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CHARACTERIZATION OF INTEGERS OF TRIQUADRATIC NUMBER FIELDS

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Abstract

By using a method of Chatelain, we have been able to construct an integral basis for the Triquadratic Number Fields $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, and this, in all possible 3 occurring cases

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covered by: $(dm, dn, d'm'n'l) \equiv (1, 1, 1), (1, 1, 2, \text{ or } 3)$ and $(1, 2, 3)$ $(\text{mod } 4)$.

From that point, let us take any element:

$$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4\sqrt{mn} + \omega_5\sqrt{d'm'n'l} + \omega_6\sqrt{\frac{dm}{d'm'}n'l}$$

$$+ \omega_7\sqrt{\frac{dn}{d'n'}m'l} + \omega_8\sqrt{\frac{mn}{m'n'}d'l} \in K_3, \text{ where } \omega_i \in \mathbb{Q}.$$

Then, to say if θ is an integer of K_3 or not, we give for the each 3 previous cases, necessary and sufficient conditions that involve only congruences modulo 2, 4 or 8 on certain summations supported by the $a_i \in \mathbb{Z}$, such that: $\omega_i = \pm\frac{1}{8}a_i$ in the first case and $\omega_i = \pm\frac{1}{4}a_i$ in the 2 others.

(The signs $\pm$ are easy to determine and depend on K_3).

In conclusion, one can say that these results give practical methods to determine if an element of K_3 is an algebraic integer or not, but also, it can be seen like a generalization of those well known, for the integer elements of biquadratic fields $K_2 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn})$.

1. Introduction, Recalls and Notations

Let us put $K_n = \mathbb{Q}(\sqrt{A_{2^0}}, \sqrt{A_{2^1}}, \ldots, \sqrt{A_{2^{n-3}}}, \mu\sqrt{A_{2^{n-2}}}, \mu'\sqrt{A_{2^{n-1}}})$, for a general n-quadratic number field, where $(\mu, \mu') \in \{(1, 1), (1, \sqrt{\pm 2}), (1, \sqrt{-1}), (\sqrt{\pm 2}, \sqrt{-1})\}$, and the $A_{2^k} \equiv 1 \pmod 4 : 0 \le k \le n-1$, are taken square free into $\mathbb{Z}^* \setminus \{1\}$, except in certain cases for which the quantities $A_{2^{n-2}}$ and $A_{2^{n-1}}$ can be equal to 1.

For such number fields K_n, D. Chatelain (cf. Ref. [1]) gave a method for the construction of an integral $\mathbb{Z}$-basis $\mathfrak{B}_{K_n}$ for its integral ring $\mathbb{Z}_{K_n}$, as well as for the calculations of $\mathfrak{D}_{K_n/\mathbb{Q}}$, the discriminant of the number field K_n relative to $\mathbb{Q}$. If this method is effective and algorithmic, for giving integral basis, in the practice we do not have any explicit formulae.

It is one of the reasons why, in this paper, we study and apply Chatelain's method to the Triquadratic Number Fields $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, to find such formulas which do not exist at this time.

Let us remark that Park et al. (cf. Ref. [5]) and Gábor Nyul (cf. Ref. [4]) gave also bases of triquadratic number fields, but they did not explain the way they obtained them.

Let us recall, that we use, which we agreed once and for all, (cf. Ref. [2]) the following notations and conventions.

Notations 1.1. (i) Let us put $\gcd(a, b) = (a, b)$, $\forall a, b \in \mathbb{Z}$. When we write a triquadradic number field: $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, this implies that d, m, n are square free rational integers, such that:

dm, dn, mn, $d'm'n'l$ are $\neq 0$ and 1, with $(d, m) = (d, n) = (m, n) = (dmn, l) = 1$ and d' / d, m' / m, n' / n, and satisfying the conditions:

$(dm, dn, d'm'n'l) \equiv (1, 1, 1), (1, 1, 2, \text{ or } 3)$ and $(1, 2, 3) \pmod 4$. (That we call, respectively, the cases 1, 2 and 3.

(ii) $s(a)$ is the sign of $a \in \mathbb{Z}^{\star}$.

(iii) For $a \equiv 1 \pmod 2$, we put $\lambda_a \in \{-1; 1\}$, such that $a \equiv \lambda_a \pmod 4$.

(iv) $s_1 = \lambda_d s(d) = \pm 1$, $s_2 = \lambda_{d'm'} s(d'm') = \pm 1$, $s_3 = \lambda_{d'n'} s(d'n') = \pm 1$, $s_4 = \lambda_{dm'n'} s(dm'n') = \pm 1$.

(v) $\gamma = \begin{cases} \lambda_{d'm'n'l} = -1, \text{ if } d'm'n'l \equiv -1 \pmod 4, \\ \lambda_{d'm'n'\frac{l}{2}} = \pm 1, \text{ if } d'm'n'l \equiv 2 \pmod 4 \text{ (that is when } l \text{ is even).} \end{cases}$

(vi) $\delta = \lambda_{d\frac{n}{2}}$ when n is even.

(vii) The Galois group acts on K_3 (which is a $\mathbb{Q}$-space vector set generated by 1 and the square roots just below) like this, and will work on the same way, hereafter, on the Chatelain's β-basis of K_3 :

$$M_3 = \begin{array}{c} \begin{matrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \end{array} \begin{array}{r} \alpha_i \\ 1 \\ \sqrt{dm} \\ \sqrt{dn} \\ \sqrt{mn} \\ \sqrt{d'm'n'l} \\ \sqrt{\frac{d}{d'}\frac{m}{m'}n'l} \\ \sqrt{\frac{d}{d'}\frac{n}{n'}m'l} \\ \sqrt{\frac{m}{m'}\frac{n}{n'}d'l} \end{array}$$

The results we have found (cf. Ref. [2]) on the Chatelain's written form (unique when K_3 satisfies Notations 1.1 (i)) of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, on Chatelain's β-basis, and on Chatelain's integral $\mathbb{Z}$-basis $\mathfrak{B}_{K_3}$, respectively, in the 3 occurring cases: $(dm, dn, d'm'n'l) \equiv (1, 1, 1), (1, 1, 2, \text{ or } 3)$ and $(1, 2, 3) (\bmod 4)$, are summarized in the following ones:

Remark 1.1. Once $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ is given satisfying Notations 1.1 (i), then the Chatelain's written form of K_3 is unique.

