Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 13, Issue 2, 2020, Pages 43-78
This paper is available online at http://www.frdint.com/
Published online March 31, 2020

ALL LINEAR THEOREMS OF THE ALTERNATIVE HAVE A COMMON FATHER. AN ADDENDUM TO A PAPER OF C. T. PERNG

GIORGIO GIORGI

Department of Economics and Management
Via S. Felice, 5 - 27100 Pavia
Italy
e-mail: giorgio.giorgi@unipv.it

Abstract

It is shown that almost all theorems of the alternative for linear systems can be obtained from a unique general theorem of the alternative, which is in turn a formal modification of the well-known theorem of the alternative due to Farkas (Farkas' theorem or Farkas-Minkowski's theorem). Furthermore, we prove that all these theorems of the alternative are in fact equivalent statements. Other remarks on the various proofs of Farkas' theorem are made.

1. Introduction

The literature on theorems of the alternative, both for linear and nonlinear systems, is indeed huge. Quite recently C. T. Perng [67] has given a list of 28

Keywords and phrases: theorems of the alternative for linear systems, Farkas' theorem, Farkas' lemma, Farkas-Minkowski's theorem.

2020 Mathematics Subject Classification: 15A39, 15A03, 15A04, 06F25.

Received January 29, 2020; Accepted February 26, 2020

theorems which are all equivalent to the well-known Farkas' theorem or Farkas' lemma (or Farkas-Minkowski's theorem). Some years ago De Giuli et al. [17] obtained a "mega-theorem" of the alternative for linear systems which generates 225 (!) special cases, among which the said Farkas' theorem. The aim of the present paper is to show "step by step" how to obtain from Farkas' theorem all known theorems of the alternative for linear systems and to obtain, as a corollary, the so-called *Tucker's theorem or key theorem*, which in turn generates Farkas' theorem. So, it is proved that all theorems of the alternative for linear systems have a "common father": Farkas' theorem; moreover, they are in fact all equivalent. However, it is possible to show that not only the first published result concerning a theorem of the alternative for linear systems, i.e., *Gordan's theorem* [44], can be obtained from Farkas' theorem, but that also the vice versa holds. Therefore, this first published result, due to Gordan, is perhaps indeed the true ancestor of all theorems of the alternative for linear systems.

The paper is organized as follows. In the Introduction, we recall the basic structure of a theorem of the alternative and recall Farkas' theorem. In Section 2, we obtain, by algebraic manipulations of Farkas' theorem, two theorems which in turn generate a long list of theorems of the alternative for linear systems. In the same section, we show that all these theorems are in fact equivalent. Section 3 will be concerned with some approaches in proving Farkas' theorem.

We shall use the following notations:

• $A \geqq B$; $A \geq B$; $A > B$, where A and B are (real) matrices of the same order (m, n), mean, respectively, $a_{ij} \geqq b_{ij}$, $\forall i, j$; $A \geqq B$, but $A \neq B$; $a_{ij} > b_{ij}$, $\forall i, j$. Similarly for the notations $A \leqq B$, $A \leq B$, $A < B$.

• $A \geqq [0]$; $A \geq [0]$, $A > [0]$, where $[0]$ is the (m, n) matrix with all zero elements, will denote, respectively, a *nonnegative matrix*, a *semipositive matrix*, a *positive matrix*.

• The same convention is used to compare two vectors x and y of $\mathbb{R}^n$. If $x \geqq [0]$, being $[0]$ the vector of $\mathbb{R}^n$ with all zero elements, we speak of *nonnegative vectors*; if $x \geq [0]$, we speak of *semipositive vectors*; if $x > [0]$, we speak of *positive vectors*. The notations $x < [0]$, $x \leq [0]$, $x \leqq [0]$ are obvious.

• $A^\top$ and $x^\top$ denote, respectively, the transpose of A and x. The identity matrix is denoted by I and the vector $u^\top = [1, 1, ..., 1]$ denotes the summation vector of $\mathbb{R}^n$.

We recall that in general a theorem of the alternative is a result having the following structure: between two given propositions (usually systems of linear or also nonlinear relations), say a “primal” system S and a “dual” system S^*, one and only one admits solutions, i.e., S admits solutions if and only if S^* does not admit solutions (equivalently: S^* admits solutions if and only if S does not admit solutions).

Perhaps Farkas’ theorem or Farkas’ lemma (or Farkas-Minkowski’s theorem) is the most well-known and quoted theorem of the alternative concerning linear systems.

Theorem 1 (Farkas’ Theorem)**.** *Let be given a matrix A of order (m, n) and a vector $b \in \mathbb{R}^m$. Then the system*

$$S_1 \equiv \{Ax = b;\ x \geqq [0]\}$$

admits solutions $x \in \mathbb{R}^n$ if and only if the system

$$S_1^* \equiv \{y^\top A \geqq [0];\ y^\top b < 0\}$$

does not admit solutions $y \in \mathbb{R}^m$.

In the next section, we shall derive from this result, by simple algebraic manipulations, almost all theorems of the alternative for linear systems quoted in the literature and many other new ones. We shall postpone to Section 3, the question of the proof of Farkas’ theorem.

Remark 1. Obviously, sistem S_1^* in Theorem 1 can be equivalently written in the form

$$S_1^* \equiv \{y^\top A \leqq [0];\ y^\top b > 0\}.$$

Moreover, S_1 and S_1^* of Theorem 1 can be equivalently written as

$$S_1 \equiv \{x^\top A^\top = b^\top;\ x \geqq [0]\},$$

$$S_1^* \equiv \{A^\top y \geqq [0];\ b^\top y < 0\}.$$

Another equivalent formulation of Farkas' theorem (not in an "alternative form") is the following one:

• A necessary and sufficient condition for S_1 to have solutions $x \in \mathbb{R}^n$ is the validity of the implication

$$y^\top A \geqq [0] \Rightarrow y^\top b \geqq [0].$$

Finally, note that for $b = [0]$, Farkas' theorem becomes trivial.

2. Farkas' Theorem as a Father of Several Theorems of the Alternative

The following result is a simple formal variant of Farkas' theorem and generates a first list of theorems of the alternative for linear systems.

