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AN ANALYTICAL AND NUMERICAL INVESTIGATION OF THE QUARKONIUM SPECTROSCOPY USING THE GENERALIZED KLEIN-GORDON EQUATION: A COMPUTATIONAL STUDY

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Abstract

In this article, we have investigated quarkonium spectrum using the generalized Klein-Gordon equation. This has been done using suitable potentials and applying them in the Klein-Gordon equation mass term analytically and numerically. The simplicity of this model and the high accuracy of the achieved results, compared to other methods, for a relativistic computation of the quarkonium spectrum (qq^{-}) are matters of consideration.

1. Introduction

Since the discovery of the heavy quark in 1974, many investigations have been

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reported on the quarkonium spectrum (qq^{-}). The theory, which deals with the quarks interaction, is termed Quantum Chromo Dynamics (QCD), and was pioneered by Murray Gellman at Caltech in the 1960s. Due to the magnitude of the coupling constant in QCD, the probability of perturbative computations does not exist in this theory. Fortunately, in high energies due to the non-Abelian of the QCD symmetrical group, the coupling constant becomes smaller (asymptotic freedom) and, therefore, the probability of using perturbative methods does exist. Currently, the most successful method for investigating the hadronic spectrum and quarks confinement is the lattice gauge method [1, 2].

This theory has its root in space and time latticing and, consequently, provides a proper method for the numerical solution of QCD. However, since it requires inordinately long times for the numerical computations of even relatively simple problems, this theory lacks enough efficiency to attract applications.

One of the simplest and most obvious ways of investigating the hadronic spectrum is using the non-relativistic potential model. In this case, the Schrödinger equation for the expected potentials in QCD is solved in the same fashion as the linear potential, in addition to the Coulomb potential and, or, the totally phenomenological potentials like $r^{0.1}$ potential. The fact that such computations agree with the empirical values especially in the case of heavy quarkoniums $(cc^{-}, bb^{-}, ...)$ is worth noting [3-7]. However, one cannot assume the above-mentioned hadrons in the excited states and, also, the hadrons with light components (including the light quarks u, d, s) as non-relativistic. Therefore, the proper and correct method for solving this problem is using the Bethe-Salpeter equation. However, this equation possesses many complexities, and its accurate solution, in most cases, is impossible [8]. For the sake of avoiding such difficulties and, at the same time, considering the relativistic corrections, one must apply other methods. In this article, while ignoring the spinal correction, we will compute the light and heavy quarkonium's spectrum by solving the single time Klein-Gordon equation for two particles with a scalar interaction which couples with mass [9-16].

2. Introducing the Model

In the case of two non-relativistic particles, the Schrödinger equation, by considering the concept of the mass center system, transforms into an equation for a

particle with a reduced mass. However, for the relativistic particles and the Klein-Gordon equation this concept possesses many complexities. Nevertheless, the following simplifications can be implemented for cases of two particles with equal masses [17]. For the sake of generalizing the Klein-Gordon equation, one can use the relation $p^2 = W^2$ in which W is the invariance of mass and p is the quadra vector of the whole momentum. Consequently, the generalized single-time Klein-Gordon equation for two particles will be:

$$(p^2 - W^2)\phi(r_1, r_2) = 0 \tag{1}$$

in which

$$p^2 = (E_1 + E_2)^2 - (p_1 + p_2)^2 \tag{2}$$

and

$$W = (p_1^2 + m_1^2)^{\frac{1}{2}} + (p_2^2 + m_2^2)^{\frac{1}{2}} \tag{3}$$

the energy will be in the mass center system. For two particles with equal mass, one can reformulate equation (3) as:

$$W^2 = 2(p_1^2 + p_2^2 + m_1^2 + m_2^2). \tag{4}$$

Now by choosing

$$r = r_1 - r_2 \tag{5}$$

and

$$R = \alpha r_1 + \beta r_2, \tag{6}$$

one can easily show that:

$$\begin{aligned}(p^2 - W^2)\phi(r_1, r_2) = [\varepsilon_0^2 + [(\alpha + \beta)^2 + 2\alpha^2 + 2\beta^2]\nabla_R^2 + 4\nabla_r^2 \\ - 2(m_1^2 + m_2^2) + 4(\alpha - \beta)\nabla_r\nabla_R]\phi(r, R)\end{aligned} \tag{7}$$

in which $\varepsilon_0 = E_1 + E_2$, but for $m_1 = m_2 = m$ and $\alpha = \beta$ by choosing

$$\phi(r, R) = \phi(r)e^{i\frac{Q.R}{\alpha+\beta}} = \phi(r)e^{i\frac{Q.R}{2\alpha}} \tag{8}$$