Lemma 1.1. *For Case* 1. $(dm, dn, d'm'n'l) \equiv (1, 1, 1) \pmod 4$

(i) *A Chatelain's written form of* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *is*:

$$K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right).$$

(ii) *The corresponding Chatelain's* β*-basis of* K_3 *is given by*:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \sqrt{d'm'n'l}, s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}, s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}.$$

(iii) *The corresponding Chatelain's* $\mathbb{Z}$*-base* $\mathfrak{B}_{K_3}$ *of* $\mathbb{Z}_{K_3}$ *is the following*:

$$\varepsilon_1 = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn} + \sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$\left. + s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

$$\varepsilon_2 = \frac{1}{8}\left(1 - \sqrt{dm} + \sqrt{dn} - \lambda_d s(d)\sqrt{mn} + \sqrt{d'm'n'l} - \lambda_{d'm'}s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$\left. + \lambda_{d'n'}s(d'n')\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - \lambda_{dm'n'}s(dm'n')\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

$$\varepsilon_3 = \frac{1}{8}\left(1 + \sqrt{dm} - \sqrt{dn} - \lambda_d s(d)\sqrt{mn} + \sqrt{d'm'n'l} + \lambda_{d'm'}s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$\left. - \lambda_{d'n'}s(d'n')\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - \lambda_{dm'n'}s(dm'n')\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

$$\varepsilon_4 = \frac{1}{8}\left(1 - \sqrt{dm} - \sqrt{dn} + \lambda_d s(d)\sqrt{mn} + \sqrt{d'm'n'l} - \lambda_{d'm'}s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$\left. - \lambda_{d'n'}s(d'n')\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + \lambda_{dm'n'}s(dm'n')\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

$$\varepsilon_5 = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + \lambda_d s(d)\sqrt{mn} - \sqrt{d'm'n'l} - \lambda_{d'm'}s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$\left. - \lambda_{d'n'}s(d'n')\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - \lambda_{dm'n'}s(dm'n')\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

$$\varepsilon_6 = \frac{1}{8}\left(1 - \sqrt{dm} + \sqrt{dn} - \lambda_d s(d)\sqrt{mn} - \sqrt{d'm'n'l} + \lambda_{d'm'}s(d'm')\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right.$$

$$- \lambda_{d'n'} s(d'n') \sqrt{\frac{d}{d'} \frac{n}{n'} m'l} + \lambda_{dm'n'} s(dm'n') \sqrt{\frac{m}{m'} \frac{n}{n'} d'l} \Bigg),$$

$$\varepsilon_7 = \frac{1}{8} \Bigg(1 + \sqrt{dm} - \sqrt{dn} - \lambda_d s(d) \sqrt{mn} - \sqrt{d'm'n'l} - \lambda_{d'm'} s(d'm') \sqrt{\frac{d}{d'} \frac{m}{m'} n'l}$$

$$+ \lambda_{d'n'} s(d'n') \sqrt{\frac{d}{d'} \frac{n}{n'} m'l} + \lambda_{dm'n'} s(dm'n') \sqrt{\frac{m}{m'} \frac{n}{n'} d'l} \Bigg),$$

$$\varepsilon_8 = \frac{1}{8} \Bigg(1 - \sqrt{dm} - \sqrt{dn} + \lambda_d s(d) \sqrt{mn} - \sqrt{d'm'n'l} + \lambda_{d'm'} s(d'm') \sqrt{\frac{d}{d'} \frac{m}{m'} n'l}$$

$$+ \lambda_{d'n'} s(d'n') \sqrt{\frac{d}{d'} \frac{n}{n'} m'l} - \lambda_{dm'n'} s(dm'n') \sqrt{\frac{m}{m'} \frac{n}{n'} d'l} \Bigg).$$

Lemma 1.2. *For Case* 2. $(dm, dn, d'm'n'l) \equiv (1, 1, 2 \text{ or } 3) \pmod 4$

(i) *A Chatelain's written form of* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *is*:

$$K_3 = \begin{cases} \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{2\lambda_{d'm'n'\frac{l}{2}}} \sqrt{\lambda_{d'm'n'\frac{l}{2}} d'm'n' \frac{l}{2}}\right), & \text{if } l \text{ is even}, \\ \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{-1}\sqrt{-d'm'n'l}\right), & \text{either}. \end{cases}$$

(ii) *The corresponding Chatelain's* β-*basis of* K_3 *is given by*:

$$\beta = \Bigg\{ 1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \gamma\sqrt{d'm'n'l}, \gamma s_2 \sqrt{\frac{d}{d'} \frac{m}{m'} n'l},$$

$$\gamma s_3 \sqrt{\frac{d}{d'} \frac{n}{n'} m'l}, \gamma s_4 \sqrt{\frac{m}{m'} \frac{n}{n'} d'l} \Bigg\}.$$

(iii) *The corresponding Chatelain's* $\mathbb{Z}$-base $\mathfrak{B}_{K_3}$ *of* $\mathbb{Z}_{K_3}$ *is the following*:

$$\varepsilon_1 = \frac{1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn}}{4}, \ \varepsilon_2 = \frac{1 - \sqrt{dm} + \sqrt{dn} - s_1\sqrt{mn}}{4},$$

$$\varepsilon_3 = \frac{1 + \sqrt{dm} - \sqrt{dn} - s_1\sqrt{mn}}{4}, \ \varepsilon_4 = \frac{1 - \sqrt{dm} - \sqrt{dn} + s_1\sqrt{mn}}{4},$$

$$\varepsilon_5 = \gamma\frac{\sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} + s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_6 = \gamma\frac{\sqrt{d'm'n'l} - s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} + s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_7 = \gamma\frac{\sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_8 = \gamma\frac{\sqrt{d'm'n'l} - s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4}.$$

Lemma 1.3. *For Case* 3. $(dm,\ dn\ d'm'n'l) \equiv (1,\ 2,\ 3) \pmod 4$

(i) *A Chatelain's written form of* $K_3 = \mathbb{Q}\left(\sqrt{dm},\ \sqrt{dn},\ \sqrt{d'm'n'l}\right)$ *is*:

$$K_3 = \mathbb{Q}\left(\sqrt{dm},\ \sqrt{2\delta}\sqrt{\delta d\,\frac{n}{2}},\ \sqrt{-1}\sqrt{-\,d'm'n'l}\right).$$

(ii) *The corresponding Chatelain's* β-*basis of* K_3 *is given by*:

$$\beta = \left\{1,\ \sqrt{dm},\ \delta\sqrt{dn},\ \delta s_1\sqrt{mn},\ -\sqrt{d'm'n'l},\ -s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l},\right.$$

$$\left. -\,\delta s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l},\ -\,\delta s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}.$$

(iii) *The corresponding de Chatelain's* $\mathbb{Z}$-*base* $\mathfrak{B}_{K_3}$ *of* $\mathbb{Z}_{K_3}$ *is the following*:

$$\varepsilon_1 = \frac{1+\sqrt{dm}}{2},\ \varepsilon_2 = \delta\frac{\sqrt{dn} + s_1\sqrt{mn}}{2},\ \varepsilon_3 = \frac{-\sqrt{d'm'n'l} - s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}}{2},$$

$$\varepsilon_4 = \delta\frac{\sqrt{dn} + s_1\sqrt{mn} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},\ \varepsilon_5 = \frac{1-\sqrt{dm}}{2},$$

$$\varepsilon_6 = \delta \frac{\sqrt{dn} - s_1\sqrt{mn}}{2}, \ \varepsilon_7 = \frac{-\sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}}{2} \ \textit{and}$$

$$\varepsilon_8 = \delta \frac{\sqrt{dn} - s_1\sqrt{mn} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4}.$$

