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AVERAGE VALUE OF VOLUMES OF BALLS IN FINSLER MANIFOLDS

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Abstract

In this paper, we give the Berger-Kazdan inequality and Santaló's formula in Finsler geometry. Based on these, we derive the average value of volumes of balls in compact reversible Finsler manifolds.

1. Introduction

In this paper, we consider the average of Holmes-Thompson volume $d\mu$ of balls in a compact reversible Finsler manifold M. Let $\mu(B(x, r))$ be the volume of a metric ball of radius r in M centered at a point x. For any function f on M, we will let $\text{ave}[f(x)]$ be the average of f with respect to the volume $d\mu$ on M, i.e., $\text{ave}[f(x)] := \frac{1}{\mu(M)}\int_M f(x)d\mu$. We will $\mathfrak{c}_{n-1}$ to represent the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

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Theorem. *If* M *is an n-dimensional compact reversible Finsler manifold, then for all* $r \leq \mathrm{inj}(M)$, *the injective radius of* M, *we have*

$$\mathrm{ave}[\mu(B(x, r))] \leq e^{2S_{\max}} \mathfrak{c}_n \left(\frac{r}{\pi}\right)^n,$$

where $S_{\max} = \sup_{y \in SM} S(y)$. *Equality holds if and only if* M *is isometric to standard n-sphere of constant sectional curvature* $(\pi/\mathrm{inj}(M))^2$.

This theorem is a generalization of the theorem of [3] that

$$\mathrm{vol}(M) \geq e^{-2S_{\max}} \mathfrak{c}_n \left(\frac{\mathrm{inj}(M)}{\pi}\right)^n,$$

with equality holding if and only if M is isometric the unit n-sphere $\mathbb{S}^n$. We could ask if main theorem is true for every point $x \in M$. That is: Is

$$\mu(B(x, r)) \geq e^{-2S_{\max}} \mathfrak{c}_n \left(\frac{r}{\pi}\right)^n,$$

where $r \leq \mathrm{inj}(M)$? This is an open question; however, it is known that $\mu(B(x, r)) \geq C(M, n) r^n$ for some $C(M, n)$ (see [9, Proposition 6.3]).

2. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [8], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties:

- F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;
- $F(ty) = tF(y)$, for all $t > 0$, $v \in T_xM$;
- F^2 is strongly convex, i.e.,

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

is positive definite for all $(x, y) \in \widetilde{TM}$.

A Finsler structure F is called *reversible* if $F(-y) = F(y)$ for all $y \in T_xM$. For a fixed $y \in T_xM$, let $\gamma_y(t)$ be the geodesic from x with $\gamma'_y(0) = y$. Along $\gamma_y(t)$, we have the osculating Riemannian metrics

$$g_{\gamma'_y(t)} := g(\gamma_y(t), \gamma'_y(t))$$

in $T_{\gamma_y(t)}M$. With the Chern connection, we can define the covariant derivative $D_{\gamma'_y(t)}J(t, y)$ of a vector field $J(t, y)$ along a geodesic $\gamma_y(t)$. A vector field $J(t, y)$ along $\gamma_y(t)$ is called a *Jacobi field* if it satisfies

$$D_{\gamma'_y(t)}D_{\gamma'_y(t)}J(t, y) + R_{\gamma'_y(t)}(J(t, y)) = 0.$$

There are Jacobi tensors, $A(t, y)$, which satisfy the Sturm-Liouville equation

$$A''(t, y)(t) + R_y(t) \cdot A(t, y)(t) = 0, \quad A(0, y) = 0, \quad A'(0, y) = I, \tag{2.1}$$

where $R_y(t)$ is the linear representation of $R_{\gamma'_y(t)}$ with respect to $g_{\gamma'_y(t)}$.

Now we recall the Berger-Kazdan inequality.

Theorem 2.2 [1]**.** *Let* $A(t, y)$ *be a solution of the Sturm-Liouville equation* (2.1). *If* $A(t, y)$ *has no conjugate points for* $0 \le s \le t \le \pi$, *then for any* $0 \le s \le \pi/2$,

$$\int_0^\pi \int_s^\pi [\det A(t - s, \varphi_s(y))] dt ds \ge \pi \int_0^{\pi/2} \sin^{n-1} s ds,$$

with equality if and only if $R = I$, *i.e.*, $A(t, y) = \sin tI$.