Theorem 2. *Let be given the positive integers* $m_1,\ m_2,\ m_3,\ n_1,\ n_2$; *the matrices* A_{ij} *of order* $(m_i,\ n_j)$, $i = 1, 2, 3$; $j = 1, 2$; *the vectors* $b^i \in \mathbb{R}^m$, $i = 1, 2, 3$. *The system*

$$\begin{cases} A_{11}x^1 + A_{12}x^2 \leqq b^1, \\ A_{21}x^1 + A_{22}x^2 = b^2, \\ A_{31}x^1 + A_{32}x^2 \geqq b^3, \\ x^1 \in \mathbb{R}^{n_1},\ x^2 \in \mathbb{R}^{n_2},\ x^1 \geqq [0], \end{cases} \tag{1}$$

admits solutions $(x^1,\ x^2)$ *if and only if the system*

$$\begin{cases} (y^1)^\top A_{11} + (y^2)^\top A_{21} + (y^3)^\top A_{31} \geqq [0], \\ (y^1)^\top A_{12} + (y^2)^\top A_{22} + (y^3)^\top A_{32} = [0], \\ (y^1)^\top b^1 + (y^2)^\top b^2 + (y^3)^\top b^3 < 0, \\ y^1 \in \mathbb{R}^{m_1},\ y^2 \in \mathbb{R}^{m2},\ y^3 \in \mathbb{R}^{m_3},\ y^1 \geqq [0],\ y^3 \leqq [0], \end{cases} \tag{2}$$

does not admit solutions $(y^1,\ y^2,\ y^3)$.

Proof. The above result comes out at once from Farkas' theorem: we put $x^2 = v^1 - v^2$, with $v^1 \geqq [0]$, $v^2 \geqq [0]$ and then we transform inequalities into equalities by means of the "slack vectors" $s^1 \geqq [0]$, $s^2 \geqq [0]$. System (1) can be therefore rewritten in the form

$$\begin{bmatrix} A_{11} & A_{12} & -A_{12} & I & [0] \\ A_{21} & A_{22} & -A_{22} & [0] & [0] \\ A_{31} & A_{32} & -A_{32} & [0] & -I \end{bmatrix} \begin{bmatrix} x^1 \\ v^1 \\ v^2 \\ s^1 \\ s^2 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \tag{3}$$

with $x^1 \geqq [0]$, $v^1 \geqq [0]$, $v^2 \geqq [0]$, $s^1 \geqq [0]$, $s^2 \geqq [0]$. Applying to (3) Farkas' theorem we obtain the thesis.

From Theorem 2 it is possible to obtain easily a first list of theorems of the alternative. In the following list we use the short convention

$$S_k \equiv \{...\};$$

$$S_k^* \equiv \{...\},$$

in order to specify that the "primal" system S_k admits solutions if and only if the "dual" system S_k^* does not admit solutions.

(1)

$$S_2 \equiv \{Ax = b\};$$

$$S_2^* \equiv \{y^\top A = [0],\ y^\top b \neq 0\}.$$

Note that this result gives necessary and sufficient conditions for the existence of solutions of a non-homogeneous system of linear equations: system S_2 admits solutions if and only if it holds $y^\top b = 0$ for any vector y such that $y^\top A = [0]$. This result is sometimes called the *Fredholm theorem of the alternative*.

(2)

$$S_3 \equiv \{Ax \leqq b\};$$

$$S_3^* \equiv \{y^\top A = [0],\ y \geqq [0],\ y^\top b < 0\}.$$

(3)

$$S_4 \equiv \{Ax \leqq b,\ x \geqq [0]\};$$

$$S_4^* \equiv \{y^\top A \geqq [0],\ y \geqq [0],\ y^\top b < 0\}.$$

This result has been used by Gale [30] and by Los [55] to prove the existence of equilibrium solutions in the classical von Neumann balanced growth model.

(4) (*Theorem of the alternative of Ky Fan*; see Fan [19], Giannessi [32]).

$$S_5 \equiv \{Ax \leqq b^1,\ Bx = b^2\};$$

$$S_5^* \equiv \{(y^1)^\top A + (y^2)^\top B = [0],\ y^1 \geqq [0],\ (y^1)^\top b^1 + (y^2)^\top b^2 < 0\}.$$

(5)

$$S_6 \equiv \{Ax \leqq b^1,\ Bx = b^2,\ x \geqq [0]\};$$

$$S_6^* \equiv \{(y^1)^\top A + (y^2)^\top B \geqq [0],\ y^1 \geqq [0],\ (y^1)^\top b^1 + (y^2)^\top b^2 < 0\}.$$

(6)

$$S_7 \equiv \{Ax + Bz = b,\ x \geqq [0]\};$$

$$S_7^* \equiv \{y^\top A \geqq [0],\ y^\top B = [0],\ y^\top b < 0\}.$$

(7)

$$S_8 \equiv \{Ax + Bz \leqq b,\ x \geqq [0]\};$$

$$S_8^* \equiv \{y^\top A \geqq [0],\ y^\top B = [0],\ y^\top b < 0,\ y \geqq [0]\}.$$

(8)

$$S_9 \equiv \{Cx \geqq c\};$$

$$S_9^* \equiv \{y^\top C = [0],\ y \geqq [0],\ y^\top c > 0\}.$$

(9)

$$S_{10} \equiv \{Cx \geqq c,\ x \geqq [0]\};$$

$$S_{10}^* \equiv \{y^\top C \leqq [0],\ y \geqq [0],\ y^\top c > 0\}.$$

(10)

$$S_{11} \equiv \{Cx \geqq c,\ Bx = b,\ x \geqq [0]\};$$

$$S_{11}^* \equiv \{(y^1)^\top C + (y^2)^\top B \leqq [0],\ y^1 \geqq [0],\ (y^1)^\top c + (y^2)^\top b > 0\}.$$

(11)

$$S_{12} \equiv \{Cx + Dz \geqq c,\ x \geqq 0\};$$

$$S_{12}^* \equiv \{y^\top C \leqq [0],\ y^\top D = [0],\ y \geqq [0],\ y^\top c > 0\}.$$

(12)

$$S_{13} \equiv \{Ax \geqq [0],\ b^\top x < 0\};$$

$$S_{13}^* \equiv \{y^\top A = b,\ y \geqq [0]\}.$$

This result is nothing but Farkas' theorem, where the "primal" and the "dual" problems have been interchanged.