Let us recall that there is particular suitable and useful integral bases $\mathfrak{B}'_{K_3} = \{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4, \varepsilon'_5, \varepsilon'_6, \varepsilon'_7, \varepsilon'_8\}$ obtained from the previous $\mathfrak{B}_{K_3}$ ones, and we will use in the proof of Theorem 2.1, which is:

Lemma 1.4. (1) *For Case* 1.

$$\varepsilon'_1 = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn} + \sqrt{d'm'n'l} + s_2\sqrt{\frac{dm}{d'm'}n'l} + s_3\sqrt{\frac{dn}{d'n'}m'l} + s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_2 = \frac{1}{8}\left(-2\sqrt{dm} - 2s_1\sqrt{mn} - 2s_2\sqrt{\frac{dm}{d'm'}n'l} - 2s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_3 = \frac{1}{8}\left(-2\sqrt{dn} + 2s_1\sqrt{mn} - 2s_3\sqrt{\frac{dn}{d'n'}m'l} + 2s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_4 = \frac{1}{8}\left(-4s_1\sqrt{mn} - 4s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_5 = \frac{1}{8}\left(-2\sqrt{d'm'n'l} - 2s_2\sqrt{\frac{dm}{d'm'}n'l} + 2s_3\sqrt{\frac{dn}{d'n'}m'l} + 2s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_6 = \frac{1}{8}\left(4s_2\sqrt{\frac{dm}{d'm'}n'l} - 4s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_7 = \frac{1}{8}\left(-4s_3\sqrt{\frac{dn}{d'n'}m'l} + 4s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_8' = \frac{1}{8}\left(-8s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

(2) *For Case* 2.

$$\varepsilon_1' = \frac{1+\sqrt{dm}+\sqrt{dn}+s_1\sqrt{mn}}{4}, \ \varepsilon_2' = \frac{-\sqrt{dm}-\sqrt{dn}}{2},$$

$$\varepsilon_3' = \frac{-\sqrt{dn}-s_1\sqrt{mn}}{2}, \quad \varepsilon_4' = -s_1\sqrt{mn},$$

$$\varepsilon_5' = \gamma\frac{\sqrt{d'm'n'l}+s_2\sqrt{\frac{dm}{d'm'}n'l}+s_3\sqrt{\frac{dn}{d'n'}m'l}+s_4\sqrt{\frac{mn}{m'n'}d'l}}{4},$$

$$\varepsilon_6' = \frac{-\gamma}{2}\left(s_2\sqrt{\frac{dm}{d'm'}n'l}+s_3\sqrt{\frac{dn}{d'n'}m'l}\right),$$

$$\varepsilon_7' = \frac{-\gamma}{2}\left(s_3\sqrt{\frac{dn}{d'n'}m'l}+s_4\sqrt{\frac{mn}{m'n'}d'l}\right), \quad \varepsilon_8' = -\gamma s_4\sqrt{\frac{mn}{m'n'}d'l}.$$

(3) *For Case* 3.

$$\varepsilon_1' = \frac{1+\sqrt{dm}}{2}, \ \varepsilon_2' = \sqrt{dm},$$

$$\varepsilon_3' = \frac{\delta}{4}\left(\sqrt{dn}+s_1\sqrt{mn}-s_3\sqrt{\frac{dn}{d'n'}m'l}-s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_4' = \frac{\delta}{2}\left(s_1\sqrt{mn}-s_4\sqrt{\frac{mn}{m'n'}d'l}\right), \ \varepsilon_5' = \frac{1}{2}\left(-\sqrt{d'm'n'l}-s_2\sqrt{\frac{dm}{d'm'}n'l}\right),$$

$$\varepsilon_6' = \frac{1}{4}\left(-s_2\sqrt{\frac{dm}{d'm'}n'l}\right), \ \varepsilon_7' = \frac{\delta}{2}\left(-s_3\sqrt{\frac{dn}{d'n'}m'l}-s_4\sqrt{\frac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_8' = \frac{\delta}{2}s_4\sqrt{\frac{mn}{m'n'}d'l}.$$

To obtain such basis, let us remark that in the $\mathbb{Z}$-module $\mathbb{Z}_{K_3}$ of rank 8, the matrix that changes the basis $\mathfrak{B}_{K_3}$ to the basis $\mathfrak{B}'_{K_3}$ in the Cases 1, 2 and 3 is, respectively:

$$P_1 = \left(\begin{array}{cccccccc|l}
\varepsilon'_1 & \varepsilon'_2 & \varepsilon'_3 & \varepsilon'_4 & \varepsilon'_5 & \varepsilon'_6 & \varepsilon'_7 & \varepsilon'_8 & \\ \hline
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & \varepsilon_1 \\
0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & \varepsilon_2 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & \varepsilon_3 \\
0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & \varepsilon_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \varepsilon_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \varepsilon_6 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & \varepsilon_7 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & \varepsilon_8
\end{array}\right);$$

$$P_2 = \left(\begin{array}{cccccccc|l}
\varepsilon'_1 & \varepsilon'_2 & \varepsilon'_3 & \varepsilon'_4 & \varepsilon'_5 & \varepsilon'_6 & \varepsilon'_7 & \varepsilon'_8 & \\ \hline
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \varepsilon_1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \varepsilon_2 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \varepsilon_3 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \varepsilon_4 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & \varepsilon_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \varepsilon_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \varepsilon_7 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon_8
\end{array}\right)$$

and

$$P_3 = \left(\begin{array}{cccccccc|l}
\varepsilon'_1 & \varepsilon'_2 & \varepsilon'_3 & \varepsilon'_4 & \varepsilon'_5 & \varepsilon'_6 & \varepsilon'_7 & \varepsilon'_8 & \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & \varepsilon_2 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \varepsilon_3 \\
0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 & \varepsilon_4 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \varepsilon_6 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \varepsilon_7 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & \varepsilon_8
\end{array}\right),$$

with $|P_i|^2 = 1$, for $i = 1, 2, 3$.

Now in the next paragraph, we will give the main theorem of this paper.

2. Characterization of Integers of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$

Let us take any element:

$$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4\sqrt{mn} + \omega_5\sqrt{d'm'n'l}$$

$$+ \omega_6\sqrt{\frac{dm}{d'm'}n'l} + \omega_7\sqrt{\frac{dn}{d'n'}m'l} + \omega_8\sqrt{\frac{mn}{m'n'}d'l} \in K_3, \text{ where } \omega_i \in \mathbb{Q}.$$

As said in the introductive part, the problem which is set, is to know if θ is an integer of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ or not.

