Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 4, Issue 1, 2015, Pages 35-39
This paper is available online at http://www.frdint.com/
Published online November 29, 2015

AN INTERESTING RECTANGULAR PROBLEM

M. A. GOPALAN*, S. VIDHYALAKSHMI and N. THIRUNIRAISELVI

Department of Mathematics
SIGC, Trichy-620002, Tamilnadu
India
e-mail: mayilgopalan@gmail.com
vidhyasigc@gmail.com
nts.maths.ig@gmail.com

Abstract

We search for non-zero distinct integers x, y representing the length and breadth of a rectangle which are such that each of the expressions $x + y$ and $xy + k(x + y) + k^2$ is a perfect square. Few interesting observations are also presented.

1. Introduction

The problems of Heron of Alexandria and Maximus Planude on pairs of rectangles have engaged the attention of Mathematicians since antiquity as can be seen from [1, 2]. In [3], a problem dealing with two non-zero integral pairs (x, y) and (u, v) representing the length and breadth, respectively, of two rectangles which are related such that $u + v = \alpha(x + y)$, $xy = \beta uv; \alpha, \beta \in z^{+} - \{0\}$, some interesting

Keywords and phrases: rectangular problem, system of linear equations, Pythagorean equation, integer solutions.

2010 Mathematics Subject Classification: 11D04, 11D99.

*Corresponding author

Received October 19, 2015; Accepted November 18, 2015

relations among the breadths and lengths of the rectangles has been analyzed. In [4], infinitely many rectangles, where, in each of which, the sum of the area and perimeter is expressed as the difference of squares of the sides, are obtained. In [5], infinitely many rectangles, where, in each of which, the ratio area / perimeter is a perfect square, are obtained. In [6], different pairs of rectangles and Pythagorean triangle, where, each pair satisfies the relation that the sum of the perimeters of a rectangle and a Pythagorean triangle is equal to twice the area of the rectangle added with two, are obtained. In [7], an attempt has been made to search for two non-zero distinct integers such that each of the expressions $na_0,\ a_0 + a_1$ and $na_0 + a_1,\ n > 1$ are perfect squares. Also, the process of obtaining the $(n+1)$ tuple $s = \{a_0, a_0, \ldots\ldots, a_0, a_1\}$, where the sum of its elements and the sum of any two of its elements is a perfect square, is illustrated.

This paper concerns with the problem of obtaining non-zero distinct integers x, y representing the length and breadth of a rectangle which are such that each of the expressions $x + y$ and $xy + k(x + y) + k^2$ is a perfect square. A few interesting observations are also presented.

2. Method of Analysis

Let x, y represents the length and breadth of a rectangle. The problem under consideration is represented by the system of double equations

$$x + y = q^2, \tag{1}$$

$$xy + k(x + y) + k^2 = p^2, \tag{2}$$

where $k,\ p$ and q are any non-zero integers.

Eliminating 'y' between (1) and (2), we have

$$x^2 - q^2 x + (p^2 - kq^2 - k^2) = 0.$$

Treating the above equation as a quadratic in x and solving for x, we have

$$x = \frac{1}{2}\left[q^2 + \sqrt{(q^2 + 2k)^2 - (2p)^2}\right]. \tag{3}$$

Let

$$\alpha^2 = (q^2 + 2k)^2 - (2p)^2 \tag{4}$$

which is in the form of the well-known Pythagorean equation. Employing the most cited solutions of the Pythagorean equation, (4) is satisfied by

$$p = rs, \quad \alpha = r^2 - s^2, \tag{5}$$

$$q^2 = r^2 + s^2 - 2k, \quad r > s > 0. \tag{6}$$

In view of (5) and (6), from (3) and (1), one obtains $x = r^2 - k$ and $y = s^2 - k$ which represents the length and breadth of the rectangle satisfying (1) and (2).