(13)

$$S_{14} \equiv \{Ax \geqq [0],\ x \geqq [0],\ b^\top x < 0\};$$

$$S^*_{14} \equiv \{y^\top A \leqq b,\ y \geqq [0]\}.$$

(14)

$$S_{15} \equiv \{Cx \leqq [0],\ Dx = [0],\ c^\top x < 0\};$$

$$S^*_{15} \equiv \{y^\top C + v^\top D + c^\top = [0],\ y \geqq [0]\}.$$

(15)

$$S_{16} \equiv \{Cx + Dz = [0],\ a^\top x + b^\top z > 0,\ z \geqq [0]\};$$

$$S^*_{16} \equiv \{y^\top C + a^\top = [0],\ y^\top D + b^\top \leqq [0]\}.$$

It is easy to reformulate Theorem 2 by allowing in (1) also nonpositive vectors. For example, it is possible to obtain the following systems "in alternative":

$$\begin{cases} A_{11}x^1 + A_{12}x^2 + A_{13}x^3 \geqq b^1, \\ A_{21}x^1 + A_{22}x^2 + A_{23}x^3 = b^2, \\ A_{31}x^1 + A_{32}x^2 + A_{33}x^3 \leqq b^3, \\ x^1 \geqq [0],\ x^2 \text{ arbitrary},\ x^3 \leqq [0], \end{cases}$$

and

$$\begin{cases} (y^1)^\top A_{11} + (y^2)^\top A_{21} + (y^3)^\top A_{31} \leqq [0], \\ (y^1)^\top A_{12} + (y^2)^\top A_{22} + (y^3)^\top A_{32} = [0], \\ (y^1)^\top A_{13} + (y^2)^\top A_{23} + (y^3)^\top A_{33} \geqq [0], \\ (y^1)^\top b^1 + (y^2)^\top b^2 + (y^3)^\top b^3 > 0, \\ y^1 \geqq [0],\ y^2 \text{ arbitrary},\ y^3 \leqq [0]. \end{cases}$$

In order to obtain a theorem which generates a second list of theorems of the alternative for linear systems, we need three preliminary results, which are in turn simple formal variants of Farkas' theorem. We continue to use the convention to denote by S_k and S_k^* the "primal" and the "dual" propositions in alternative. As usual, the order and the conformability of the various matrices and vectors have to be respected.

Lemma 1.

$$S_{17} \equiv \{Ax + Bv + Cz = [0],\ v \geqq [0],\ z > [0]\};$$

$$S_{17}^* \equiv \{y^\top A = [0],\ y^\top B \geqq [0],\ y^\top C \geq [0]\}.$$

Proof. If S_{17}^* does not admit solution, also the following system

$$S_{17}^* \equiv \{y^\top A = [0],\ y^\top B \geqq [0],\ y^\top C \geq [0],\ y^\top Cu > 0\}$$

does not admit solution and vice versa. But if S_{17}^* does not admit solution, by the result sub (6), the system

$$\tilde{S}_{17} \equiv \{Ax + Bv + C\bar{z} = -Cu,\ v \geqq [0],\ \bar{z} \geqq [0]\}$$

will admit solutions.

Therefore, system S_{17} admits solutions: put $z = (\bar{z} + u) > [0]$. It is then easy to perform the inverse reasoning: if S_{17}^* admits solutions, then S_{17} does not admit solutions.

Lemma 2.

$$S_{18} \equiv \left\{ A_1 x^1 + A_2 x^2 + A_3 x^3 + \sum_{j=4}^{q} A_j x^j = [0], \right.$$

$$\left. x^2 \geqq [0],\ x^3 > [0],\ x^j \geq [0],\ j = 4, \ldots, q \right\};$$

(A_i *denotes here the* i-*th matrix*, *not the* i-*th row of* A).

$$S_{18}^* \equiv \left\{ \begin{matrix} y^\top A_1 = [0],\ y^\top A_j \geqq [0],\ j = 2, ..., q;\ and\ y^\top A_3 \geq [0] \\ or\ y^\top A_j > [0]\ for\ at\ least\ an\ index\ j,\ 4 \leqq j \leqq q \end{matrix} \right\}.$$

Proof. We note first that $x^j \geq [0]$ is equivalent to $\{x^j \geqq [0], (u^j)^\top x^j - w_j = 0, w_j > 0\}$, where $w_j \in \mathbb{R}$ and $(u^j)^\top = (1, 1, ..., 1)$. Here u^j is of the same order of x^j, for $j = 4, ..., q$. Then system S_{18} can be rewritten in the equivalent form

$$\begin{bmatrix} A_1 \\ [0] \\ [0] \\ \vdots \\ [0] \end{bmatrix} x^1 + \begin{bmatrix} A_2 & A_4 & A_5 & \cdots & A_q \\ [0] & u^4 & [0] & \cdots & [0] \\ [0] & [0] & u^5 & \cdots & [0] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ [0] & [0] & [0] & \cdots & u^q \end{bmatrix} \begin{bmatrix} x^2 \\ x^4 \\ x^5 \\ \vdots \\ x^q \end{bmatrix} + \begin{bmatrix} A_3 & [0] \\ [0] & -I \end{bmatrix} \begin{bmatrix} x^3 \\ w \end{bmatrix} = [0],$$

$$(x^2, x^4, x^5, ..., x^q)^\top \geqq [0];\ (x^3, w)^\top > [0].$$

If this system does not admit solutions, by Lemma 1 it will exist vectors y and $v = (v_j)$, $j = 4, ..., q$, such that

(a) $y^\top A_1 = [0]$;

(b) $y^\top A_2 \geqq [0]$;

(c) $y^\top A_j + (u^j)^\top v_j \geqq [0]$, $j = 4, ..., q$;

(d) $[y^\top A_3; -v] \geq [0]$.

From (d), we get $y^\top A_3 \geqq [0]$ and $v \leqq [0]$; from (c), we get $y^\top A_j \geqq -(u^j)^\top v_j \geqq [0]$. Moreover: if $v = [0]$, then $y^\top A_3 \geq [0]$, but if there exists an index j_0 such that $v_{j_0} < 0$, then $y^\top A_{j_0} \geqq -u^{j_0} v_{j_0} > 0$. Again by Lemma 1, we can affirm that if S_{18} admits solutions, then S_{18}^* does not admit solutions.

Lemma 3.

$$S_{19} \equiv \left\{ A_1 x^1 + A_2 x^2 + A_3 x^3 + \sum_{j=4}^{q} A_j x^j = b, \right.$$

$$\left. x^2 \geqq [0],\ x^3 > [0],\ x^j \geq [0],\ j = 4, \ldots, q \right\};$$

(A_k *is the* k*-th matrix*).

$$S_{19}^* \equiv \left\{ \begin{array}{l} y^\top A_1 = [0],\ y^\top A_j \geqq [0],\ j = 2, \ldots, q;\ y^\top b \leqq 0 \\ and\ \ y^\top b < 0\ or\ y^\top A_3 \geq [0]\ or\ y^\top A_j > [0] \\ for\ at\ least\ an\ index\ j,\ 4 \leqq j \leqq q \end{array} \right\}.$$

Proof. S_{19} can be rewritten in the form

$$\left\{ A_1 x^1 + A_2 x^2 + (A_3 \mid -b) \begin{bmatrix} x^3 \\ 1 \end{bmatrix} + \sum_{j=4}^{q} A_j x^j = [0]; \right.$$

$$\left. x^2 \geqq [0],\ (x^3; 1)^\top > [0],\ x^j \geq [0],\ j = 4, \ldots, q \right\}.$$

By means of Lemma 2, we obtain the desired result.

