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BOX TOPOLOGY PRODUCT SPACES AND SEMIREGULAR SPACES

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Abstract

Within this paper, recent discoveries concerning semiregular spaces are used to show semiregular is a box topology product property, i.e., a property simultaneously shared by a product space with the box topology and each of its factor spaces.

1. Introduction

In this paper, all spaces are topological spaces and the topology on all product spaces will be the box topology. Below definitions and results used in this paper are given.

Regularly open sets were introduced in 1937 [7] and used to define the semiregularization process and semiregular spaces.

Definition 1.1. Let (X, T) be a space and let $O \subseteq X$. Then O is regularly open, denoted by $O \in RO(X, T)$, iff $O = Int(Cl(O))$.

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Within a 1987 paper [1], it was shown that for a space (X, T), $RO(X, T) = \{Int(Cl(O)) | O \in T\}$, which is used in this paper.

In the 1937 paper [7], it was shown that the regularly open set for a space (X, T) is a base for a topology Ts on X coarser than T, and the space (X, Ts) was called the semiregularization space of (X, T).

Definition 1.2. A space (X, T) is semiregular iff $RO(X, T)$ is a base for the topology T.

As long known, a regular space is semiregular [8] and the product space of regular spaces with the Tychonoff topology is regular [8]. Thus, in the special case where each factor space is regular, the product space with the Tychonoff topology is semiregular iff each factor space is semiregular raising the question of whether the result could be extended to all semiregular spaces. If so, could the result be extended to product spaces with the box topology? Within the paper [2], a positive answer was obtained for the first question leaving only the second question to be investigated in this paper.

Within the recent paper [2], feebly open set generated spaces played an important role when considering product spaces and semiregular spaces. Feebly open sets were introduced in 1978 [6].

Definition 1.3. Let (X, T) be a space and let $A \subseteq X$. Then A is feebly open, denoted by $A \in FO(X, T)$, iff there exists an $O \in T$ such that $O \subseteq A \subseteq scl(O)$.

In 1987 paper [1], the following useful characterization was given for feebly open sets.

Definition 1.4. Let (X, T) be a space and let $A \subseteq X$. Then A is feebly open iff there exists an $O \in T$ such that $O \subseteq A \subseteq Int(Cl(O))$.

In 1985 [3], it was shown that for a space (X, T), $FO(X, T)$ is a topology on X and $FO(X, FO(X, T)) = FO(X, T)$. Within the 1987 paper [1], it is shown that for a space (X, T), $RO(X, T) = RO(X, FO(X, T))$. In a 2014 paper [4], it was shown that for a space (X, T), the semiregularization process creates at most one new topology on X, which was combined with results above in a 2015 paper [5] to prove for each space (X, T), $(X, Ts) = (X, (FO(X, T))s)$ is semiregular, (X, T) is semiregular iff $T = Ts = (FO(X, T))s$, and $(X, (FO(X, T))$ is semiregular iff $FO(X, T) = (FO(X, T))s$, in which case $(X, T) = (X, FO(X, T)) = (X, (FO(X, T))s = (X, Ts)$.

Efforts to resolve the question concerning product spaces with the box topology and semiregular spaces led to questions concerning the behavior of a topological base in the semiregularization process, which are resolved below.

2. The Behavior of a Base in the Semiregularization Process

Theorem 2.1. *Let (X, T) be a space and let $\mathcal{B}$ be a base for T. Then $\mathcal{B}s = \{Int(Cl(B)) \mid B \in \mathcal{B}\}$ is a base for Ts.*

Proof. Since $\mathcal{S} = \{Int(Cl(O)) \mid O \in T\}$ is a base for Ts and $\mathcal{B} \subseteq T$, then $\mathcal{B}s \subseteq \mathcal{S} \subseteq Ts$. Let $U \in Ts$. Let $x \in U$. Let $V \in T$ such that $x \in Int(Cl(V)) \subseteq U$. Let $B \in \mathcal{B}$ such that $x \in B \subseteq Int(Cl(O))$. Then $Int(Cl(B)) \in \mathcal{B}s$ and $x \in Int(Cl(B)) \subseteq Int(Cl(V)) \subseteq U$. Thus $\mathcal{B}s$ is a base for Ts.