The first thing to do is to distinguish among the 3 generic cases, the one we are dealing with. To do this, let us take a look at the 7 quantities $dm, dn, mn, d'm'n'l,$ $\frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l.$ One remarks that:

Remarks 2.1. (1) Case 1 occurs if and only if more than 4 of the elements: $dm, dn, mn, d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$ are $\equiv 1 \pmod 4$, (exactly all terms are $\equiv 1 \pmod 4$).

(2) Case 2 occurs if and only if exactly 3 of the elements: $dm, dn, mn,$ $d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$ are $\equiv 1 \pmod 4$, (that are dm, dn and $mn \equiv 1$ $\pmod 4$, and for the others, either $d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$ are $\equiv 2$ $\pmod 4$), or these same 4 ones are $\equiv 3 \pmod 4$.

(3) Case 3 occurs if and only if exactly 1 of the elements: $dm, dn, mn,$ $d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$ is $\equiv 1 \pmod 4$, (that is $dm \equiv 1 \pmod 4$, and for the others: $dn, mn, \frac{mn}{m'n'}d'l, \frac{dn}{d'n'}m'l$ are $\equiv 2 \pmod 4$, and $d'm'n'l, \frac{dm}{d'm'}n'l$ are $\equiv 3 \pmod 4$).

Then let us remark that we will write θ in Theorem 2.1, on the corresponding Chatelain's β-basis for cases 1, 2 and 3. We obtain:

Remarks 2.2. (1) In Case 1. $\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4 s_1(s_1\sqrt{mn}) + \omega_5 \sqrt{d'm'n'l} + \omega_6 s_2\left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \omega_7 s_3\left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right) + \omega_8 s_4\left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right)$, $\omega_i \in \mathbb{Q}$.

(2) In Case 2. $\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4 s_1(s_1\sqrt{mn}) + \omega_5\gamma\left(\gamma\sqrt{d'm'n'l}\right) + \omega_6\gamma s_2\left(\gamma s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \omega_7\gamma s_3\left(\gamma s_3\sqrt{\frac{dn}{d'n'}m'l}\right) + \omega_8\gamma s_4\left(\gamma s_4\sqrt{\frac{mn}{m'n'}d'l}\right)$, $\omega_i \in \mathbb{Q}$.

(3) In Case 3. $\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\delta\left(\delta\sqrt{dn}\right) + \omega_4\delta s_1(\delta s_1\sqrt{mn}) - \omega_5\sqrt{d'm'n'l} - \omega_6 s_2\left(s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right) - \omega_7\delta s_3\left(\delta s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}\right) - \omega_8\delta s_4\left(\delta s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right)$, $\omega_i \in \mathbb{Q}$.

From that, because we are able to write $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ in a Chatelain's written form, it is clear that the following quantities are well known, once θ is given:

$$s_1 = \lambda_d s(d) = \pm 1,\ s_2 = \lambda_{d'm'} s(d'm') = \pm 1,\ s_3 = \lambda_{d'n'} s(d'n') = \pm 1,$$

$$s_4 = \lambda_{dm'n'} s(dm'n') = \pm 1,\ \gamma = \begin{cases} \lambda_{d'm'n'l} = -1, \text{ if } d'm'n'l \equiv -1 \pmod 4, \\ \lambda_{d'm'n'\frac{l}{2}} = \pm 1, \text{ if } d'm'n'l \equiv 2 \pmod 4, \end{cases}$$

and $\delta = \lambda_{d\frac{n}{2}}$ when n is even.

From all that we get the following main theorem:

Theorem 2.1. *Let* $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, *a triquadratic number field, satisfying Notations* 1.1 (i), *where* $s_1, s_2, s_3, s_4, \gamma$ *and* δ *are the signs coming from its Chatelain's written form. Then there are equivalencies between propositions* (i) *and* (ii).

(i) $\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4\sqrt{mn} + \omega_5\sqrt{d'm'n'l} + \omega_6\sqrt{\frac{dm}{d'm'}n'l} + \omega_7$ $\sqrt{\frac{dn}{d'n'}m'l} + \omega_8\sqrt{\frac{mn}{m'n'}d'l}$, $\omega_i \in \mathbb{Q}$, *is an integer of* K_3 (*i.e., belongs to the ring of integral elements* $\mathbb{Z}_{K_3}$ *of* K_3).

(ii) (A) *Case* 1. $\exists\,(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8$, *such that*:

(A_1) $\theta = \frac{1}{8}(a_1 + a_2\sqrt{dm} + a_3\sqrt{dn} + s_1a_4\sqrt{mn} + a_5\sqrt{d'm'n'l} + s_2a_6\sqrt{\frac{dm}{d'm'}n'l}$ $+ s_3a_7\sqrt{\frac{dn}{d'n'}m'l} + s_4a_8\sqrt{\frac{mn}{m'n'}d'l})$, *and satisfying*:

$$(A_2)\begin{cases} a_1 \in \mathbb{Z}, \\ a_1 - a_2 \equiv 0 \pmod 2, \\ a_3 - a_1 \equiv 0 \pmod 2, \\ a_1 + a_2 - a_3 - a_4 \equiv 0 \pmod 4, \\ a_1 - a_5 \equiv 0 \pmod 2, \\ a_1 - a_2 - a_5 + a_6 \equiv 0 \pmod 4, \\ a_1 + a_3 - a_5 - a_7 \equiv 0 \pmod 4, \\ a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8 \equiv 0 \pmod 8. \end{cases}$$

(B) *Case* 2. $\exists\,(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8$, *such that*:

(B_1) $\theta = \frac{1}{4}(a_1 + a_2\sqrt{dm} + a_3\sqrt{dn} + s_1a_4\sqrt{mn} + \gamma a_5\sqrt{d'm'n'l} + \gamma s_2a_6\sqrt{\frac{dm}{d'm'}n'l}$ $+ \gamma s_3a_7\sqrt{\frac{dn}{d'n'}m'l} + \gamma s_4a_8\sqrt{\frac{mn}{m'n'}d'l})$, *and satisfying*:

$$(\mathrm{B}_2)\begin{cases} a_1 \in \mathbb{Z}, \\ a_1 - a_2 \equiv 0 \pmod 2, \\ a_2 - a_3 \equiv 0 \pmod 2, \\ a_1 - a_2 + a_3 - a_4 \equiv 0 \pmod 4, \\ a_5 \in \mathbb{Z}, \\ a_5 - a_6 \equiv 0 \pmod 2, \\ a_6 - a_7 \equiv 0 \pmod 2, \\ a_5 - a_6 + a_7 - a_8 \equiv 0 \pmod 4. \end{cases}$$

