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BANACH CONTRACTION MAPPING THEOREM IN η-CONE RECTANGULAR METRIC SPACE

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Abstract

The concept of η-cone metric space appeared in [1]. On the other hand the concept of cone rectangular metric space appeared in [2]. In the present paper, we combine the notions of cone rectangular metric space and η-cone metric space to form η-cone rectangular metric space and prove the Banach contraction mapping theorem in this setting

1. Introduction and Preliminaries

Definition 1.1 (Huang and Zhang [3]). Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. P is called a cone if and only if

(a) P is closed, nonempty, and $P \neq \{\theta\}$, where θ is the zero vector in E;

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(b) For any nonnegative real numbers a and b, and $x, y \in P$, we have $ax + by \in P$;

(c) for $x \in P$, if $-x \in P$, then $x = \theta$.

Definition 1.2 (Huang and Zhang [3]). Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \quad \Leftrightarrow \quad y - x \in \operatorname{int}(P).$$

We shall write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{Int}(P)$, where $\operatorname{Int}(P)$ designates the interior of P.

Definition 1.3 (Huang and Zhang [3]). The cone P is said to be normal if there is a real number $C > 0$ such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. In particular, we will say that P is a K-normal cone to indicate the fact that the normal constant is K

Definition 1.4 (Azam et al. [2]). Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space.

Example 1.5 (Azam et al. [2]). Let $X = \mathbb{N}$, $E = \mathbb{R}^2$, and

$$P = \{(x, y) : x, y \geq 0\}.$$

Define $d : X \times X \mapsto E$ by

$$d(x, y) = \begin{cases} (0, 0), & \text{if } x = y, \\ (3, 9), & \text{if } x \text{ and } y \text{ are in } \{1, 2\} \text{ and } x \neq y, \\ (1, 3), & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone rectangular metric space, but (X, d) is not a cone metric space, since it lacks the triangular property:

$$(3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 3) + (1, 3) = (2, 6)$$

as $(3, 9) - (2, 6) = (1, 3) \in P$.

Definition 1.6 (Gaba [1])**.** Let X be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an η-cone metric on X if

(a) $\theta \preceq d_\eta(x, y)$ for all $x \in X$ and $d_\eta(x, y) = \theta$ iff $x = y$;

(b) $d_\eta(x, y) = d_\eta(y, x)$ for all $x, y \in X$;

(c) $d_\eta(x, z) \preceq \eta(x, z)[d_\eta(x, y) + d_\eta(y, z)]$ for all $x, y, z \in X$.

Moreover, the pair (X, d_η) is called an η-cone metric space.

Remark 1.7 (Gaba [1])**.** If for all $x, y \in X$

(a) $\eta(x, y) = 1,$ then we obtain the definition of cone metric space (Huang and Zhang [3]).

(b) $\eta(x, y) = L,$ where $L \geq 1,$ then we obtain the definition of cone metric type space (Cvetkovic et al. [4]).

(c) $\eta(x, y) = C,$ where $C \geq 1, E = \mathbb{R}$ and $P = [0, \infty),$ then we obtain the definition of metric type space (Khamsi [5]).

Example 1.8 (Gaba [1]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$ and $X = \{1, 2, 3\}$. Let $\alpha \geq 0$ be a constant and define $\eta : X \times X \mapsto [1, \infty)$ and $d_\eta : X \times X \mapsto E$ by

$$\eta(x, y) = 1 + x + y,$$

$$d_\eta(1, 1) = d_\eta(2, 2) = d_\eta(3, 3) = (0, 0),$$

$$d_\eta(1, 2) = d_\eta(2, 1) = 80(1, \alpha),$$

$$d_\eta(1, 3) = d_\eta(3, 1) = 1000(1, \alpha),$$

$$d_\eta(2, 3) = d_\eta(3, 2) = 600(1, \alpha).$$

Then (X, d_η) is an η-cone metric space.

Now we introduce the following

Definition 1.9. Let X be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an η-cone rectangular metric on X if

(a) $\theta \preceq d_\eta(x, y)$ for all $x, y \in X$ and $d_\eta(x, y) = \theta$ iff $x = y$;

(b) $d_\eta(x, y) = d_\eta(y, x)$ for all $x, y \in X$;

(c) $d_\eta(x, y) \leq \eta(x, y)[d_\eta(x, w) + d_\eta(w, z) + d_\eta(z, y)]$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$.

