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ANOTHER LOOK AT TOPOLOGICAL PRODUCT SPACE PROPERTIES AND EXAMPLES

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Abstract

Within this paper, the needed correction for product properties is revisited, additional corrections are given, and additional product properties and not product properties are given.

1. Introduction and Preliminaries

Product spaces with the Tychonoff topology were introduced in 1930 [6].

Definition 1.1. A topological property P is a product property iff both a product space with the Tychonoff topology and all of its factor spaces simultaneously share property P [6].

Every since their introduction, product properties have been greatly studied. In 1930, with the many diverse known topological properties, the existence of a least

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topological property was not even imagined and, as a consequence, received no consideration in the definition or the investigation of product properties until 2016 [1], when the existence of the least topological property was established. In a 2015 paper [2], the need and use of "not-T_0" revealed "not-T_0" as a strong, useful property motivating the addition of "not-T_0" and all other "not-*P*" properties, where *P* is a topological property for which "not-*P*" exists, as important, useful properties in the study of topology, opening a never before imagined fertile topological territory.

Theorem 1.1. *$L = (T_0$ or "not-$T_0)$ is the least topological property* [1].

Also, *L* can also be given by (*P* or "not-*P*"), where *P* is a topological property for which "not-*P*" exists [1].

Since every topological space has property *L* [3], then every product space and all its factor spaces, regardless of the diversity of their properties, simultaneously share property *L* and, by the 1930 definition, *L* is a product property, totally destroying the 1930 intent of product properties. Since the least topological property *L* had not been considered in the earlier studies, a simple, meaningful solution to the dilemma created by the existence of *L* was the exclude *L* from the product properties.

Definition 1.2. A topological property *P* is a product property iff $P \neq L$ and a product space satisfied property *P* iff each factor space satisfies property *P* [1].

Within classical topology, there are few examples of a topological property *P* and its negative "not-*P*", including "not-*P*", where *P* is a product property. With the addition of "not-*P*" into the body of topology, where *P* is a product property, there is a need to define "not-*P*".

Definition 1.3. Let *P* be a product property. Then a product space is "not-*P*" iff there exists a factor space of the product space that is "not-*P*" [1].

With the given definition of "not-*P*", where *P* is a product property, care must be taken to ensure that for each product property *P*, "not-*P*" exists. That concern was addressed in the 2016 paper [4].

Theorem 1.2. *Let P be a topological property. Then each of the following are* *equivalent*: (a) “*not-P*”, *the negation of P*, *exists*, (b) “*not-P*” *is a topological property*, *P is stronger than L*, *and* $P \neq$ “*not-P*”, (c) $P \neq L$ *and* $P \neq$ “*not-P*”, (d) *P is stronger than L*, *and* (e) “*not-P*” *is stronger than L* [4].

Thus *L* is the only topological property *P* for which “not-*P*” does not exist. Since each product property *P* is not *L* and *L* is the least topological property, then *P* is stronger than *L*, and “not-*P*” exists. Also, for the definition of “not-*P*”, where *P* is a product property, to be creditable, (“not-(“not-*P*”)”) would have to be *P*, which it is.

Within a 2016 paper [1], it was established that for a product property *P*, “not-*P*” is not a product property. In past studies of classic topology, topological properties that are product properties were studied first, with topological properties that fail to be product properties studied much later; but now, using the result above, product properties and not product properties can be studied simultaneously with many new, never before imagined not product properties quickly and easily added in the study. In addition, for each product property *P*, “not-*P*” itself is a never before imagined not product property. If each factor space for a product space is “not-*P*”, where *P* is a product property, then the product space is “not-*P*”. However, a product space is “not-*P*” if only one of its factor spaces is “not-*P*” regardless of how diverse the properties of the remaining factor spaces.

Also, in classical topology, except for topological properties *P* and *Q* where (*P* or *Q*) was known to exist or (*P* and *Q*) was known to exist, is there inclusion of (*P* or *Q*) or (*P* and *Q*). Thus, with the work on product properties above, a natural question to pose was whether for product properties *P* and *Q*, is (*P* or *Q*) or (*P* and *Q*) a product property? Within the paper [3], it was shown that for topological properties *P* and *Q*, (*P* or *Q*) is a topological property, but (*P* and *Q*) need not be a topological property as seen by (*P* and “not-*P*”), where *P* is a topological property for which “not-*P*” exists, which was used later to prove that there is no strongest topological property [5]. Within the 2016 paper [1], it was believed that for product properties *P* and *Q*, (*P* or *Q*) is a product property, which was then used in a proof that for product properties *P* and *Q*, (*P* and *Q*) is a product property. However, a second look at the proof in the paper [1] that for product properties *P* and *Q*, (*P* or *Q*) is a product property revealed, at best, that the proof is incomplete raising questions about the claims that (P or Q) and (P and Q) are product properties. Below these questions are further investigated and resolved.

2. Resolution for (*P* or *Q*)

Theorem 2.1. *Let P and Q be product properties. Then* (P *or* Q) *is a product property iff* $P = Q$.

Proof. Clearly, if $P = Q$, then (P or Q) is a product property. Suppose there exist distinct product properties P and Q for which (P or Q) is a product property. Let (X_1, T_1) be a space with property P, let (X_2, T_2) be a space with property Q, let X be the product of X_1 and X_2, and let W be the Tychonoff topology on X. Since the factor spaces are (P or Q), then (X, W) is (P or Q). Thus, (X, W) is P or (X, W) is Q, but (X, W) is not P since (X_2, T_2) is $Q \neq P$ and (X, W) is not Q since (X_1, T_1) is $P \neq Q$. Thus for distinct product properties P and Q, (P or Q) is not a product property.

Theorem 2.2. *Let P and Q be distinct product properties and let (X, W) be a product space. If (X, W) is* (P *or* Q), *then each factor space is* (P *or* Q).

Proof. Since (X, W) is (P or Q), then (X, W) has property P or (X, W) has property Q. If (X, W) has property P, then each factor space has property P and if (X, W) has property Q, then each factor space has property Q. Thus, each factor space has property (P or Q).

Hence (P or Q), where P and Q are distinct product properties, can be used to give unique examples in the study of product properties.

3. Resolution for (*P* and *Q*)

Theorem 3.1. *Let P and Q be product properties. Then* (P *and* Q) *is a product property.*

Proof. By the results above, each of "not-P" and "not-Q" exists and is different from *L*. Thus (“not-*P*” or “not-*Q*”) is a topological property stronger than *L* and, by the results above, (“not-(“not-*P*” or “not-*Q*”)”) $=$ (*P* and *Q*) exists. The remainder of the proof is straightforward and omitted.

Thus, many more product properties can be quickly and easily added to the study of product properties.

Within the 2016 paper [1], the then believed, and now known, fact that for product properties *P* and *Q*, (*P* and *Q*) is a product property was used to prove that for product properties *P* and *Q* for which (*P* and “not-*Q*”) exist, (*P* and “not-*Q*”) is not a product property. Thus many more, never before imagined not product properties are quickly and easily added to the study of product spaces.

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