(C) *Case* 3. $\exists\,(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8$, *such that*:

$$(\mathrm{C}_1)\ \theta = \frac{1}{4}\Big(a_1 + a_2\sqrt{dm} + \delta a_3\sqrt{dn} + \delta s_1 a_4\sqrt{mn} - a_5\sqrt{d'm'n'l} - s_2 a_6\sqrt{\frac{dm}{d'm'}n'l} - \delta s_3 a_7\sqrt{\frac{dn}{d'n'}m'l} - \delta s_4 a_8\sqrt{\frac{mn}{m'n'}d'l}\Big),$$ *and satisfying*:

$$(\mathrm{C}_2)\begin{cases} a_1 \equiv 0 \pmod 2, \\ a_2 - a_1 \equiv 0 \pmod 4, \\ a_3 \in \mathbb{Z}, \\ a_4 - a_3 \equiv 0 \pmod 2, \\ a_5 \equiv 0 \pmod 2, \\ a_6 - a_5 \equiv 0 \pmod 4, \\ a_7 - a_3 \equiv 0 \pmod 2, \\ a_3 - a_4 - a_7 + a_8 \equiv 0 \pmod 2. \end{cases}$$

Remark 2.3. One can notice that the element θ is written in its usual way (that is not in a Chatelain's written form).

Proof 2.1. • (A) Case 1.

(i) $\Rightarrow$ (ii)

Let us suppose that θ is an integer of K_3. So:

$$\theta=\omega_1+\omega_2\sqrt{dm}+\omega_3\sqrt{dn}+\omega_4\sqrt{mn}+\omega_5\sqrt{d'm'n'l}+\omega_6\sqrt{\frac{dm}{d'm'}n'l}+\omega_7\sqrt{\frac{dn}{d'n'}m'l}$$

$$+\omega_8\sqrt{\frac{mn}{m'n'}d'l}\in\mathbb{Z}_{K_3},\text{ with }\omega_i\in\mathbb{Q}.$$

And following Remark 2.2. (1), we can rewrite θ on the Chatelain's β-basis of K_3 as following:

$$\theta=\omega_1+\omega_2\sqrt{dm}+\omega_3\sqrt{dn}+\omega_4 s_1(s_1\sqrt{mn})+\omega_5\sqrt{d'm'n'l}+\omega_6 s_2\left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right)$$

$$+\omega_7 s_3\left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right)+\omega_8 s_4\left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right)\in\mathbb{Z}_{K_3}.$$

We can too rewrite the integer θ on the basis $\mathfrak{B}'_{K_3}$ of $\mathbb{Z}_{K_3}$ (cf. Lemma 1.4. (1)). So there exist $(x_i)_{1\le i\le 8}\in\mathbb{Z}$ such that:

$$\theta=x_1\varepsilon'_1+x_2\varepsilon'_2+x_3\varepsilon'_3+x_4\varepsilon'_4+x_5\varepsilon'_5+x_6\varepsilon'_6+x_7\varepsilon'_7+x_8\varepsilon'_8.$$

Let us develop and factorize this last expression on Chatelain's β-basis of K_3, then we get:

$$\theta=\frac{x_1}{8}+\frac{1}{8}(x_1-2x_2)\sqrt{dm}+\frac{1}{8}(x_1-2x_3)\sqrt{dn}+\frac{1}{8}(x_1-2x_2+2x_3-4x_4)(s_1\sqrt{mn})$$

$$+\frac{1}{8}(x_1-2x_5)\sqrt{d'm'n'l}+\frac{1}{8}(x_1-2x_2-2x_5+4x_6)\left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right)$$

$$+\frac{1}{8}(x_1-2x_3+2x_5-4x_7)\left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right)$$

$$+\frac{1}{8}(x_1-2x_2+2x_3-4x_4+2x_5-4x_6+4x_7-8x_8)\left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

Let us put $(a_i)_{1\le i\le 8}\in\mathbb{Z}$, such that:

$$\begin{cases} a_1 = x_1, \\ a_2 = x_1 - 2x_2, \\ a_3 = x_1 - 2x_3, \\ a_4 = x_1 - 2x_2 + 2x_3 - 4x_4, \\ a_5 = x_1 - 2x_5, \\ a_6 = x_1 - 2x_2 - 2x_5 + 4x_6, \\ a_7 = x_1 - 2x_3 + 2x_5 - 4x_7, \\ a_8 = x_1 - 2x_2 + 2x_3 - 4x_4 + 2x_5 - 4x_6 + 4x_7 - 8x_8. \end{cases}$$

Replace by these $(a_i)_{1\le i\le 8}$ the values $(x_i)_{1\le i\le 8}$ in the expression of θ, let us identify these 2 expressions of θ, we get our first point, that is:

$\exists\,(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8$, such that:

$$\theta = \frac{1}{8}\Big(a_1 + a_2\sqrt{dm} + a_3\sqrt{dn} + s_1a_4\sqrt{mn} + a_5\sqrt{d'm'n'l} + s_2a_6\sqrt{\frac{dm}{d'm'}n'l}$$

$$+ s_3a_7\sqrt{\frac{dn}{d'n'}m'l} + s_4a_8\sqrt{\frac{mn}{m'n'}d'l}\Big).$$

That demonstrates the first point (A_1) of (A).

Moreover, to demonstrate the second point (A_2) of (A), let us solve $(x_i)_{1\le i\le 8}$ from the $(a_i)_{1\le i\le 8}$, we get:

$$\begin{cases} x_1 = a_1 \in \mathbb{Z}, \\ x_2 = \dfrac{a_1 - a_2}{2} \in \mathbb{Z}, \\ x_3 = \dfrac{a_1 - a_3}{2} \in \mathbb{Z}, \\ x_4 = \dfrac{a_1 + a_2 - a_3 - a_4}{4} \in \mathbb{Z}, \\ x_4 = \dfrac{a_1 - a_5}{2} \in \mathbb{Z}, \\ x_5 = \dfrac{a_1 - a_2 - a_5 + a_6}{4} \in \mathbb{Z}, \\ x_7 = \dfrac{a_1 + a_3 - a_5 - a_7}{4} \in \mathbb{Z}, \\ x_8 = \dfrac{a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8}{8} \in \mathbb{Z}. \end{cases}$$

That is the requested system of congruences on the $(a_i)_{1\le i\le 8}$, of point (A_2).

(ii) $\Rightarrow$ (i)

Let us suppose that there exists $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8$, such that (ii) is realized. We have to show that θ is an integer of K_3.

From the congruences system, let us put the $(x_i)_{1\le i\le 8}$ in the function $(a_i)_{1\le i\le 8}$, as done above. Clearly the $(x_i)_{1\le i\le 8}$ belongs to $\mathbb{Z}$.