Moreover, the pair (X, d_η) will be called an η-cone rectangular metric space.

Example 1.10. Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$. Define $\eta : X \times X \mapsto [1, \infty)$ and $d_\eta : X \times X \mapsto E$ by

$$d_\eta(1, 1) = d_\eta(2, 2) = d_\eta(3, 3) = d_\eta(4, 4) = (0, 0),$$

$d_\eta(1, 2) = d_\eta(2, 1) = (3, 6),$

$d_\eta(2, 3) = d_\eta(3, 2) = d_\eta(1, 3) = d_\eta(3, 1) = (1, 2),$

$d_\eta(1, 4) = d_\eta(4, 1) = d_\eta(2, 4) = d_\eta(4, 2) = d_\eta(3, 4) = d_\eta(4, 3) = (2, 4),$

$\eta(x, y) = x + y - 2.$

Then (X, d_η) is an η-cone rectangular metric space but not an η-cone metric space.

Proof. By definition of d_η, it is trivial to check Definition 1.9(a), and Definition 1.9(b). Now we check Definition 1.9(c)

$(3, 6) = d_\eta(1, 2) < \eta(1, 2)[d_\eta(1, 3) + d_\eta(3, 4) + d_\eta(4, 2)] = [(1, 2) + (2, 4) + (2, 4)]$
$= (5, 10),$

$(1, 2) = d_\eta(1, 3) < \eta(1, 3)[d_\eta(1, 2) + d_\eta(2, 4) + d_\eta(4, 3)] = 2[(3, 6) + (2, 4) + (2, 4)]$
$= (14, 28),$

$(2, 4) = d_\eta(1, 4) < \eta(1, 4)[d_\eta(1, 2) + d_\eta(2, 3) + d_\eta(3, 4)] = 3[(3, 6) + (1, 2) + (2, 4)]$
$= (18, 36),$

$(1, 2) = d_\eta(2, 3) < \eta(2, 3)[d_\eta(2, 1) + d_\eta(1, 4) + d_\eta(4, 3)] = 3[(3, 6) + (2, 4) + (2, 4)]$
$= (21, 42),$

$(2, 4) = d_\eta(2, 4) < \eta(2, 4)[d_\eta(2, 1) + d_\eta(1, 3) + d_\eta(3, 4)] = 4[(3, 6) + (1, 2) + (2, 4)]$
$= (24, 48),$

$(2, 4) = d_\eta(3, 4) < \eta(3, 4)[d_\eta(3, 1) + d_\eta(1, 2) + d_\eta(2, 4)] = 5[(1, 2) + (3, 6) + (2, 4)]$
$= (30, 60).$

It follows that (X, d_η) is an η-cone rectangular metric space. Finally (X, d_η) is not an η-cone metric space, since it lacks Definition 2.2(d3) [1] as:

$$(3, 6) = d_\eta(1, 2) > \eta(1, 2)[d_\eta(1, 3) + d_\eta(3, 2)] = (2, 4)$$

as $(3, 6) - (2, 4) = (1, 2) \in P$.

Definition 1.11. Let (X, d_η) be an η-cone rectangular metric space. If for every $c \in E$, with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to $x \in X$, and $x \in X$ is the limit of the sequence $\{x_n\}$. We shall write $\lim_n x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.12. Let (X, d_η) be an η-cone rectangular metric space. If for every $c \in E$, with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ will be called a Cauchy sequence.

Definition 1.13. Let (X, d_η) be an η-cone rectangular metric space. If every Cauchy sequence in X converges in X, then (X, d_η) will be called a complete η-cone rectangular metric space.

2. Main Result

In a similar way as Lemma 2.9 [1], we have the following, whose proof can be completed in a similar fashion as Lemma 1 [3] .

Lemma 2.1. *Let (X, d_η) be an η-cone rectangular metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x iff*

$$d_\eta(x_n, x) \to \theta \text{ as } n \to \infty.$$

In a similar way as Lemma 2.12 [1], we have the following, whose proof can be completed in a similar fashion as Lemma 4 [3].