equation (7) transforms into

$$\left[\frac{E^2}{4}+\nabla_r^2-m^2\right]\phi(r)=0 \tag{9}$$

in which $E^2=\varepsilon_0^2-2Q^2$. Equation (9) is quite similar to the Klein-Gordon equation for one particle. In this article, after considering this equation as the base, we arrive at quarkonium energy spectrum by incorporating the potential. In the stationary state, for the sake of investigating the bound state, the Klein-Gordon equation with scalar and vector potentials is generally written as [18, 19]

$$\left[\nabla_r^2+\frac{(E-v(r))^2}{4}-(m+s(r))^2\right]\phi(r)=0 \tag{10}$$

in which $s(r)$ is the scalar potential and $v(r)$ is the fourth component of a vector potential. Then equation (10) can be investigated in the following three forms:

(1) $v(r)\neq 0$, $s(r)=0$,

(2) $v(r)=0$, $s(r)\neq 0$,

(3) $v(r)\neq 0$, $s(r)\neq 0$.

Since the first form has already been investigated [20], we focus attention in this article on the second and third forms. In the second form, considering the fact that the potential is purely scalar, equation (10) transforms into

$$\left[\nabla_r^2-(m+s(r))^2+\frac{E^2}{4}\right]\phi(r)=0 \tag{11}$$

which, for the sake of simplicity, by defining $u^2(r)=2ms+s^2$, equation (11) transforms into

$$\left[\nabla_r^2-u^2(r)+\frac{E^2-4m^2}{4}\right]\phi(r)=0. \tag{12}$$

Now, by defining $\varepsilon=\dfrac{E^2-4m^2}{8m}$ and choosing $\dfrac{u^2}{2m}=kr+v_0$, equation (12) transforms into the Schrödinger equation for one particle in linear potential

$$\left[-\frac{1}{2m}\nabla_r^2+kr\right]\phi(r)=(\varepsilon-v_0)\phi(r). \tag{13}$$

Therefore, considering the answers of equation (13), the energy eigenvalues in $l = 0$ can be computed as the following form:

$$E_n^2 = 4m^2 + (4m^2k^2)^{\frac{1}{3}}(-x_n) + 8mv_0^2 \tag{14}$$

in which x_n are the Airy function negative zeros. When $l \neq 0$, equation (13) does not yield an analytical solution and must be solved numerically. However, with appropriate approximations, one can arrive at the equation (13) eigenvalues for *l*s, which do not equal zero by the following equation [21]

$$\varepsilon_{N,l} = v_0 + \left(\frac{1}{2m}\right)^{\frac{n}{n+2}} k^{\frac{2}{n+2}} \left(1+\frac{n}{2}\right) \times \left[\frac{2l(l+1)}{n}\right]^{\frac{n}{n+2}} \left[1 + \frac{n(2N+1)}{\sqrt{l(l+1)(n+2)}}\right] \tag{15}$$

in which $N = 0, 1, 2, \ldots$ and for the linear potential in equation (13), $n = 1$.

Consequently, we will be able to arrive at a different quarkonium spectrum by employing equations (14) and (15). To achieve this, we select the following parameter values: $k = 0.13\ \text{GeV}^2$, $m_b = 5\,\text{GeV}$, and $m_c = \frac{mb}{3} = 1.67\,\text{GeV}$ and if we know $\psi_{1s}(3097)$ and $\phi_{1s}(1020)$, the value of $m_s = 0.4\,\text{GeV}$ and $v_0 = -0.518\,\text{GeV}$ will be computable. The computed values for ss^-, cc^-, and bb^- are given in Tables 1 to 3.

By comparing the calculated and empirical values, we can see that in spite of the simplicity of the chosen potential, the accommodation of the results with experience is fairly good.

In the next section, for improving the achieved results, we will use more complex potentials which provide greater accuracy.

3. Computing Quarkonium Spectrum (qq^-) using more Complex Potentials $(s(r) \neq 0,\ v(r) = 0)$

To date, many potentials have been adopted in the investigation of quarkonium spectrum with the aid of the Schrödinger equation. Such potentials are either completely phenomenological such as the Martin potential [3-7], or consistent with the QCD framework, such as the Cornell potential [3-7]. Here, we will investigate

two potentials which are more complex than those investigated in the previous section, for producing the qq^- spectrum using the Klein-Gordon equation:

(A) The Indiana potential [22, 23]

$$s(r) = \frac{4}{3}\frac{6\pi}{33-2n_f}\frac{\left(1-\frac{r}{r_0}\right)^2}{r\mathrm{Ln}\frac{r}{r_0}} \tag{16}$$

in which n_f is the number of flavor and r_0 is a constant number.