This is the fundamental curvature free estimate. It was developed for the proof of the Blaschke problem for sphere. For a proof see [1] or [9].

3. Volume forms on Finsler Manifolds

In this section, we will investigate the volume forms on an n-dimensional Finsler manifold (M, F). Let $\{dx^i, dy^i\}_{i=1}^n$ be the dual basis for $T^*\widetilde{TM}$. The Holmes-Thompson volume form $d\mu$ on M is defined by

$$d\mu(x) = \sigma(x)dx^1 \wedge \cdots \wedge dx^n := \sigma(x)dx,$$

where

$$\sigma(x) = \frac{1}{c_{n-1}} \int_{S_xM} \det(g_{ij}(x, y)) \times \left(\sum_{n-1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n \right).$$

For a tangent vector $(x, y) \in \widetilde{TM}$, define the *distortion* τ by

$$\tau(y) := \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)},$$

and the *S-curvature* $S : \widetilde{TM} \to \mathbb{R}$ is defined by

$$S(y) := \frac{d}{dt}\bigg|_{t=0} [\tau(\gamma'_y(t))].$$

An important property is that $S = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $S = 0$ for Berwald spaces. Locally Minkowski spaces and Riemannian spaces are all Berwald spaces.

By the Chern connection, we obtain the decomposition

$$T^*(\widetilde{TM}) = \text{span}\{dx^i\} \oplus \text{span}\{\delta y^i\},$$

where δy^i is the vertical component dy^i and is given by $\delta y^i = dy^i + N^i_j dx^j$ for some N_l^s determined by the Chern connection. Then there is a naturally induced Sasaki metric $\hat{g}$ on $\widetilde{TM}$ defined by

$$\hat{g}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j \oplus g_{ij}(x, y)\delta y^i \otimes \delta y^j,$$

and the volume form dV of $\hat{g}$ on $\widetilde{TM}$ is given by

$$dV(x, y) := \sqrt{\det(g_{ij}(x, y))}dx^1 \wedge \cdots \wedge dx^n \wedge \sqrt{\det(g_{ij}(x, y))}\delta y^1 \wedge \cdots \wedge \delta y^n$$

$$= \det(g_{ij}(x, y))dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n.$$

Let $\omega = \frac{\partial F}{\partial y^i} dx^i$ be the Hilbert 1-form on $\widetilde{TM}$. In local coordinates, we have $dV = (d\omega)^n / n!$.

Define

$$S_xM := \{y \in T_xM : F(x, y) = 1\} \text{ and } SM := \bigcup_{x \in M} S_xM.$$

For any $y \in SM$, we denote by $\varphi_t(y)$ the *geodesic flows* on SM with $\varphi_0(y) = y$. It is obvious that $\mathfrak{p} \circ \varphi_t(y) = \gamma_y(t)$ and $\gamma'_y(t) = \varphi_t(y)$, where $\mathfrak{p} : SM \to M$ is the bundle projection. Then there is another interpretation of this volume on tangent space. Let $i : SM \to \widetilde{TM}$ the natural embedding, and X_ω the Reeb field of the Hilbert 1-form ω. It is uniquely determined by the conditions $\omega(X_\omega) = 1$, $i_{X_\omega}(d\omega) = 0$. In particular, we have $L_{X_\omega}\omega = 0$ and the geodesic flow of F, i.e., the flow with infinitesimal generator X_ω, consists of contact diffeomorphisms and the volume form $i^*(dV)$ on SM is

$$dV(x, y) = \frac{1}{(n-1)!}\omega \wedge (d\omega)^{n-1}.$$

Since $L_{X_\omega}\omega = 0$, this volume form is also invariants under the geodesic flow φ_t of F.