It is worth mentioning here that the values of r, s and k should satisfy (6). After performing a few numerical calculations it is observed that (6) is satisfied by the following three sets of values for r, s and k as presented in the table below.

k	r	s	q
n	$2\beta^2 + 6\beta + 4 - n$	$2\beta + 3$	$n - [2\beta^2 + 6\beta + 5]$
$-2n$	$2\beta^2 - 2\beta + 2n$	$2\beta - 1$	$2\beta^2 - 2\beta + 2n + 1$
$-(2n-1)$	$2\beta^2 + 2\beta + 2n - 1$	$2\beta + 1$	$2\beta^2 + 2\beta + 2n$

In the above table $\beta, n = 1, 2, 3, \ldots$.

A few numerical illustrations are presented below

n	β	k	r	s	x	y	$x + y$	$xy + k(x + y) + k^2$
1	1	1	11	5	120	24	12^2	55^2
		–2	2	1	6	3	3^2	2^2
		–1	5	3	26	10	6^2	15^2

2	1	2	10	5	98	23	11^2	50^2
		–4	4	1	20	5	5^2	4^2
		–3	7	3	52	12	8^2	21^2
3	2	3	21	7	438	46	22^2	147^2
		–6	10	3	106	15	11^2	30^2
		–5	17	5	294	30	18^2	85^2

3. Remarkable Observations

(1) When $k > 0,$ the L.H.S. of (2) is expressed as the sum of two perfect squares minus $(2k + 2)$ times a perfect square. In other words, L.H.S. of (2) $= \{[2\beta + 3]^2 + [4\beta^3 + 18\beta^2 + 24\beta - k(2\beta + 3) + 9]^2 - (2k + 2)[2\beta + 3]^2\}.$

(2) When $k = -2n,$ it is seen that the area of the rectangle minus even multiple of its semi-perimeter added with $4n^2$ is represented as the sum of two perfect squares added with $(4n - 2)$ times a perfect square.

(3) When $k = -(2n - 1),$ it is seen that the area of the rectangle minus odd multiple of its semi-perimeter added with $(2n - 1)^2$ is the sum of two perfect squares added with $(n - 1)$ times a perfect square.

In particular, when $n = 1,\ k = -1,$ it is seen that the area of the rectangle minus its semi-perimeter and added with unity obtained from (2) represents the hypotenuse of the Pythagorean triangle, whose generators are $4\beta^2 + 6\beta^2 + 2\beta,\ 4\beta^2 + 4\beta + 1.$

When $n = 2,\ k = -3,$ it is seen that the area of the rectangle minus three times its semi-perimeter and added with nine obtained from (2) is written as sum of three squares, namely, $[4\beta^2 + 4\beta + 1]^2 + [(4\beta + 2)(\beta^2 + \beta + 1)]^2 + [4\beta + 2]^2.$

Conclusion

In this paper, we have presented infinitely many rectangles with dimensions x, y such that each of the expressions $x + y = q^2$, $xy + k(x + y) + k^2 = p^2$ is a perfect square. To conclude, one may search for other characterizations of rectangles.

Acknowledgement

The financial support from the UGC, New Delhi (F.MRP-5123/14 (SERO/UGC) dated March 2014) for a part of this work is gratefully acknowledged.

References

[1] L. E. Dickson, History of Theory of Numbers, Vol. 11, Chelsea Publishing Company, New York, 1952.

[2] B. L. Van der waerden, Geometry and Algebra in Ancient Civilizations, Springer Verlag, Berlin, Heidelberg, New York, 1983.

[3] M. A. Gopalan and G. Janaki, On pairs of rectangles, Impact J. Sci. Tech. 3(3) (2009), 39-43.

[4] M. A. Gopalan, V. Pandichelvi and R. Srikanth, Integral solutions of $2(x + y) + xy = x^2 - y^2$, Impact J. Sci. Tech. 2(3) (2008), 119-123.

[5] M. A. Gopalan and B. Sivakami, An interesting rectangular problem, Archimedes J. Math. 1(1) (2011), 67-72.

[6] M. A. Gopalan and B. Sivakami, An interesting rectangle-Pythagorean triangle problem, Impact J. Sci. Tech. 6(1) (2012), 67-76.

[7] M. A. Goplan, S. Vidhyalakshmi and N. Thiruniraiselvi, On two interesting Diophantine problems, Impact J. Sci. Tech. 9(3) (2015), 51-55.