)\| &\leq a\|d(hx, Tx)\| + b\|d(hy, gy)\| + c\|d(hx, gy)\| \\
&\quad +e\|d(hy, Tx)\| + f\|d(hx, hy)\|
\end{aligned}$$

for all $x, y \in X$; $a, b, c, e, f \geq 0$; $a + b + c + e + f < 1$. If $T(X) \cup g(X) \subset h(X)$ *and* $h(X)$ *is a complete subspace of X, then the maps* T, g, h *have a unique point of coincidence in X. Moreover, if* (T, h) *and* (g, h) *are weakly compatible pairs, then* T, g, h *have a unique common fixed point.*

Remark 3.3. From the above Corollary we have the following

(a) If $c = e = 0$, then we get Corollary 2.2 [5].

(b) If $a = b = c = e = 0$, then we get Theorem 2.1 [1].

(c) If $c = e = f = 0$ and $a = b$, then we get Theorem 3.1 [1].

Remark 3.4. Taking $E = B$, $P = C$, and defining $\phi : P \mapsto P$ by $\phi[a] = a$ for all $a \in P$ in Theorem 3.1, then we get the following

(a) Corollary 2.3 [5], provided $c = e = 0$.

(b) Result of Vetro [3], provided $c = e = 0$ and $h = I_X$.

(b) Generalization of Theorems 1 and 3 of Huang and Zhang [2], provided $c = e = 0$, $T = g$ and $h = I_X$.

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