The following theorem generates easily a second series of theorems of the alternative for linear systems, among which some "classical" theorems of the alternative.

Theorem 3. *Let be given the folllowing two partitions of integers*

$$M_i,\ i = 1, 2, \ldots, p;\ \bigcup_{i=1}^{p} M_i = (1, 2, \ldots, m);$$

$$N_j,\ j = 1, 2, \ldots, q;\ \bigcup_{j=1}^{q} N_j = (1, 2, \ldots, n).$$

Let be given the matrices

$$A_{ij} = (a_{kr}),\ k \in M_i,\ r \in N_j,$$

and the vectors

$$b^i = (b_k),\ k \in M_i;\ x^j = (x_r),\ r \in N_j;\ y^i = (y_k),\ k \in M_i.$$

Then the system

$$\begin{cases} A_{11}x^1 + A_{12}x^2 + A_{13}x^3 + \sum_{j=4}^{q} A_{1j}x^j = b^1, \\ A_{21}x^1 + A_{22}x^2 + A_{23}x^3 + \sum_{j=4}^{q} A_{2j}x^j \leqq b^2, \\ A_{31}x^1 + A_{32}x^2 + A_{33}x^3 + \sum_{j=4}^{q} A_{3j}x^j < b^3, \\ A_{i1}x^1 + A_{i2}x^2 + A_{i3}x^3 + \sum_{j=4}^{q} A_{ij}x^j \leq b^i,\ (i = 4, ..., p), \\ x^1 \ \textit{arbitrary},\ x^2 \geqq [0],\ x^3 > [0],\ x^j \geq [0],\ j = 4, ..., q, \end{cases} \tag{4}$$

admits solutions $(x^1, x^2, x^3, x^j, j = 4, ..., q)$ *if and only if the system*

$$\begin{cases} (y^1)^\top A_{11} + \sum_{i=1}^{p} (y^i)^\top A_{i1} = [0], \\ E^j \equiv (y^1)^\top A_{1j} + \sum_{i=2}^{p} (y^i)^\top A_{ij} \geqq [0],\ j = 2, ..., q, \\ y^\top b \equiv (y^1)^\top b^1 + \sum_{i=2}^{p} (y^i)^\top b^i \leqq 0, \\ y^1 \ \textit{arbitrary},\ y^i \geqq [0],\ i = 2, ..., p, \end{cases} \tag{5}$$

does not admit solutions y *such that*

$$y^\top b < 0,$$

or

$$y^3 \geq [0],$$

or

$$y^4 > [0], \ ...\ \textit{or}\ ...,\ y^p > [0],$$

or

$$E^3 \geq [0],$$

or

$$E^4 > [0], \ldots \textit{ or } \ldots, E^q > [0].$$

Proof. Let us introduce in the inequalities of system (4) the following "slack vectors": $w^2 \geqq [0]$, $w^3 > [0]$, $w^i \geq [0]$, $i = 4, \ldots, p$. Then (4) can be rewritten in the form:

$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \\ A_{p1} \end{bmatrix} x^1 + \begin{bmatrix} A_{12} & [0] \\ A_{22} & I \\ A_{32} & [0] \\ \vdots & \vdots \\ A_{p2} & [0] \end{bmatrix} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} + \begin{bmatrix} A_{13} & [0] \\ A_{23} & [0] \\ A_{33} & I \\ \vdots & \vdots \\ A_{p3} & [0] \end{bmatrix} \begin{bmatrix} x^3 \\ w^3 \end{bmatrix}$$

$$+ \sum_{j=4}^{q} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \\ \vdots \\ A_{pj} \end{bmatrix} x^j + \sum_{i=4}^{p} \begin{bmatrix} [0] \\ [0] \\ \vdots \\ I \\ \vdots \\ [0] \end{bmatrix} w^i = b,$$

with x^1 arbitrary, $(x^2, w^2)^\top \geqq [0]$, $(x^3, w^3)^\top > [0]$, $x^j \geq [0]$, $j = 4, \ldots, q$, $w^i \geq [0]$, $i = 4, \ldots, p$.

Then, by application of Lemma 3, we get that (4) admits solutions if and only if (5) does not admit solutions.

By simple considerations and algebraic manipulations it is easily possible to obtain from Theorem 3 a quite long second list of theorems of the alternative. We give only some relevant examples of this second list, always under the notations and conventions previously adopted.

1.2 (Gordan's Theorem)**.**

$$S_{20} \equiv \{Ax = [0],\ x \geq [0]\};$$

$$S_{20}^{*} \equiv \{y^{\top} A > [0]\}.$$

The above result is perhaps the first published theorem of the alternative for linear algebraic systems, being established as early as 1873 by Gordan [44]. It can be obtained directly form Farkas' theorem, but also the vice versa holds (see further in Section 3). Gordan's theorem has also a "geometrical" version (see, e.g., Murata [60]): let L be the linear subspace generated by the rows of A, $(A_1, \ldots, A_m)$:

$$L = \{z : z^{\top} = y^{\top} A,\ y \in \mathbb{R}^m\}$$

and let

$$L^{\perp} = \{x \in \mathbb{R}^n,\ z^{\top} x = 0,\ \forall z \in L\}.$$

Then, exactly one of the following alternatives holds: either L contains a positive vector, or $L^{\perp}$ contains a semipositive vector.

2.2 (Stiemke's Theorem)**.**

$$S_{21} \equiv \{Ax = [0],\ x > [0]\};$$

$$S_{21}^{*} \equiv \{y^{\top} A \geq [0]\}.$$

Stiemke's theorem is equivalent to Gordan's theorem, in the sense that it is another algebraic form of Gordan's theorem. This fact was recognized by Antosiewicz [2], with reference to a more general theorem, and by Giorgi [36]. See also Giorgi [35, 38, 39), Nikaido [62].