Now let us develop this following expression which is an integer of K_3:

$$x_1\varepsilon_1' + x_2\varepsilon_2' + x_3\varepsilon_3' + x_4\varepsilon_4' + x_5\varepsilon_5' + x_6\varepsilon_6' + x_7\varepsilon_7' + x_8\varepsilon_8'.$$

We find that:

$$\begin{aligned} & x_1\varepsilon_1' + x_2\varepsilon_2' + x_3\varepsilon_3' + x_4\varepsilon_4' + x_5\varepsilon_5' + x_6\varepsilon_6' + x_7\varepsilon_7' + x_8\varepsilon_8' \\ &= \frac{1}{8}\Big(a_1 + a_2\sqrt{dm} + a_3\sqrt{dn} + s_1a_4\sqrt{mn} + a_5\sqrt{d'm'n'l} + s_2a_6\sqrt{\frac{dm}{d'm'}n'l} \\ &\quad + s_3a_7\sqrt{\frac{dn}{d'n'}m'l} + s_4a_8\sqrt{\frac{mn}{m'n'}d'l}\Big) \\ &= \theta. \end{aligned}$$

In consequence, as announced in the point (ii) of case 1:

$$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4\sqrt{mn} + \omega_5\sqrt{d'm'n'l} + \omega_6\sqrt{\frac{dm}{d'm'}n'l} + \omega_7\sqrt{\frac{dn}{d'n'}m'l}$$

$$+ \omega_8\sqrt{\frac{mn}{m'n'}d'l},\ \omega_i \in \mathbb{Q},\ \text{is an integer of } K_3.$$

(Exactly with $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8) \in \mathbb{Q}^8$, defined by the equalities: $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8) = (\frac{1}{8}a_1, \frac{1}{8}a_2, \frac{1}{8}a_3, \frac{1}{8}s_1a_4, \frac{1}{8}a_5, \frac{1}{8}s_2a_6, \frac{1}{8}s_3a_7, \frac{1}{8}s_4a_8))$.

• (B) Case 2 and (C) Case 3.

(i) $\Rightarrow$ (ii)

The proof remains the same, using respectively the corresponding Chatelain's β-basis.

For case 2.

$$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4 s_1(s_1\sqrt{mn}) + \omega_5\gamma\left(\gamma\sqrt{d'm'n'l}\right)$$

$$+ \omega_6\gamma s_2\left(\gamma s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \omega_7\gamma s_3\left(\gamma s_3\sqrt{\frac{dn}{d'n'}m'l}\right)$$

$$+ \omega_8\gamma s_4\left(\gamma s_4\sqrt{\frac{mn}{m'n'}d'l}\right) \in \mathbb{Z}_{K_3},$$

and for case 3:

$$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\delta\left(\delta\sqrt{dn}\right) + \omega_4\delta s_1(\delta s_1\sqrt{mn}) - \omega_5(\sqrt{-d'm'n'l})$$

$$- \omega_6 s_2\left(-s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}\right) - \omega_7\delta s_3\left(-\delta s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}\right)$$

$$- \omega_8\delta s_4(-\delta s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}) \in \mathbb{Z}_{K_3}.$$

And using the suitable bases $(\varepsilon_i')_{1\le i\le 8}$ of Lemma 1.4. (2) and (3) for the Cases (2) and (3), then rewrite θ on them, and let us factorize on corresponding Chatelain's β-basis, we find:

For case 2.

$$\theta = \frac{1}{4}(x_1 + (x_1 - 2x_2)\sqrt{dm} + (x_1 - 2x_2 - 2x_3)\sqrt{dn} + (x_1 - 2x_3 - 4x_4)(s_1\sqrt{mn})$$

$$+ x_5(\gamma\sqrt{d'm'n'l}) + (x_5 - 2x_6)(\gamma s_2\sqrt{\frac{dm}{d'm'}n'l}) + (x_5 - 2x_6 - 2x_7)(\gamma s_3\sqrt{\frac{dn}{d'n'}m'l})$$

$$+ (x_5 - 2x_7 - 4x_8)(\gamma s_4 \sqrt{\frac{mn}{m'n'} d'l}),$$

$$\theta = \frac{1}{4} a_1 + \frac{1}{4} a_2 \sqrt{dm} + \frac{1}{4} a_3 \sqrt{dn} + \frac{1}{4} a_4 (s_1 \sqrt{mn}) + \frac{1}{4} a_5 (\gamma \sqrt{d'm'n'l})$$

$$+ \frac{1}{4} a_6 (\gamma s_2 \sqrt{\frac{dm}{d'm'} n'l}) + \frac{1}{4} a_7 (\gamma s_3 \sqrt{\frac{dn}{d'n'} m'l}) + \frac{1}{4} a_8 (\gamma s_4 \sqrt{\frac{mn}{m'n'} d'l}).$$

And which assures us that, what was requested:

$\exists\, (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8,$ such that point (B_1)) is true:

$$\theta = \frac{1}{4}(a_1 + a_2 \sqrt{dm} + a_3 \sqrt{dn} + s_1 a_4 \sqrt{mn} + \gamma a_5 \sqrt{d'm'n'l} + \gamma s_2 a_6 \sqrt{\frac{dm}{d'm'} n'l}$$

$$+ \gamma s_3 a_7 \sqrt{\frac{dn}{d'n'} m'l} + \gamma s_4 a_8 \sqrt{\frac{mn}{m'n'} d'l}).$$

Now extracting the $(x_i)_{1 \le i \le 8}$ from the $(a_i)_{1 \le i \le 8}$, we get the congruences of point (B_2):

$$\begin{cases} a_1 = x_1, \\ \dfrac{a_1 - a_2}{2} = x_2 \in \mathbb{Z}, \\ \dfrac{a_2 - a_3}{2} = x_3 \in \mathbb{Z}, \\ \dfrac{a_1 - a_2 + a_3 - a_4}{4} = x_4 \in \mathbb{Z}, \\ a_5 = x_5 \in \mathbb{Z}, \\ \dfrac{a_5 - a_6}{2} = x_6 \in \mathbb{Z}, \\ \dfrac{a_6 - a_7}{2} = x_7 \in \mathbb{Z}, \\ \dfrac{a_5 - a_6 + a_7 - a_8}{4} = x_8 \in \mathbb{Z}. \end{cases}$$

That concludes Case 2's necessary conditions.

