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AN ELEMENTARY PROOF FOR POWER-SUM DECOMPOSITION OF MONOMIALS

PISHENG DING

Mathematics Department
Illinois State University
Normal, IL 61790-4520
U.S.A.
e-mail: pding@ilstu.edu

Abstract

This note gives an elementary proof for the known fact that every monomial of n variables can be expressed as a linear combination of powers of linear forms in these n variables.

1. Introduction

It is known that a monomial in the variables $x_1, x_2, \cdots, x_n$ can be expressed as a linear combination of powers of linear forms in the x_i's as stated in the following theorem.

Theorem. *A monomial of the form* $\prod_{i=1}^{n} x_i^{r_i}$, *where each* r_i *is a positive integer, can be expressed as*

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$$\sum_{k=1}^{m} c_k y_k^r$$

with $c_k \in \mathbb{R}$, $r = \sum_{i=1}^{n} r_i$, *and each* y_k *being a linear form in the* x_i's.

This note offers a very elementary proof for this fact. It should be noted that the elementary proof given here only shows the existence of power-sum decomposition of any monomial; it does not address the "Waring problems" of minimal decomposition, which [1], among others, addresses.

2. Proof

We apply induction on n, the number of variables, to reduce the general case to the two-variable case. The base case $n = 2$, which is the most substantial part of the proof, will be established later. We first execute the inductive step.

2.1. The inductive step

Suppose that the result is true for all $n \in \{2, \cdots, N-1\}$; we argue the result for $n = N$.

Consider $\prod_{i=1}^{N} x_i^{r_i}$, where r_i are positive integers. Now

$$\prod_{i=1}^{N} x_i^{r_i} = \left(\prod_{i=1}^{N-1} x_i^{r_i} \right) \cdot x_N^{r_N}.$$

By the induction hypothesis, the factor $\prod_{i=1}^{N-1} x_i^{r_i}$, having only $N-1$ variables, can be expressed as

$$\sum_{k=1}^{m} c_k y_k^{r-r_N},$$

where $c_k \in \mathbb{R}$, $r = \sum_{i=1}^{n} r_i$, and each y_k is a linear form in $x_1, \cdots, x_{N-1}$. Then

$$\prod_{i=1}^{N} x_i^{r_i} = \left(\sum_{k=1}^{m} c_k\, y_k^{r-r_N} \right) \cdot x_N^{r_N}$$

$$= \sum_{k=1}^{m} c_k \left(y_k^{r-r_N} \cdot x_N^{r_N} \right).$$

Each product $y_k^{r-r_N} \cdot x_N^{r_N}$ is a monomial in two variables, which, according to the base case of the induction, can be written as a linear combination of rth powers of linear forms in y_k and x_N. As a linear form in y_k and x_N equals a linear form in $x_1, \cdots, x_N$, the desired result follows.

2.2. The base case $n = 2$

We now address the base case $n = 2$. For simpler notation, let the two variables be x and y. Let d be greater than 2. We are to show that a monomial of the form $x^j y^{d-j}$ with $0 < j < d$ is expressible as

$$\sum_k c_k (\alpha_k x + \beta_k y)^d,$$

where $c_k, \alpha_k, \beta_k \in \mathbb{R}$.

For any integer i, consider the binomial expansion of $(ix + y)^d$:

$$(ix + y)^d = (ix)^d + y^d + \sum_{j=1}^{d-1} \binom{d}{j} (ix)^j y^{d-j}.$$

It follows that

$$\sum_{j=1}^{d-1} i^j \binom{d}{j} x^j y^{d-j} = (ix + y)^d - (ix)^d - y^d. \tag{1}$$

The right side of Equation (1) is a linear combination of dth powers of linear forms in x and y; call it $R(ix, y)$. Now let i range over the set $\{1, 2, \cdots, (d-1)\}$ and we obtain a system of $(d-1)$ equations:

$$\sum_{j=1}^{d-1} i^j \left(\binom{d}{j} x^j y^{d-j} \right) = R(ix, y), \tag{2}$$

which can be viewed as a linear system in the $(d-1)$ variables $\binom{d}{j} x^j y^{d-j}$. The coefficient matrix A_d of this linear system is $(i^j)_{i,j=1}^{d-1}$;

$$\begin{aligned} \det A_d &= \det(i^j)_{i,j=1}^{d-1} \\ &= (d-1)! \det V_{d-1}(1, 2, \cdots, (d-1)), \end{aligned}$$

where $V_{d-1}(1, 2, \cdots, (d-1))$ is the $(d-1) \times (d-1)$ Vandermonde matrix on the $(d-1)$-tuple $(1, 2, \cdots, (d-1))$:

$$\begin{bmatrix} 1^0 & 1^1 & \cdots & 1^{d-2} \\ 2^0 & 2^1 & \cdots & 2^{d-2} \\ \vdots & \vdots & \vdots & \vdots \\ (d-1)^0 & (d-1)^1 & \cdots & (d-1)^{d-2} \end{bmatrix}.$$

It is a standard fact that this Vandermonde matrix is invertible. It follows that the matrix A_d is invertible. The linear system (2) thus has a unique solution:

$$\begin{bmatrix} \binom{d}{1} x^1 y^{d-1} \\ \binom{d}{2} x^2 y^{d-2} \\ \vdots \\ \binom{d}{d-1} x^{d-1} y^1 \end{bmatrix} = A_d^{-1} \begin{bmatrix} R(x, y) \\ R(2x, y) \\ \vdots \\ R((d-1)x, y) \end{bmatrix}.$$

Hence, for each $j \in \{1, 2, \cdots, (d-1)\}$, the monomial $x^j y^{d-j}$ is a linear combination of $R(ix, y)$'s, all already seen to be linear combinations of dth powers of linear forms in x and y. The base case is now established.

References

[1] R. Oldenburger, Polynomials in several variables, Ann. Math. 41 (1940), 694-710.