Let us denote by $R(A)$ the *column space* of the matrix A and by $N(A)$ the *null space* of A (i.e., the set of vectors x such that $Ax = [0]$). Recall then that the subspaces $R(A)$, $N(A)$, $R(A^{\top})$ and $N(A^{\top})$ satisfy the relations

$$R(A^{\perp}) = N(A^{\top}); \quad R(A^{\top})^{\perp} = N(A);$$

$$R(A) = N(A^\top)^\perp; \quad R(A^\top) = N(A)^\perp.$$

From Gordan's theorem it follows that exactly one of the following statements must be true (Stiemke's theorem):

I. $N(A)$ has a positive vector;

II. $N(A)^\perp = R(A^\top)$ has a semipositive vector.

3.2 (Ville's Theorem)**.**

$$S_{22} \equiv \{Ax \leqq [0],\ x \geq [0]\};$$

$$S_{22}^* \equiv \{y^\top A > [0],\ y \geqq [0]\}.$$

Ville's theorem of the alternative is essentially the same theorem proved by von Neumann and Morgenstern [61] and by Gale [27]. See also the result sub (6.2). Also Ville's theorem can be obtained directly from Farkas' theorem. It must be remarked that system S_{22}^* in Ville's theorem can be equivalently rewritten in the form

$$S_{22}^{**} \equiv \{y^\top A > [0],\ y \geq [0]\}$$

or also in the form

$$S_{22}^{***} \equiv \{y^\top A > [0],\ y > [0]\}.$$

Following a current terminology in matrix theory (see, e.g., Fiedler and Ptàk [22]), we say that a matrix A of order (m, n) is an S-*matrix* or belongs to the S-*class* if and only if the system

$$\begin{cases} Ax > [0], \\ x \geq [0] \end{cases}$$

admits solutions. The matrix A belongs to the S_0-*class* or is an S_0-*matrix* if and only if the system

$$\begin{cases} Ax \geqq [0], \\ x \geq [0] \end{cases}$$

admits solutions.

Therefore, Ville's theorem can be restated as follows:

• Let A be of order (m, n). Then $A \in S$ if and only if $-A^{\top} \notin S_0$.

Ville's theorem is important in the analysis of certain linear economic models (see, e.g., Giorgi and Meriggi [41, 42], De Giuli et al. [18], Giorgi [40]).

(4.2)

$$S_{23} \equiv \{Ax \leqq [0],\ x > [0]\};$$

$$S_{23}^{*} \equiv \{y^{\top} A \geq [0],\ y \geqq [0]\}.$$

This theorem of the alternative is attributed by Marlow [57] to Tucker [79].

(5.2) (Second form of Gordan's theorem)**.**

$$S_{24} \equiv \{Bx < [0]\};$$

$$S_{24}^{*} \equiv \{y^{\top} B = [0],\ y \geq [0]\}.$$

(6.2) (First theorem of the alternative of Gale [29])**.**

$$S_{25} \equiv \{Bx < [0],\ x \geqq [0]\};$$

$$S_{25}^{*} \equiv \{y^{\top} B \geqq [0],\ y \geq [0]\}.$$

This result is a restatement of Ville's theorem sub 3.2), with $A \equiv (-B^{\top})$ and where the roles of S_{22} and S_{22}^{*} have been interchanged.

(7.2)

$$S_{26} \equiv \{Ax < [0],\ x > [0]\};$$

$$S_{26}^{*} \equiv \{y^{\top} A \geqq [0],\ y \geq [0] \text{ or } y^{\top} A \geq [0],\ y \geqq [0]\}.$$

This result may be considered as a variant of Ville's theorem.

(8.2)

$$S_{27} \equiv \{Bx \leq [0]\};$$

$$S_{27}^{*} \equiv \{y^{\top}B = [0],\ y > [0]\}.$$

This result is only a formal variant of Stiemke's theorem.

(9.2)

$$S_{28} \equiv \{Bx \leq [0],\ x \geqq [0]\};$$

$$S_{28}^{*} \equiv \{y^{\top}B \geqq [0],\ y > [0]\}.$$

This result is only a formal variant of the one sub 4.2).

(10.2)

$$S_{29} \equiv \{Ax < b\};$$

$$S_{29}^{*} \equiv \{y^{\top}A = [0]\}; \quad y^{\top}[I;\ -b] \geq [0].$$

This result is attributed to Carver [14]. Note that S_{29}^{*} may be rewritten as $\{y^{\top}A = [0],\ y^{\top}b \leqq 0,\ y \geq [0]\}$.

(11.2)

$$S_{30} \equiv \{Ax \leqq b,\ x > [0]\};$$

$$S_{30}^{*} \equiv \{y^{\top}[A;\ -b] \geq [0],\ y \geqq [0]\}.$$

(12.2)

$$S_{31} = \{Ax = b,\ x > [0]\};$$

$$S_{31}^{*} \equiv \{y^{\top}[A;\ -b] \geq [0]\}.$$

The above result may be considered a non-homogeneous version of Stiemke's theorem.

(13.2)

$$S_{32} \equiv \{Ax \geqq [0],\ Bx > [0],\ x \geqq [0]\};$$

$$S_{32}^{*} \equiv \{y^{\top}A + v^{\top}B \leqq [0],\ y \geqq [0],\ v \geq [0]\}.$$

(14.2) (Motzkin's theorem of the alternative or Motzkin's transposition theorem)**.**

$$S_{33} \equiv \{Ax = [0],\ Bx \geqq [0],\ Dx > [0]\};$$

$$S_{33}^{*} \equiv \{y^{\top}A + v^{\top}B + w^{\top}D = [0],\ v \geqq [0],\ w \geq [0]\}.$$

(A and B may be missing. It seems that this result has been anticipated by J. B. J. Fourier [24]).

(15.2) (Tucker's theorem of the alternative)**.**

$$S_{34} \equiv \{Ax = [0],\ Bx \geqq [0],\ Cx \geq [0]\};$$

$$S_{34}^{*} \equiv \{y^{\top}A + v^{\top}B + w^{\top}C = [0],\ v \geqq [0],\ w > [0]\}.$$

(A and B may be missing).

(16.2) (Slater's theorem of the alternative)**.**

$$S_{35} \equiv \{Ax = [0],\ Bx \geqq [0],\ Cx \geq [0],\ Dx > [0]\};$$

$$S_{35}^{*} \equiv \left\{ \begin{array}{l} \{y^{\top}A + v^{\top}B + (w^{1})^{\top}C + (w^{2})^{\top}D = [0],\ v \geqq [0],\ w^{1} \geqq [0], \\ w^{2} \geq [0]\} \text{ or } \{y^{\top}A + v^{\top}B + (w^{1})^{\top}C + (w^{2})^{\top}D = [0], \\ v \geqq [0],\ w^{1} > [0],\ w^{2} \geqq [0]\} \end{array} \right\}.$$

(A and B may be missing).