Now for Case 3, we get:

$$\theta = \frac{1}{4}[2x_1 + (2x_1 + 4x_2)\sqrt{dm} + x_3 \delta \sqrt{dn} + (x_3 + 2x_4)\delta s_1 \sqrt{mn} + 2x_5(-\sqrt{d'm'n'l})$$

$$+ (2x_5 + 4x_6)\left(-s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + (x_3 + 2x_7)\left(-\delta s_3\sqrt{\frac{dn}{d'n'}m'l}\right)$$

$$+ (x_3 + 2x_4 + 2x_7 + 2x_8)\left(-\delta s_4\sqrt{\frac{mn}{m'n'}d'l}\right)\Big],$$

$$\theta = \frac{1}{4}a_1 + \frac{1}{4}a_2\sqrt{dm} + \frac{1}{4}a_3\delta\sqrt{dn} + \frac{1}{4}a_4\delta s_1\sqrt{mn} + \frac{1}{4}a_5\left(-\sqrt{d'm'n'l}\right)$$

$$+ \frac{1}{4}a_6\left(-s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \frac{1}{4}a_7\left(-\delta s_3\sqrt{\frac{dn}{d'n'}m'l}\right) + \frac{1}{4}a_8\left(-\delta s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

Then the point (C_1) is proved:

$$\exists\,(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \in \mathbb{Z}^8, \text{ such that:}$$

$$\theta = \frac{1}{4}\Bigg(a_1 + a_2\sqrt{dm} + \delta a_3\sqrt{dn} + \delta s_1 a_4\sqrt{mn} - a_5\sqrt{d'm'n'l} - s_2 a_6\sqrt{\frac{dm}{d'm'}n'l}$$

$$- \delta s_3 a_7\sqrt{\frac{dn}{d'n'}m'l} - \delta s_4 a_8\sqrt{\frac{mn}{m'n'}d'l}\Bigg),$$

and by calculation of the $(x_i)_{1 \le i \le 8}$, we get the congruences (C_2) requested in this last case:

$$\begin{cases} a_1 = 2x_1 \in 2\mathbb{Z}, \\ \dfrac{a_2 - a_1}{4} = x_2 \in \mathbb{Z}, \\ a_3 = x_3 \in \mathbb{Z}, \\ \dfrac{a_4 - a_3}{2} = x_3 \in \mathbb{Z}, \\ a_5 = 2x_5 \in 2\mathbb{Z}, \\ \dfrac{a_6 - a_5}{4} = x_6 \in \mathbb{Z}, \\ \dfrac{a_7 - a_3}{2} = x_7 \in \mathbb{Z}, \\ \dfrac{a_3 - a_4 - a_7 + a_8}{2} = x_8 \in \mathbb{Z}. \end{cases}$$

And that concludes too the Case 3's necessary conditions.

Now the converse (ii) $\Rightarrow$ (i); for the Cases 2 and 3. The proof remains the same that in the first case.

3. Example

We are now giving an example in the following case.

Example 3.1. Case 2; subcase: $(dm, dn, d'm'n'l) \equiv (1, 1, 2) \pmod 4$.

Let us try to build non trivial integers of: $K_3 = \mathbb{Q}\left(\sqrt{-42}, \sqrt{2310}, \sqrt{-154}\right)$ of the type:

$$\theta = \frac{1}{4}(\epsilon_1 + \epsilon_2\sqrt{33} + \epsilon_3\sqrt{-15} + \epsilon_4\sqrt{-55} + \epsilon_5\sqrt{-154} + \epsilon_6\sqrt{-42} + \epsilon_7\sqrt{2310}$$

$$+ \epsilon_8\sqrt{70}) \text{ where } \epsilon_i = \pm 1.$$

First let us show that we are effectively in this subcase mentioned. To see that, let us do the products modulo squares, we find:

$$(33, -15, -55, -154, -42, 2310, 70) \equiv (1, 1, 1, 2, 2, 2, 2) \pmod 4.$$

So

$$K_3 = \mathbb{Q}\left(\sqrt{33}, \sqrt{-15}, \sqrt{-154}\right) = \mathbb{Q}\left(\sqrt{3\times 11}, \sqrt{3\times(-5)}, \sqrt{1\times 11\times 1\times(2\times(-7))}\right).$$

And so:

$$dm = 33,\ dn = -15,\ mn = -55,\ d'm'n'l = -154,\ \frac{dm}{d'm'}n'l = -42,$$

$$\frac{dn}{d'n'}m'l = 2310,\ \frac{mn}{m'n'}d'l = 70.$$

Then:

$$(d, m, n, d', m', n', l) = (3, 11, -5, 1, 11, 1, -14).$$

The signs we need are:

$$s_1 = \lambda_d s(d) = -1,\ s_2 = \lambda_{d'm'} s(d'm') = -1,\ s_3 = \lambda_{d'n'} s(d'n') = 1,$$

$$s_4 = \lambda_{dm'n'} s(dm'n') = 1, \text{ with: } \gamma = \lambda_{d'm'n'\frac{l}{2}} = \lambda_{-77} = -1.$$

And the Chatelain's written form is:

$$K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{2\lambda_{d'm'n'\frac{l}{2}}}\sqrt{\lambda_{d'm'n'\frac{l}{2}} d'm'n'\frac{l}{2}}\right)$$

$$= \mathbb{Q}\left(\sqrt{3\times 11}, \sqrt{3\times(-5)}, \sqrt{-2}\sqrt{-(1\times 11\times 1\times(-7))}\right),$$

where $(\mu, \mu') = (1, \sqrt{-2})$.

The Chatelain's β-basis is:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \gamma\sqrt{d'm'n'l}, \gamma s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l},\right.$$

$$\left.\gamma s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, \gamma s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}$$

$$= \{1, \sqrt{33}, \sqrt{-15}, -\sqrt{-55}, -\sqrt{-154}, \sqrt{-42}, -\sqrt{2310}, -\sqrt{70}\}.$$

Let us come back to our elements θ.

$$\theta = \frac{1}{4}(\epsilon_1 + \epsilon_2\sqrt{33} + \epsilon_3\sqrt{-15} + \epsilon_4\sqrt{-55} + \epsilon_5\sqrt{-154}$$

$$+ \epsilon_6\sqrt{-42} + \epsilon_7\sqrt{2310} + \epsilon_8\sqrt{70}),$$

where $\epsilon_i = \pm 1$, we have:

$$\theta = \frac{1}{4}(\epsilon_1 + \epsilon_2\sqrt{33} + \epsilon_3\sqrt{-15} + (-\epsilon_4)(-\sqrt{-55}) + (-\epsilon_5)(-\sqrt{-154}) + \epsilon_6\sqrt{-42}$$

$$+ (-\epsilon_7)(-\sqrt{2310}) + (-\epsilon_8)(-\sqrt{70})).$$

This means that: $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (\epsilon_1, \epsilon_2, \epsilon_3, -\epsilon_4, -\epsilon_5, \epsilon_6, -\epsilon_7, -\epsilon_8)$.

