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AN ALMOST CONTRACTION MAPPING THEOREM IN METRIC SPACES WITH UNIQUE FIXED POINT

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Abstract

Recall from Berinde [1] that if (X, d) is a metric space, a map $T : X \mapsto X$ is called an almost contraction if there exists $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$. Observe that if $L = 0$, then T is a Banach contraction, and by the Banach contraction mapping theorem, T has a unique fixed point. In the present paper, we show that for some fixed $L > 0$ and $\delta \in (0, 1)$, we can guarantee uniqueness of the fixed point. An example is given to illustrate the main result.

1. Introduction

In Berinde [1], the following was obtained:

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Theorem 1.1. *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be an almost contraction, that is, a mapping for which there exists a constant* $\delta \in [0, 1)$ *and some* $L \geq 0$ *such that* $d(Tx, Ty) \leq \delta d(x, y) + Ld\ (y, Tx)$ *for all* $x, y \in X$. *Then*

(a) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(b) *For any* $x_0 \in X$, *the Picard sequence* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$ *converges to some* $x^* \in Fix(T)$.

(c) *The following estimates hold*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1})$$

for $n = 0, 1, 2, \cdots; i = 1, 2, \cdots$.

Observe that the above is an existence, but not a uniqueness theorem about the fixed point. In this paper, we show for some fixed $L > 0$ and $\delta \in (0, 1)$, we can guarantee uniqueness of the (δ, L)-weak contraction if the underlying space is complete. This paper is organized as follows. Section 2 contains the main result, and we give an example to illustrate it.

2. Main Results

Definition 2.1. Let (X, d) be a metric space. A map $T : X \mapsto X$ will be called a $(\delta, 1-\delta)$-weak contraction mapping if it is not the identity mapping and there exists $\delta \in (0, 1)$ such that for all $x, y \in X$, the following holds

$$d(Tx, Ty) \leq \delta d(x, y) + (1-\delta) d(y, Tx).$$

Theorem 2.2. *Let* (X, d) *be a metric space, and* $T : X \mapsto X$ *be a* $(\delta, 1-\delta)$-*weak contraction mapping. T has a unique fixed point, provided* (X, d) *is complete.*

Proof. Let $\{x_n\}$ be a sequence defined by $x_n = Tx_{n-1}$ for all $n = 1, 2, \cdots$. Observe that

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$

$$\leq \delta d(x_n + x_{n+1}) + (1-\delta) d(x_{n+1}, Tx_n)$$

$$= \delta d(x_n + x_{n+1}) + (1-\delta) d(x_{n+1}, x_{n+1})$$

$$= \delta d(x_n + x_{n+1}).$$

From the above, by induction, we obtain $d(x_{n+1}, x_{n+2}) \leq \delta^n d(x_1, x_2)$ for all $n = 1, 2, \cdots$. Since $\delta \in (0, 1)$, consequently $\{x_n\}$ is Cauchy. By completeness of X, $\lim_{n\to\infty} x_n = x^*$ for some $x^* \in X$. We show x^* is a fixed point of *T*. Observe

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \delta d(x_n, x^*) + (1-\delta) d(x^*, x_{n+1}). \end{aligned}$$

Taking limits in the above as $n \to \infty$, we deduce $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. For uniqueness, suppose $y^* = Ty^*$ but $y^* \neq x^*$, then observe

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \delta d(x^*, y^*) + (1-\delta) d(y^*, Tx^*) \\ &= \delta d(x^*, y^*) + (1-\delta) d(y^*, x^*) \\ &= d(x^*, y^*) \end{aligned}$$

which is impossible. So $x^* = y^*$ and uniqueness follows.

Example 2.3. Let $X = [1, 2]$ and define $d : X \times X \mapsto \mathbb{R}^+$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $T : X \mapsto X$ be defined by $Tx = \frac{x+3}{4}$ for all $x \in X$, then all the conditions of the previous theorem are satisfied and $x = 1$ is the unique fixed point. Moreover with $\delta = \frac{1}{2}$, we get the following:

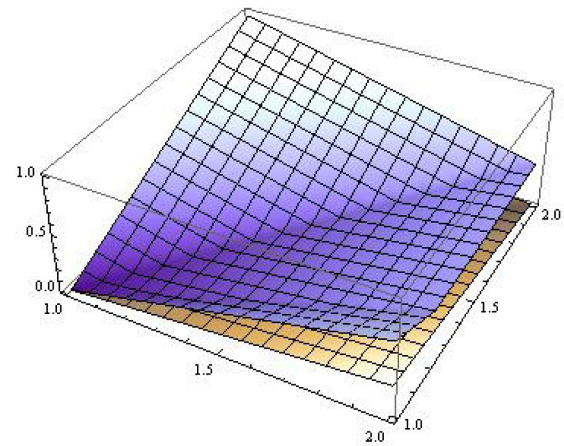

Figure 1. The graph of $\frac{1}{4}|x-y| \leq \frac{1}{2}|x-y| + \frac{1}{2}\left|y - \frac{x+3}{4}\right|$ for all $x, y \in X$ with left hand side being in "Coffee Tones (bottom)" and right hand side being in "Lake Colors (top)".

References

[1] V. Berinde, Approximation fixed points of weak contractions using the Picard iteration, Nonlinear Analysis Forum 9(1) (2004), 43-53.