Definition 2.1. For each $\alpha \in A$, let (X_α, T_α) be a space, and let $X = \prod_{\alpha \in A} X_\alpha$. Then the box topology B on X is the topology on X with base $\mathcal{B} = \left\{\prod_{\alpha \in A}(O_\alpha) \mid O_\alpha \in T_\alpha \text{ for all } \alpha \in A\right\}$.

Theorem 2.2. *For each $\alpha \in A$, let (X_α, T_α) be a space, let B be the box topology on $\prod_{\alpha \in A}(X_\alpha, T_\alpha)$, let $X = \prod_{\alpha \in A} X_\alpha$, and $\mathcal{O} = \prod_{\alpha \in A} O_\alpha$ such that $O_\alpha \in T_\alpha$ for each $\alpha \in A$. Then $Int(Cl(\mathcal{O})) = \prod_{\alpha \in A} Int(Cl(O_\alpha))$, where for $Int(Cl(\mathcal{O}))$, both the Int and Cl are in (X, B).*

Proof. If for each $\alpha \in A$, $Cl(O_\alpha) = X_\alpha$, then $Cl(\mathcal{O}) \subseteq \prod_{\alpha \in A} Cl(O_\alpha)$. Thus consider the case that $D = \{\alpha \in A \mid Cl(O_\alpha) \neq X_\alpha\} \neq \phi$.

For each $\beta \in D$, let $U_{(\beta, \beta)} = X_\beta \setminus Cl(O_\beta)$ and for each $\alpha \in A$, $\alpha \neq \beta$. Let

$U_{(\alpha,\beta)} = X_\alpha$. Let $x \notin \prod_{\alpha\in A} Cl(O_\alpha)$. Let $\mu \in A$ such that $x_\mu \notin Cl(O_\mu)$. Then $x \in \prod_{\alpha\in A} U_{(\alpha,\mu)} = \mathcal{U}_\mu \in B$ and $\mathcal{U}_\mu \cap \prod_{\alpha\in A} Cl(O_\alpha) = \phi$. Thus $\prod_{\alpha\in A} Cl(O_\alpha)$ is closed in (X, B). Hence, in both cases, $Cl(\mathcal{O}) = Cl\left(\prod_{\alpha\in A}(O_\alpha)\right) \subseteq \prod_{\alpha\in A} Cl(O_\alpha)$.

Let $x \in \prod_{\alpha\in A} Cl(O_\alpha)$. Let $\prod_{\alpha\in A} W_\alpha$ be a B-base element containing x. Then $x_\alpha \in W_\alpha \in T_\alpha$ and $x_\alpha \in W_\alpha \cap Cl(O_\alpha)$, which implies $W_\alpha \cap O_\alpha \neq \phi$ for all $\alpha \in A$. Thus $x \in Cl(\mathcal{O}) = Cl\left(\prod_{\alpha\in A} O_\alpha\right)$. Hence $\prod_{\alpha\in A} Cl(O_\alpha) \subseteq Cl(\mathcal{O})$. Therefore $\prod_{\alpha\in A} Cl(O_\alpha) = Cl(\mathcal{O})$.

Since $\prod_{\alpha\in A} Int(Cl(O_\alpha)) \in B$ and $\prod_{\alpha\in A} IntCl(O_\alpha) \subseteq Cl(\mathcal{O})$, then $\prod_{\alpha\in A} Int(Cl(O_\alpha)) \subseteq Int(Cl(\mathcal{O}))$.

Let $y \in Int(Cl(\mathcal{O})) = Int\left(\prod_{\alpha\in A} Cl(O_\alpha)\right)$. Let $\prod_{\alpha\in A} Z_\alpha$ be a B-base element containing y such that $\prod_{\alpha\in A} Z_\alpha \subseteq Int\left(\prod_{\alpha\in A} Cl(O_\alpha)\right)$. Then $\prod_{\alpha\in A} Z_\alpha \subseteq \prod_{\alpha\in A} Cl(O_\alpha)$, which implies $Z_\alpha \subseteq Cl(O_\alpha)$ for each $\alpha \in A$. Thus $y_\alpha \in Z_\alpha \subseteq Int(Cl(O_\alpha))$ and $y \in \prod_{\alpha\in A} Int(Cl(O_\alpha))$. Hence $Int(Cl(\mathcal{O})) \subseteq \prod_{\alpha\in A} Int(Cl(O_\alpha))$. Therefore $Int(Cl(\mathcal{O})) = \prod_{\alpha\in A} Int(Cl(O_\alpha))$