(B)

$$s(r) = \frac{4}{3}\frac{\alpha_s}{r} + k\mathrm{Ln}\frac{r}{r_0} + cr^2, \tag{17}$$

α_s, k, c, and r_0 are constant numbers.

The above-mentioned potential behaviors are given in Figures 1 and 2. Now, by using equation (11) and inserting potentials (16) and (17), one can compute the quarkonium spectrum by using numerical analysis. The achieved results are given in Tables 4-6. One must note that the computed spectrum for ss^- (Table 6) is only executed with the potential defined in (17).

In the numerical computations, $m_b = 4.4\,\mathrm{GeV}$, $m_c = 1.5\,\mathrm{GeV}$, and $m_s = 0.5\,\mathrm{GeV}$. Also, the constant value r_0 in potential (16) for cc^- equals $1.440\frac{1}{\mathrm{GeV}}$ and for bb^- equals $1.567\frac{1}{\mathrm{GeV}}$. The values (r_0, c, k, α_0) in potential (17) for cc^-, bb^-, and ss^- are $(1, 0.0057, 0.567, 0.03)$, $(1, 0.005, 0.674, 0.03)$, and $(1, 0.0057, 0.353, 0.03)$, respectively. The units of all numbers are on the basis of some powers of GeV.

4. Conclusions

In this article, to circumvent the complexity of solving the Bethe-Salpeter equation, a simple alternative model has been introduced. The analysis constitutes a relativistic investigation of qq^- system without considering the quark spin and this

model results in equation (9). This two-particle equation is similar to the Klein-Gordon equation for a free particle. Considering the scalar potential for qq^{-} system, the analysis leads to an equation of the form of equation (11) which, in a non-relativistic limit, transforms into the Schrödinger equation with $s(r)$ potential. This equation does not yield closed-form solutions for a general potential, and therefore in most cases, must be solved numerically. By choosing $s(r)+\frac{s^2(r)}{2m}=kr+v_0$ one can, analytically, arrive at energy levels in $l=0$ (equation (14)). Also, for non-zero *l*s one can, approximately, compute the eigenvalues in an analytical form (equation (15)). The achieved spectrum, by considering the computed eigenvalues, is not so close to empirical values (Tables 1 and 2) and this is due to the simplicity of the chosen potential. To improve the answers, two other potentials have been used (equations (16) and (17)). These potentials, which are qualitatively similar (Figures 1 and 2), lead to greatly improved accuracy (Tables 4 and 5). In these two forms, the computed answers have a less-than-one percent error compared to the empirical values, in most cases. Also, the achieved answers are comparable to the Quasi Potential Equation (QPE) model which is a relativistic equation for the two particles in the momentum space [24] (Tables 4 and 5). It is pertinent to note, that with the assistance of the Schrödinger equation, comparatively accurate solutions can be generated for different potentials (e.g., the Indiana potential). Furthermore we note that in the present article, the quarkonium system is modeled as a relativistic system. One must consider the relativistic corrections and these corrections are considered in equation (11).

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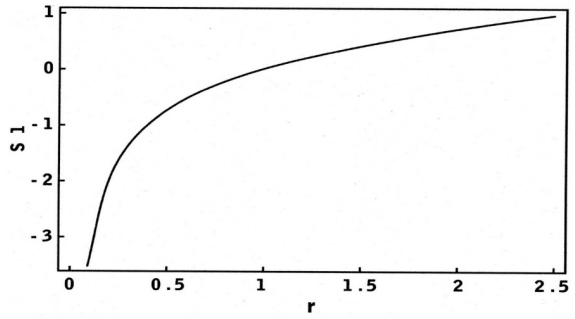

Figure 1. The Indiana potential changes curve on the basis of *r*.

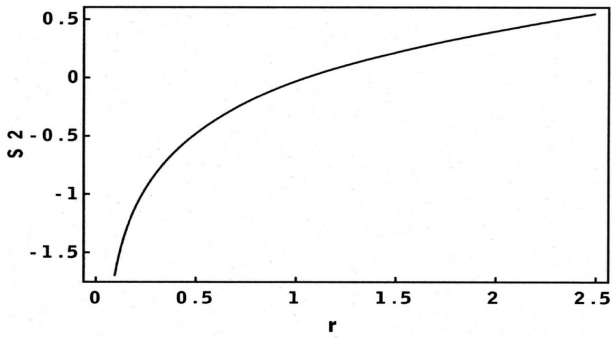

Figure 2. The (17) potential changes curve on the basis of *r*.