(17.2) (First Mangasarian's theorem of the alternative)**.**

$$S_{36} \equiv \left\{ \begin{array}{l} Ax \geq [0],\ Bx \geqq [0],\ Cx \geqq [0],\ Dx = 0 \\ \text{or } Ax \geqq [0],\ Bx > [0],\ Cx \geqq [0],\ Dx = [0] \end{array} \right\};$$

$$S_{36}^{*} \equiv \left\{ \begin{array}{l} \{(y^1)^\top A + (y^2)^\top B + (y^3)^\top C + (y^4)^\top D = [0], \\ y^1 > [0],\ y^2 \geq [0],\ y^3 \geqq [0]\} \end{array} \right\}.$$

(18.2)

$$S_{37} \equiv \{Ax \leq [0],\ x > [0]\};$$

$$S_{37}^{*} \equiv \{\{y^\top A \geq [0],\ y \geqq [0]\} \quad \text{or} \quad \{y^\top A \geqq [0],\ y > [0]\}\}.$$

(19.2) (Second Mangasarian's theorem of the alternative)**.**

$$S_{38} \equiv \{Ax \leq b\};$$

$$S_{38}^{*} \equiv \left\{ \begin{array}{l} \{y^\top A = [0],\ y \geqq [0],\ b^\top y = -1\} \\ \text{or } \{y^\top A = [0],\ y > [0],\ b^\top y \leqq [0]\} \end{array} \right\}.$$

(20.2) (Second Gale's theorem of the alternative)**.**

$$S_{39} \equiv \{Ax \leqq b,\ x \geqq [0]\};$$

$$S_{39}^{*} \equiv \{y^\top A \geqq [0],\ y \geqq [0],\ y^\top b < 0\}.$$

(21.2)

$$S_{40} \equiv \{Ax \geqq [0],\ Bx > [0],\ x > [0]\};$$

$$S_{40}^{*} \equiv \{\{y^\top A + v^\top B \leqq [0],\ y \geqq [0],\ v \geq [0]\}$$

$$\text{or } \{y^\top A + v^\top B \leq [0],\ y \geqq [0],\ v \geqq [0]\}\}.$$

(22.2) (First Fenchel's theorem of the alternative)**.**

$$S_{41} \equiv \{Ax + Bz = [0],\ x \geqq [0],\ z \geq [0]\};$$

$$S_{41}^{*} \equiv \{y^\top A \geqq [0],\ y^\top B > [0]\}.$$

(23.2) (Second Fenchel's theorem of the alternative)**.**

$$S_{42} \equiv A\{x + Bz = [0],\ x \geqq [0],\ z > [0]\};$$

$$S_{42}^{*} \equiv \{y^{\top}A \geqq [0],\ y^{\top}B \geq [0]\}.$$

(24.2)

$$S_{43} \equiv \{Ax + Bz = [0],\ x \geq [0],\ z \geq [0]\};$$

$$S_{43}^{*} \equiv \{\{y^{\top}A > [0],\ y^{\top}B \geqq [0]\} \quad \text{or} \quad \{y^{\top}A \geqq [0],\ y^{\top}B > [0]\}\}.$$

(25.2)

$$S_{44} \equiv \{Ax + Bz = [0],\ Cx + Dz > [0],\ z \geqq [0]\};$$

$$S_{44}^{*} \equiv \{y^{\top}A + v^{\top}C = [0],\ y^{\top}B + v^{\top}D \leqq [0],\ v \geq [0]\}.$$

The list could be made longer, but we think that the reader has realized the techniques to get from Theorem 3 the theorems of the alternative needed for his purposes.

Remark 2. From Theorem 3 it is also possible to obtain various *non-homogeneous versions* of theorems of the alternative for linear systems. We report only the *non-homogeneous Farkas' theorem or Duffin's theorem of the alternative* (see, e.g., Mangasarian [56]).

$$S_{45} \equiv \{Ax \leqq b,\ c^{\top}x > \gamma\}.$$

$$S_{45}^{*} \equiv \{\{y^{\top}A = [0],\ y^{\top}B < 0\ \ y \geqq [0]\}$$

$$\text{or } \{y^{\top}A = c^{\top},\ y^{\top}B \leqq \gamma,\ y \geqq [0]\}\}.$$

This result can be obtained at once from Motzkin's theorem. For a general treatment of non-homogeneous theorems of the alternative for linear systems, the reader is referred to Szilagyi [74].

From the previous results it is also possible to obtain in a simple way two theorems due to A. W. Tucker [79].

Theorem 4 (Key Theorem)**.** *Let be given the matrix* A *of order* (m, n)*. Then, the two systems*

$$\{Ax \geqq [0]\}; \qquad \{y^\top A = [0],\ y \geqq [0]\}$$

admit, *respectively*, *solutions* x^0 *and* y^0 *such that*

$$Ax^0 + y^0 > [0].$$

Proof. In order to prove that the following system

$$\begin{cases} A^\top y = [0], \\ -Ax \leqq [0], \\ -Ax - Iy < [0], \\ y \geqq [0] \end{cases}$$

admits solutions, it is necessary and sufficient to show, on the grounds of Theorem 3, that its "dual system"

$$\begin{cases} (z^2)^\top A + (z^3)^\top A = [0], \\ (z^1)^\top A^\top - (z^3)^\top \geqq [0], \\ z^1 \in \mathbb{R}^n,\ z^2 \geqq [0],\ z^3 \geq [0], \end{cases}$$

does not admit solutions. Let us suppose, on the contrary, that this dual system admits solutions: we should get the following contradiction

$$0 = (z^2 + z^3)^\top Az^1 \geqq (z^2 + z^3)^\top z^3 \geqq (z^3)^\top z^3 > 0.$$

Theorem 4, called sometimes also "key theorem", generates in its own turn several "classical" theorems of the alternative, among which Farkas' theorem. Hence, we have, so to speak, closed the "loop": all theorems of the alternative for linear systems not only have a common father but they are in fact all equivalent. Moreover, it is true that Gordan's theorem can be obtained by Farkas' theorem, as previously seen, but also the vice versa holds (see, e.g., Giorgi [35, 36, 39], Güler [45], Borwein and Lewis [11], Perng [66, 67]). We can therefore affirm that Gordan's theorem, published in 1873, is perhaps the true "common father" of all other theorems of the alternative for linear systems. However, it must not be forgotten the contributions of

J. B. J. Fourier. Theorem 4 has been proved by induction by Tucker himself (Tucker [79]), by Mangasarian [56], by Kemp and Kimura [52], etc. Other proofs, in the author's opinion not less elementary, are due to Nikaido [62], Broyden [12], Singh and Patron [69]. A short and interesting proof, inspired by Morishima [58], is given by Fujmoto [24].