Fix a point $p \in M$, F induces a Riemannian metric on $T_pM \setminus \{0\}$ by

$$g_p(y) := g_{ij}(p, y)dy^i \otimes dy^j.$$

Let $\dot{g}_p$ and $d\nu_p$ denote the Riemannian metric and the Riemannian volume form on S_pM induced by g_p, respectively. Thus

$$d\nu_p(y) = \sqrt{\det(g_{ij}(p, y))}\left(\sum_{n-1}^{n}(-1)^{i-1}y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \quad \wedge \cdots \wedge dy^n\right).$$

One of the necessary tools in our discussion is the Finslerian version of Santaló's formula (see [7]), which we now state. The proof shall be exactly same as the Riemannian case with small modification necessary for the Finsler case, which only uses the fact that the geodesic flow preserves dV. Let Ω be a relatively compact domain in a Finsler manifold M with smooth boundary $\partial\Omega$. Denote by $\mathfrak{n}$ the unit inward normal vector field along $\partial\Omega$. Thus, $g_{\mathfrak{n}}(\mathfrak{n}, y) = 0$ for all $y \in T\partial\Omega$. Set $S^+\partial\Omega := \{y \in S\Omega : g_{\mathfrak{n}}(\mathfrak{n}, y) > 0 \text{ on } \partial\Omega\}$ with measure $d\chi(y) := dA(\mathfrak{p}(y))$ $d\nu_{\mathfrak{p}(y)}(y)$, where dA denotes the induced measure on $\partial\Omega$. We again emphasize that dA is not the volume form of the induced Finsler metric on $\partial\Omega$. For any $y \in S\Omega$, we set as in the Riemannian case

$$l(y) := \sup\{t | \gamma_y(t) \in \Omega \text{ and } \gamma_y|_{[0,t]} \text{ is minimizing}\} \text{ possibly } \infty.$$

Then we have the following proposition.

Proposition 3.1 [5, Proposition 2]**.** *For all integrable function f on* $S\Omega$, *we have*

$$\int_{S\Omega} f dV = \int_{S+\partial\Omega}\left\{\int_0^{l(y)} f(\phi_t(y))dt\right\}e^{\tau(y)}g_{\mathfrak{n}}(\mathfrak{n}, y)d\chi(y).$$

In the Riemannian case, $e^{\tau(y)} = 1$, $g_{\mathfrak{n}} = g$, and therefore, Proposition 3.1 yields Santaló's formula [7].

4. Proof of Main Theorem

In this section, we prove our main theorem. We first recall the following theorem due to Kim and Min (see [4]).

Theorem 4.1 [4, Theorem 2.3]**.** *The reversible Finsler metrics with positive constant flag curvature are Riemannian*.

We can be extended to Finsler manifolds with little modification as follows (see [2, Proof of Theorem B]).

Theorem 4.2. *If* (M, F) *is an n-dimensional compact reversible Finsler manifold*, *then for all* $r \leq \operatorname{inj}(M)$, *we have*

$$\operatorname{ave}[\mu(B(x, r))] \geq e^{-2S_{\max}} \mathfrak{c}_n \left(\frac{r}{\pi}\right)^n.$$

Equality holds if and only if (M, F) *is isometric to standard n-sphere of constant sectional curvature* $(\pi/\operatorname{inj}(M))^2$.

Proof. We first scale the metric so that we are considering $B(x, \pi)$, where $\pi \leq \operatorname{inj}(M)$. We need to show

$$\operatorname{ave}[\mu(B(x, \pi))] \geq e^{-2S_{\max}} \mathfrak{c}_n$$

with equality holding if and only if M is isometric to the unit n-sphere $\mathbb{S}^n$. We consider $M \times \left[0, \frac{\pi}{2}\right]$ with the warped product Finsler metric

$$\sin^2 tF(y)^2 + dt^2 \text{ for } \left(y, \frac{\partial}{\partial t}\right) \in T\left(M \times \left[0, \frac{\pi}{2}\right]\right).$$

For all $p \in \{x\} \times \left[\varepsilon, \frac{\pi}{2}\right]$, we see

$$\left[\int_{\varepsilon}^{\frac{\pi}{2}} \sin^n(t)dt\right]\mu(B(x, \pi)) = \mu\left(B(x, \pi) \times \left[\varepsilon, \frac{\pi}{2}\right]\right)$$

$$= \int_{S_p\left(M\times\left[\varepsilon, \frac{\pi}{2}\right]\right)} \left\{\int_0^{l(y)} e^{-\tau(y)} \det(A(s, y))ds\right\} dV(y).$$