3. Product Spaces with the Box Topology and Semiregular Spaces

Theorem 3.1. *For each* $\alpha \in A$, *let* (X_α, T_α) *be a space, let B be the box topology on* $\prod_{\alpha\in A}(X_\alpha, T_\alpha)$, *let Z be the box topology on* $\prod_{\alpha\in A}(X_\alpha, (T_\alpha)s)$, *and let* $X = \prod_{\alpha\in A} X_\alpha$. *Then* $Z = Bs$ *and* $(X, Z) = (X, Bs)$, *which is semiregular.*

Proof. By Theorems 2.1 and 2.2, $\mathcal{B}s = \left\{\prod_{\alpha\in A} Int(Cl(O_\alpha)) \mid O_\alpha \in T_\alpha\right\}$ is a base for Bs. Since for each $\alpha \in A$, $Int(Cl(O_\alpha)) \in (T_\alpha))s$, then $\mathcal{B}s \subseteq$

$\{\prod_{\alpha\in A} U_\alpha \mid U_\alpha \in (T_\alpha)s\}$ and $Bs \subseteq Z$. Let $\mathcal{U} \in Z$. Let $x \in \mathcal{U}$. For each $\alpha \in A$, let $V_\alpha \in (T_\alpha)s$ such that $x_\alpha \in V_\alpha$ and $\prod_{\alpha\in A} V_\alpha \subseteq \mathcal{U}$. For each $\alpha \in A$, let $O_\alpha \in T_\alpha$ such that $x_\alpha \in Int(Cl(O_\alpha)) \subseteq V_\alpha$. Then $x \in \prod_{\alpha\in A} Int(Cl(O_\alpha)) \in \mathcal{B}s$ and $\prod_{\alpha\in A} Int(Cl(O_\alpha)) \subseteq U$. Thus $\mathcal{B}s$ is a base for Z, $Z = Bs$, and $(X, Z) = (X, Bs)$. By the results above, (X, Bs) is semiregular.

Corollary 3.1. *The semiregularization space of a product space with the box topology is the product of the semiregularization space of each factor space.*

Theorem 3.2. *Let* X, B, *and* Z *be as in Theorem* 3.1. *Then* (a) (X_α, T_α) *is semiregular for each* $\alpha \in A$ *iff* (b) (X, B) *is semiregular.*

Proof. (a) implies (b): Since $T_\alpha = (T_\alpha)s$ for each $\alpha \in A$, then $B = Z = Bs$ and (X, B) is semiregular.

(b) implies (a): Since (X, B) is semiregular, then $(X, B) = (X, Bs)$. Then for each $\alpha \in A$, $(X_\alpha, T_\alpha) = (X_\alpha, (T_\alpha)s)$ and (X_α, T_α) is semiregular for each $\alpha \in A$.

Theorem 3.3. *Let* X, B, *and* Z *be as in Theorem* 3.1, *and let* F *be the box topology on* $\prod_{\alpha\in A}(X_\alpha, FO(X_\alpha, T_\alpha))$. *Then* $Fs = Bs$ *and* $(X, Fs) = (X, Bs)$, *which is semiregular.*

Proof. By Theorem 3.2, the product topology on $\prod_{\alpha\in A}(X_\alpha, FO(X_\alpha, T_\alpha))s)$ is Fs. Since for each $\alpha \in A$, $FO(X_\alpha, T_\alpha))s = (T_\alpha)s$, then $Fs = Bs$, which is semiregular.

Combining the results above gives the last result in this paper.

Corollary 3.2. *For each* $\alpha \in A$, *let* (X_α, T_α) *be a space. Then* $(X_\alpha, FO(X_\alpha, T_\alpha))$ *is semiregular for all* $\alpha \in A$ *iff* (X, F) *is semiregular, in which case* $(X, B) = (X, Bs) = (X, (FO(X, B))s) = (X, F) = (X, Fs) = (X, (FO(X, F))s)$.

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