The following corollary to Theorem 4, also due to Tucker [79], has been used by Howe [49] and by Nikaido [62] in proving the existence of equilibrium solutions for the celebrated von Neumann growth model.

Corollary 1. *The system*

$$\{Ax \geqq [0],\ y^\top A \leqq [0],\ x \geqq [0],\ y \geqq [0]\}$$

admits solutions x *and* y *such that*

$$y + Ax > [0]; \quad x^\top - y^\top A > [0].$$

Proof. It is sufficient to apply Theorem 4 to the dual systems

$$\begin{bmatrix} I \\ A \end{bmatrix} x \geqq [0]; \qquad (w,\ y)^\top \begin{bmatrix} I \\ A \end{bmatrix} = [0],$$

with $(w,\ y) \geqq [0]$.

Theorem 5 (Tucker [79])**.** *Let be given the square skew-symmetric matrix* L *of order* n $(i.e., L^\top = -L)$*. Then the system*

$$\{Lx \geqq [0],\ x \geqq [0]\}$$

admits a solution x^0 *such that*

$$Lx^0 + x^0 > [0].$$

Proof. In order to prove that the following system

$$\{Lx \geqq [0],\ (L + I)x > [0],\ x \geqq [0]\}$$

admits solutions, it is necessary and sufficient, on the grounds of Theorem 3, to show that the dual system

$$\{y^{\top}L + v^{\top}(L+I) \leqq [0],\ y \geqq [0],\ v \geq [0]\}$$

does not admit solutions. Absurdly suppose that this dual system admits a solution. Being L skew-symmetric, it will hold $x^{\top}Lx = 0,\ \forall x \in \mathbb{R}^n,$ and, moreover,

$$(y^{\top}L + v^{\top}(L+I)) \leqq [0] \Leftrightarrow (-y^{\top}L^{\top} - v^{\top}L^{\top} + v^{\top} \leqq [0]) \Leftrightarrow$$

$$\Leftrightarrow v^{\top}(y+v)^{\top}L^{\top} = L(y+v).$$

Therefore, we obtain from the said dual system, the contradiction

$$0 < v^{\top}v \leqq v^{\top}(y+v) \leqq (y+v)^{\top}L(y+v) = 0.$$

As it is well-known, Theorem 5 is important in proving duality results in linear programming problems.

3. The Question of the Proof of Farkas' Theorem

Farkas' theorem is one of the most quoted and used theorems of the alternative for linear systems and has received attention also for what concerns its proof. Several papers have appeared, quite recently, with titles promising "short and elementary" proofs of Farkas' theorem (see the Bibliographical References at the end of the present paper). However, in most cases the proofs are not short, nor elementary. For a (not exhaustive) review of the proposed proofs of Farkas' theorem the reader may consult Giorgi [35, 38, 39] and Jacimovic [51].

Perhaps the most popular proof is the one based on a separation theorem between convex sets. In this type of proofs it is essential to demonstrate that a *finitely generated cone* is a closed set. Several authors skip this step and give for granted the above closedness. This is misleading, as Borwein [10] has proved that the closedness of a finitely generated cone is in fact *equivalent* to Farkas' theorem itself! Moreover, the question is not a trivial one, as the finitely generated cone C

$$C = \{z \in \mathbb{R}^m : z = Ax,\ x \geqq [0]\},$$

with A matrix of order (m, n), is the direct image (not the inverse image!) of a closed set. We give here a proof, quite elementary and not too long, of the closedness of C.

Theorem 6. *Let A be a matrix of order (m, n). Then the set*

$$C = \{z \in \mathbb{R}^m : z = Ax, x \geqq [0]\}$$

(*finitely generated cone or finite cone*) *is a closed convex cone of* $\mathbb{R}^m$.

Proof. First we remark that every hyperplane of $\mathbb{R}^m$ is a closed set and that every linear subspace of $\mathbb{R}^m$ is a closed set, being the intersection of hyperplanes passing through the origin (the intersection of an arbitrary number of closed sets is closed). For a direct proof that every linear subspace of $\mathbb{R}^m$ is closed one can see Bazaraa and Shetty [6]. The facts that C is a cone with vertex at the origin and that it is convex are obvious. Let us rewrite C as a (nonnegative) linear combination of the columns A^j of the matrix A:

$$C = \left\{z \in \mathbb{R}^m : z = \sum_{j=1}^{n} \mu_j A^j, \mu_j \geqq [0]\right\}.$$

We will show that the cone C is closed by an induction argument based on the number of the column vectors A^j, $J = 1, ..., k$.

When $k = 1$, C is either the origin $[0]$ or a half-line and is therefore closed. Now suppose that for $k > 1$ the cone generated by the vectors $A^1, ..., A^{k-1}$ is closed:

$$C_{k-1} = \left\{z \in \mathbb{R}^m : z = \sum_{j=1}^{k-1} \mu_j A^j, \mu_j \geqq [0]\right\}$$

is closed. We have to show that the cone

$$C_k = \left\{z \in \mathbb{R}^m : z = \sum_{j=1}^{k} \mu_j A^j, \mu_j \geqq [0]\right\}$$

is closed too. There are two cases.

(1) First, suppose that C_k contains the vectors $-A^1, -A^2, ..., -A^k$. Then C_k is a linear sub-space of dimension not exceeding k, so it is closed.

(2) Assume that at least one of the vectors $-A^1, -A^2, ..., -A^k$ does not belong to C_k, say $-A^k \notin C_k$ (renumber if necessary). Then, every $y \in C_k$ can be represented as $y = \bar{y} + \alpha A^k$, $\alpha \geqq 0$, where $\bar{y} \in C_{k-1}$. To show that C_k is closed, suppose that $\bar{z}$ is a limit point. Then, there exists a sequence $\{z^n\}_{n=1}^{\infty} \subset C_k$ such that $z^n \to \bar{z}$ as $n \to \infty$, where z^n has the form

$$z^n = \bar{y}^n + \alpha_n A^k, \quad \alpha_n \geqq 0.$$

If the sequence $\{\alpha_n\}$ is bounded, we can assume, without loss of generality, that the sequence converges to a limit α, and consequently $z^n - \alpha A^k \in C_{k-1}$, where this last set is closed. Therefore

$$z - \alpha A^k = \lim_{n\to\infty} (z^n - \alpha_n A^k) = \lim_{n\to\infty} \bar{y}^n \equiv \bar{y} \in C_{k-1},$$

since C_{k-1} is closed.