We now apply Theorem 2.2 (Berger-Kazdan inequality) and Proposition 3.1 (Santaló's formula),

$$\left[\int_\varepsilon^{\frac{\pi}{2}} \sin^n(t)dt\right]^2 \int_M \mu(B(x, \pi))d\mu$$

$$= \int_{S\left(M\times\left[0, \frac{\pi}{2}\right]\right)} \left\{\int_0^{l(y)} e^{-\tau(y)} \det(A(s, y))ds\right\} dV(y)$$

$$\geq \int_{S+\partial\left(M\times\left[\varepsilon, \frac{\pi}{2}\right]\right)} \left\{\int_0^{l(y)} \left[\int_0^{l(\phi_t(y))} e^{-\tau(\phi_t(y))} \det(A(s, \phi_t(y)))ds\right] dt g_{\mathfrak{n}}(\mathfrak{n}, y)\right\} d\chi(y)$$

$$\geq e^{-2S_{\max}} \int_{S^+\left(M\times\left\{\frac{\pi}{2}\right\}\right)} \left\{\int_0^{l(y)} \left[\int_0^{l(y)-t} \det(A(s, \phi_t(y)))ds\right] dt g_{\mathfrak{n}}(\mathfrak{n}, y)\right\} d\chi(y)$$

$$\geq e^{-2s_{\max}} \frac{\mathfrak{c}_{n+1}}{2\mathfrak{c}_n \pi^{n+1}} \int_{S^+\left(M\times\left\{\frac{\pi}{2}\right\}\right)} l(y)^{n+2} g_{\mathfrak{n}}(\mathfrak{n}, y)d\chi(y).$$

We consider the warped product metric on $M \times \left[\varepsilon, \frac{\pi}{2}\right]$ (see [6]). Since $\operatorname{inj}(M) \geq \pi$, $l(y) = \pi$ as long as γ_y does not hit $M \times \{\varepsilon\}$, that is as long as $g_{\mathfrak{n}}(\mathfrak{n}, y) < \cos\varepsilon$. Thus let

$$S_\varepsilon^+ = \left\{y \in S^+\left(M \times \left\{\frac{\pi}{2}\right\}\right) : g_{\mathfrak{n}}(\mathfrak{n}, y) < \cos\varepsilon\right\}$$

and our inequality becomes

$$\left[\int_\varepsilon^{\frac{\pi}{2}} \sin^n(t)dt\right]^2 \int_M \mu(B(x, \pi))d\mu \geq e^{-2S_{\max}} \frac{\mathfrak{c}_{n+1}\pi}{2\mathfrak{c}_n} \int_{S_\varepsilon^+} g_{\mathfrak{n}}(\mathfrak{n}, y)d\chi(y).$$

Letting ε go to 0, we get

$$\left[\frac{\mathfrak{c}_{n+1}}{2\mathfrak{c}_n}\right]^2 \int_M \mu(B(x, \pi))d\mu \geq e^{-2S_{\max}} \frac{\mathfrak{c}_{n+1}\pi}{2\mathfrak{c}_n} \frac{\mathfrak{c}_{n-1}\pi}{n} \mu(M).$$

Thus

$$\operatorname{ave}[\mu(B(x, \pi))] \geq e^{-2S_{\max}} \frac{2\mathfrak{c}_{n-1}\pi}{n\mathfrak{c}_{n+1}} = e^{-2S_{\max}} \mathfrak{c}_n.$$

In order for equality to hold, we must in particular have equality in the Berger-Kazdan inequality for all $y \in S_\varepsilon^+$ for any $\varepsilon > 0$. Since almost every $v \in S\left(M \times \left(0, \frac{\pi}{2}\right]\right)$ is tangent to a geodesic γ_y for some such y, we see that $M \times \left(0, \frac{\pi}{2}\right)$ has constant flag curvature one. Since $M \times \left\{\frac{\pi}{2}\right\}$ is totally geodesic and isometric to M, we see that M has constant flag curvature one. Theorem 4.1, then, implies that F is a Riemannian metric and the universal covering of M is $\mathbb{S}^n$. But, since we assumed that $\operatorname{inj}(M) \geq \pi$, it follows that M must be isometric to the unit n-sphere $\mathbb{S}^n$.

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