Now θ belongs to $\mathbb{Z}_{K_3}$ if and only if:

$$\begin{cases} a_1 \in \mathbb{Z} \\ a_1 - a_2 \equiv 0 \pmod 2 \\ a_2 - a_3 \equiv 0 \pmod 2 \\ a_1 - a_2 + a_3 - a_4 \equiv 0 \pmod 4 \\ a_5 \in \mathbb{Z} \\ a_5 - a_6 \equiv 0 \pmod 2 \\ a_6 - a_7 \equiv 0 \pmod 2 \\ a_5 - a_6 + a_7 - a_8 \equiv 0 \pmod 4 \end{cases} \Rightarrow \begin{cases} \epsilon_1 = \pm 1 \Rightarrow \epsilon_0 \in \mathbb{Z} \\ \epsilon_1 - \epsilon_2 \equiv 0 \pmod 2 \\ \epsilon_2 - \epsilon_3 \equiv 0 \pmod 2 \\ \epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 \equiv 0 \pmod 4 \\ \epsilon_5 = \pm 1 \Rightarrow \epsilon_4 \in \mathbb{Z} \\ -\epsilon_5 - \epsilon_6 \equiv 0 \pmod 2 \\ \epsilon_6 + \epsilon_7 \equiv 0 \pmod 2 \\ -\epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8 \equiv 0 \pmod 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} \epsilon_1 = \pm 1, \\ \epsilon_2 = \epsilon_1 - 2k_2,\ k_2 \in \{0, \epsilon_1\}, \\ \epsilon_3 = \epsilon_1 - 2(k_2 + k_3),\ (k_2, k_3) \in \{\{0\} \times \{0, \epsilon_1\}\} \cup \{\{\epsilon_1\} \times \{0, -\epsilon_1\}\}, \\ \epsilon_4 = -\epsilon_1 + 2(k_3 + 2k_4),\ (k_3, k_4) \in \{\{0\} \times \{0\}\} \cup \{\{\epsilon_1\} \times \{0\}\} \cup \{\{-\epsilon_1\} \times \{\epsilon_1\}\}. \end{cases}$$

And because the 4 last equations leave the same that the four first ones, we have:

$$(\epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = (-\epsilon_1, \epsilon_2, -\epsilon_3, \epsilon_4).$$

In conclusion:

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in E = \{(\epsilon_1, \epsilon_1, \epsilon_1, -\epsilon_1)\} \cup \{(\epsilon_1, \epsilon_1, -\epsilon_1, \epsilon_1)\} \cup \{(\epsilon_1, -\epsilon_1, -\epsilon_1, -\epsilon_1)\} \cup \{(\epsilon_1, -\epsilon_1, \epsilon_1, \epsilon_1)\}.$$

$$= \{\pm(1, 1, 1, -1)\} \cup \{\pm(1, 1, -1, 1)\} \cup \{\pm(1, -1, -1, -1)\} \cup \{\pm(1, -1, 1, 1)\}.$$

And then too:

$$(\epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) \in E = \{(-\epsilon_1, \epsilon_1, -\epsilon_1, -\epsilon_1)\} \cup \{(-\epsilon_1, \epsilon_1, \epsilon_1, \epsilon_1)\} \cup \{(-\epsilon_1, -\epsilon_1, \epsilon_1, -\epsilon_1)\} \cup \{(-\epsilon_1, -\epsilon_1, -\epsilon_1, \epsilon_1)\}.$$

This means that the solutions $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)$ of our problem, are obtained by doing: $((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), (\epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)) \in E \times E.$

So: $\theta = \frac{1}{4}(\epsilon_1 + \epsilon_2\sqrt{33} + \epsilon_3\sqrt{-15} + \epsilon_4\sqrt{-55} + \epsilon_5\sqrt{-154} + \epsilon_6\sqrt{-42} + \epsilon_7\sqrt{2310}$ $+ \epsilon_8\sqrt{70})$, where $\epsilon_i = \pm 1$, belongs to $\mathbb{Z}_{K_3} \Leftrightarrow$ The 64 elements solutions of type 8-upplets $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)$, are such that:

$((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), (\epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)) \in E \times E$, where:

$E = \{\pm(1, 1, 1, -1)\} \cup \{\pm(1, 1, -1, 1)\} \cup \{\pm(1, -1, -1, -1)\} \cup \{\pm(1, -1, 1, 1)\}.$

Let us note that these 64 solutions can be classified by using the natural action of $Gal(K_3 / \mathbb{Q})$ on this same set of solutions.

Then we find finally 8 orbits and each of them has a cardinal equal to 8; given by $Gal(K_3 / \mathbb{Q})(\theta_i)$, $i = 1, \ldots, 8$, and each of them is passing, respectively, by only one of the elements:

$$\theta_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}; \theta_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}; \theta_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \theta_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}; \theta_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}; \theta_6 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix};$$

$$\theta_7 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \theta_8 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

For instance, the orbit S_{θ_1} of θ_1 under $Gal(K_3 / \mathbb{Q}) = G$ is the following set:

$$G(\theta_1) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

And the irreducible polynomial of θ_1 over $\mathbb{Q}$, whose other roots just above is (from MAPPLE Calculator):

$$Irr(\theta_1, X) = X^8 - 2X^7 - 535X^6 + 1596X^5 + 111509X^4$$
$$- 226650X^3 - 11395285X^2 + 677800X + 649147150.$$

We get by the same way, the following irreducible polynomials over $\mathbb{Q}$ for θ_i, $i =$ 2, …, 8.

$$Irr(\theta_2, X) = X^8 - 2X^7 - 535X^6 + 1316X^5 + 118789X^4$$
$$- 343410X^3 - 11884725X^2 + 16299000X + 584808750;$$

$$Irr(\theta_3, X) = X^8 - 2X^7 - 535X^6 - 840X^5 + 118019X^4$$
$$+ 419814X^3 - 12451753X^2 - 33307172X + 568782124;$$

$$Irr(\theta_4, X) = X^8 - 2X^7 - 535X^6 + 980X^5 + 105349X^4$$
$$- 350466X^3 - 10605573X^2 + 49155768X + 676247454;$$

$$Irr(\theta_5, X) = X^8 + 2X^7 - 535X^6 - 1596X^5 + 111509X^4$$
$$+ 226650X^3 - 11395285X^2 - 677800X + 649147150;$$

$$Irr(\theta_6, X) = X^8 + 2X^7 - 535X^6 - 1316X^5 + 118789X^4$$

$$+343410X^3 - 11884725X^2 - 16299000X + 584808750;$$

$$Irr(\theta_7, X) = X^8 + 2X^7 - 535X^6 + 840X^5 + 118019X^4$$

$$- 419814X^3 - 12451753X^2 + 33307172X + 568782124;$$

$$Irr(\theta_8, X) = X^8 + 2X^7 - 535X^6 - 980X^5 + 105349X^4$$

$$+350466X^3 - 10605573X^2 - 49155768X + 676247454.$$

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