Table 1. The computed values for cc^- using equation (11) with a potential of $s + \frac{s^2}{2m} = kr + v_0$. Different states' masses are arrived at by putting the achieved eigenvalues from equation (13) in $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$

$N^{2s+1}L_J$	cc^- Meson	$m_{\text{Exp.}}$	$m_{\text{Theo.}}$
1^3S_1	J/Ψ	3097	3097
2^3S_1	$\Psi(2s)$	3686	3689
3^3S_1	$\Psi(3s)$	4040	4111
4^3S_1	$\Psi(4s)$	-	4450
1^3P_0	$\chi(1P)_{COG}$	3525	3454
1^3D_1	$\Psi(3770)$	3770	3738
2^3D_1	$\Psi(4160)$	4160	4205
3^3D_1	$\Psi(4415)$	4415	4626

Table 2. The computed values for bb^- using equation (11) with a potential of $s + \frac{s^2}{2m} = kr + v_0$. Different states' masses are arrived at by putting the achieved eigenvalues from equation (13) in $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$

$N^{2s+1}L_J$	bb^- Meson	$m_{\text{Exp.}}$	$m_{\text{Theo.}}$
1^3S_1	$\gamma(1S)$	9460	9509
1^3P_0	$\chi_b(1P)_{COG}$	9900	9760
2^3S_1	$\gamma(2S)$	10023	9938
2^3P_0	$\chi_b(2P)_{COG}$	10261	10224
3^3S_1	$\gamma(3S)$	10355	10275
4^3S_1	$\gamma(4S)$	10580	10565
5^3S_1	$\gamma(5S)$	10865	10823
6^3S_1	$\gamma(6S)$	11019	11058

Table 3. The computed values for ss^- using equation (11) with a potential of $s + \frac{s^2}{2m} = kr + v_0$. Different states' masses are arrived at by putting the achieved eigenvalues from equation (13) in $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$

		(ss^-)-State	
$L = 0$	$L = 1$	$L = 2$	$L = 3$
2266 -			
1976 -			
			1857 -
		1655 -	
1613 -			
	1396 -		
1020 -			

Table 4. The computed values for cc^- with potentials (16) and (17). Different states' masses in potential (16) have been achieved with the help of the relationship $\psi_n = E_n - 0.336\,\text{GeV}$ in which $E_n = \sqrt{8m\varepsilon_n + 4m^2}$ (Theo. 1). Also, potential (17) is achieved from $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$ in which $v_0 = -0.289\,\text{GeV}$ (Theo. 2)

$N^{2s+1}L_J$	bb^- Meson	$m_{\text{Exp.}}$	Theo. 1	Theo. 2	Ref. [24]
1^3S_1	$\gamma(1S)$	9460	9460	9460	9444
1^3P_0	$\chi_b(1P)_{COG}$	9900	9895	9871	9894
2^3S_1	$\gamma(2S)$	10023	10025	10024	9995
2^3P_0	$\chi_b(2P)_{COG}$	10261	10264	10244	10270
3^3S_1	$\gamma(3S)$	10355	10360	10357	10366
4^3S_1	$\gamma(4S)$	10580	10615	10610	10610
5^3S_1	$\gamma(5S)$	10865	10827	10836	10850
6^3S_1	$\gamma(6S)$	11019	11012	11018	11066

Table 5. The computed values for bb^- with potentials (16) and (17). Different states' masses in potential (16) have been achieved with the help of the relationship $\psi_n = E_n - 0.472\,\text{GeV}$ in which $E_n = \sqrt{8m\varepsilon_n + 4m^2}$ (Theo. 1). Also, potential (17) is achieved from $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$ in which $v_0 = -0.0869\,\text{GeV}$ (Theo. 2)

$N^{2s+1}L_J$	cc^- Meson	$m_{\text{Exp.}}$	Theo. 1	Theo. 2	Ref. [24]
1^3S_1	J/Ψ	3097	3097	3097	3136
2^3S_1	$\Psi(2S)$	3686	3677	3685	3678
3^3S_1	$\Psi(3S)$	4040	4045	4067	4066
4^3S_1	$\Psi(4S)$	-	4351	4379	-
1^3P_0	$\chi(1P)_{COG}$	3525	3493	3501	3525
1^3D_1	$\Psi(3770)$	3770	3764	3773	3737
2^3D_1	$\Psi(4160)$	4160	4108	4120	4102
3^3D_1	$\Psi(4415)$	4415	4389	4415	4404

Table 6. The computed values for ss^- with potential (17). Different states' masses have been achieved using the relationship $\psi_n = E_n = \sqrt{8m\varepsilon_n + 4m^2 + 8mv_0}$ in which $v_0 = -0.447\,\text{GeV}$

(ss^-)-State			
$L = 0$	$L = 1$	$L = 2$	$L = 3$
2412 -			
2066 -			
			1915 -
		1694 -	
1648 -			
	1420 -		
1020 -			