Lemma 2.2. *Let* (X, d_η) *be an* η*-cone rectangular metric space,* P *be a normal cone with normal constant* K. *Let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is Cauchy iff*

$$d_\eta(x_n, x_m) \to \theta \text{ as } n, m \to \infty.$$

Now our main result is as follows

Theorem 2.3. *Let* (X, d_η) *be a complete* η*-cone rectangular metric space,* P *be a normal cone with normal constant* K, *and the mapping* $T : X \mapsto X$ *satisfies*

$$d_\eta(Tx, Ty) \leq \lambda d_\eta(x, y)$$

for all $x, y \in X$, *where* $0 \leq \lambda < 1$. *Then* T *has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. Define a sequence of points in X as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0, \; n = 0, 1, 2, \cdots.$$

We suppose x_0 is not a periodic point, in fact, if $x_n = x_0$, then we obtain

$$\begin{aligned} d_\eta(x_0, Tx_0) &= d_\eta(x_n, Tx_n) \\ &= d_\eta(T^n x_0, T^{n+1}x_0) \\ &\leq \lambda d_\eta(T^{n-1}x_0, T^n x_0) \\ &\leq \lambda^2 d_\eta(T^{n-2}x_0, T^{n-1}x_0) \\ &\vdots \\ &\leq \lambda^n d_\eta(x_0, Tx_0). \end{aligned}$$

It follows that $(\lambda^n - 1)d_\eta(x_0, Tx_0) \in P$, which further implies $-d_\eta(x_0, Tx_0) =$

$\frac{\lambda^n - 1}{1 - \lambda^n} d_\eta(x_0, Tx_0) \in P$, and $d_\eta(x_0, Tx_0) = 0$, and so x_0 is a fixed point of T. In what follows, we suppose $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Now by using Definition 1.9(c), for all $y \in X$, we have

$$d_\eta(y, T^4 y) \leq \eta(y, T^4 y)[d_\eta(y, Ty) + d_\eta(Ty, T^2 y) + d_\eta(T^2 y, T^4 y)]$$

$$\leq \eta(y, T^4 y) d_\eta(y, Ty) + \lambda\eta(y, T^4 y) d_\eta(y, Ty)$$

$$+ \lambda^2 \eta(y, T^4 y) d_\eta(y, T^2 y).$$

Similarly, for all $y \in X$, we have

$$d_\eta(y, T^6 y) \leq \eta(y, T^6 y)[d_\eta(y, Ty) + d_\eta(Ty, T^2 y) + d_\eta(T^2 y, T^3 y) + d_\eta(T^3 y, T^4 y)$$

$$+ d_\eta(T^4 y, T^6 y)]$$

$$\leq \eta(y, T^4 y) d_\eta(y, Ty) + \lambda\eta(y, T^4 y) d_\eta(y, Ty) + \lambda^2 \eta(y, T^4 y) d_\eta(y, Ty)$$

$$+ \lambda^3 \eta(y, Ty) d_\eta(y, T^2 y) + \lambda^4 \eta(y, T^4 y) d_\eta(y, T^2 y)$$

$$\leq \sum_{i=0}^{3} \lambda^i \eta(y, T^4 y) d_\eta(y, Ty) + \lambda^4 \eta(y, T^4 y) d_\eta(y, T^2 y).$$

Now by induction, we obtain for each $k = 2, 3, 4, \cdots$

$$d_\eta(y, T^{2k} y) \leq \sum_{i=0}^{2k-3} \lambda^i \eta(y, T^{2k} y) d_\eta(y, Ty) + \lambda^{2k-2} \eta(y, T^{2k} y) d_\eta(y, T^2 y).$$

Moreover, for all $y \in X$,

$$d_\eta(y, T^5 y) \leq \eta(y, T^5 y)[d_\eta(y, Ty) + d_\eta(Ty, T^2 y)$$

$$+ d_\eta(T^2 y, T^3 y) + d_\eta(T^3 y, T^4 y) + d_\eta(T^4 y, T^5 y)]$$

$$\leq \sum_{i=0}^{4} \eta(y, T^5 y)\lambda^i d_\eta(y, Ty).$$

By induction, for each $k = 0, 1, 2, \cdots,$ we have

$$d_\eta(y, T^{2k+1} y) \leq \sum_{i=0}^{2k} \eta(y, T^{2k+1} y)\lambda^i d_\eta(y, Ty).$$

Using $d_\eta(y, T^{2k} y) \leq \sum_{i=0}^{2k-3} \lambda^i \eta(y, T^{2k} y) d_\eta(y, Ty) + \lambda^{2k-2}\eta(y, T^{2k} y) d_\eta(y, T^2 y),$ for $k = 1, 2, 3, \cdots,$ we have

$$d_\eta(T^n x_0, T^{n+2k} x_0) \leq \lambda^n d_\eta(x_0, T^{2k} x_0)$$

$$\leq \lambda^n \left[\sum_{i=0}^{2k-3} \lambda^i \eta(y, T^{2k} y)(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2 x_0)) \right.$$

$$\left. + \lambda^{2k-2}\eta(y, T^{2k} y)(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2 x_0)) \right]$$

$$\leq \lambda^n \sum_{i=0}^{2k-2} \lambda^i \eta(y, T^{2k} y)(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2 x_0))$$

$$\leq \frac{\lambda^n \eta(y, T^{2k} y)(1 - \lambda^{2k-1})}{1 - \lambda}(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2 x_0))$$

$$\leq \frac{\lambda^n \eta(y, T^{2k} y)}{1 - \lambda}(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2 x_0)).$$

Similarly, for $k = 0, 1, 2, \cdots,$ and from $d_\eta(y, T^{2k+1} y) \leq \sum_{i=0}^{2k} \eta(y, T^{2k+1} y)\lambda^i$ $d_\eta(y, Ty),$ we deduce the following:

$$d_\eta(T^n x_0, T^{n+2k+1}x_0) \le \lambda^n d_\eta(x_0, T^{2k+1}x_0)$$

$$\le \lambda^n\left[\sum_{i=0}^{2k}\lambda^i\eta(y, T^{2k+1}y)(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2x_0))\right]$$

$$\le \frac{\lambda^n\eta(y, T^{2k+1}y)(1-\lambda^{2k+1})}{1-\lambda}(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2x_0))$$

$$\le \frac{\lambda^n\eta(y, T^{2k+1}y)}{1-\lambda}(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2x_0)).$$

It follows that if $n < m,$ then

$$d_\eta(T^n x_0, T^m x_0) \le \frac{\lambda^n\eta(y, T^{m-n}y)}{1-\lambda}(d_\eta(x_0, Tx_0) + d_\eta(x_0, T^2x_0)).$$

Taking norm to inequality in the above, and then taking limits as $n \to \infty,$ we deduce that

$$\lim_{n,m\to\infty}\|d_\eta(T^n x_0, T^m x_0)\| = 0,$$

and Lemma 2.2 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there is $u \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 = u.$ Now we show u is a fixed point of T. Observe that

$$d_\eta(Tu, u) \le \eta(Tu, u)[d_\eta(Tu, T^n x_0) + d_\eta(T^n x_0, T^{n+1}x_0) + d_\eta(T^{n+1}x_0, u)]$$

$$\le \eta(Tu, u)[\lambda d_\eta(u, T^{n-1}x_0) + d_\eta(T^n x_0, T^{n+1}x_0) + d_\eta(T^{n+1}x_0, u)].$$

Now taking norm to inequality in the above, and then taking limits as $n \to \infty,$ we deduce that

$$\lim_{n\to\infty}\|d_\eta(Tu, u)\| = 0,$$

which implies $Tu = u,$ and so u is a fixed point of T. For uniqueness of the fixed point, suppose $w = Tw$, but $w \neq u$, then observe we have the following:

$$d_\eta(u, w) = d_\eta(Tu, Tw) \leq \lambda d_\eta(u, w) < d_\eta(u, w),$$

which is a contradiction, and uniqueness follows.

Now we have the following example in support of the main result.

Example 2.4. Let X, E, P and d_η be defined as in Example 1.10. As that example showed, (X, d_η) is an η-cone rectangular metric space, but not an η-cone metric space. Now define a mapping $T : X \mapsto X$ by

$$Tx = \begin{cases} 3, & \text{if } x \neq 4, \\ 1, & \text{if } x = 4. \end{cases}$$

Note that

$$d_\eta(T(1), T(2)) = d_\eta(T(1), T(3)) = d_\eta(T(2), T(3)) = (0, 0)$$

and in all other cases

$$d_\eta(Tx, Ty) = (1, 2), \quad d_\eta(x, y) = (2, 4).$$

Hence for $\lambda = \frac{1}{2}$, all conditions of the previous theorem hold and $3 \in X$ is the unique fixed point.

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