We can conclude that

$$z = \bar{y} + \alpha A^k \in C_k.$$

Thus, if the sequence $\{\alpha_n\}$ is bounded, the set C_k is closed.

We now assume that $\alpha_n \to \infty$ for $n \to \infty$. Then, since z^n converges, it is a bounded sequence. Hence $\{\alpha_n\}^{-1} z^n \to [0]$ as $n \to \infty$. It follows that $\{\alpha_n\}^{-1} \bar{y}^n + A^k \to [0]$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \{\alpha_n\}^{-1} \bar{y}^n = -A^k$. But, since C_{k-1} is closed, this means that $-A^k \in C_{k-1}$, which is a contradiction.

Shorter proofs of Theorem 6, but less elementary, are given, e.g., by Achmanov [1], Craven [15], Hestenes [46, 47].

Now we make some remarks and considerations on some other proofs of Farkas' theorem appeared in the literature.

(A) *The proof by induction of Gale* [28, 29] is indeed simple, not too long and does not require special algebraic or topological requirements. Perhaps this proof may be improved only with respect to the notations. See, e.g., Kurz and Salvadori [54], Giorgi [35], Bartl [4, 5]. In spite of its simplicity this proof is quite infrequent in the literature and it appears sometimes in mathematical economics textbooks.

(B) *Proofs of Farkas' theorem by means of Gordan's theorem*. The proof of Gordan's theorem by means of a separation argument *does not require* the closedness of a finitely generated cone and is therefore simpler than the direct proof of Farkas' theorem. The equivalence between Gordan's theorem and Stiemke's theorem has been proved by Antosiewicz [2], Giorgi [36], Perng [67]. In the paper of Giorgi [36] Theorem 4 (Tucker's theorem or key theorem) is then simply obtained from Stiemke's theorem. It is well-known that the key theorem generates, in its own turn, a sequence of theorems of the alternative for linear systems, among which Farkas' theorem. An interesting and new proposal to prove Gordan's theorem via variational principles is offered by Borwein and Lewis [11] and by Güler [45]. Borwein and Lewis prove in a straightforward way the following result.

Theorem 7. *Define the function*

$$f(x) = \ln\left(\sum_{i=1}^{m} e^{(a^i)^\top x}\right)$$

where $a^1, a^2, \ldots, a^m$ *are vectors of* $\mathbb{R}^n$. *Then the following statements are equivalent*.

(a) $f(x)$ *is bounded from below*.

(b) *There exists* $\lambda \in \mathbb{R}^m$, $\lambda \geq [0]$, *such that*

$$\sum_{i=1}^{m} \lambda_i a^i = [0].$$

(c) *There is no* $x \in \mathbb{R}^n$ *such that* $a^i x < 0,\ i = 1, \ldots, m.$

Note that the equivalence (b) $\Leftrightarrow$ (c) is just Gordan's theorem.

From Theorem 7 Borwein and Lewis [11] obtain then Farkas' theorem. It seems that the idea of this approach is due to Hiriart-Urruty [48]. For other considerations on Gordan's theorem see Perng [66].

(C) A new proof of Farkas' theorem is contained in the paper of Fujimoto et al. [26], proof based on a previous approach of Fujimoto [24], who in turn owes his idea to Morishima [58]. In the Metroeconomica paper various other approaches in proving Farkas' theorem are examined. See also the working paper N. 129 [26] of the same authors on www.economiaweb.it

(D) Some authors, e.g., Kuhn [53], Stoer and Witzgall [72], Webster [77], use the so-called "Fourier-Motzkin method for elimination of variables" in linear inequalities. Indeed, it seems that J. B. J. Fourier [24] has been the first to recognize that a mechanical system has a stable equilibrium state if and only if some homogeneous system of inequalities has no solution. A compact treatment of this method is contained in the book of Webster [77]. See also the interesting paper of Szilagyi [75].

(E) A new proof of Farkas' theorem by means of an appropriate separation theorem which, however, does not require the closedness of a finitely generated cone, is presented by Cambini and Martein [13]. Farkas' theorem is obtained as a corollary (Corollary 4.2.1 in Cambini and Martein [13]) of the following general result (Theorem 4.2.1 in Cambini and Martein [13]), we report for the reader's convenience.

Theorem 8. *Let* V *be a linear subspace of* $\mathbb{R}^s$ *such that* $V \cap int\,\mathbb{R}^s_- = \varnothing$. *Then the following conditions hold*:

(i) $V^* \cap \mathbb{R}^s_-$ *is a face* $F = \left\{ z = \sum_{i \in J} \gamma_j(-e^j),\ \gamma \geqq 0 \right\}$ *of* $\mathbb{R}^s_-$, *where* J *is a proper subset of the set of indices* $\{1, \ldots, s\}$, $V^* = V + \mathbb{R}^s_+$ *is the so-called conic* *extension of* V, *which is a closed and convex cone, and* e^j *is the unit vector having the* j*-th component equal to one and all others equal to zero. It is used the convention that* $F = \{[0]\}$ *when* $J = \emptyset$.

(ii) *If* $J \neq \emptyset$, *for each hyperplane of equation* $\alpha^\top z = 0$, $\alpha \geq [0]$, *which separates* V *and* $\mathbb{R}^s_-$, *we have* $\alpha_j = 0$, $\forall j \in J$. *Furthermore, there exists a separating hyperplane such that* $\alpha_i > 0$, $\forall i \notin J$.

(iii) *If* $J = \emptyset$, *i.e.*, $V \cap \mathbb{R}^s_- = \{[0]\}$, *there exists a separating hyperplane such that* $\alpha_i > 0$, $\forall i = 1, \ldots, s$.

The reader will note that the above statement is not quite elementary, and of course it is not quite elementary also its proof.

(F) Many theorems of the alternative for nonlinear systems are available in the mathematical literature. These theorems usually hold under various generalized convexity assumptions on the functions involved and some of them are formulated in an infinite-dimensional topological setting. Obviously, their proofs are in general not elementary; moreover, some of these theorems do not recover in a direct way the corresponding simpler theorems of the alternative for linear systems, such as Farkas' theorem.

A useful nonlinear theorem of the alternative is presented by Berge and Ghouila-Houri [8]. These authors prove their result by means of a generalization of Helly's theorem; their proof is elegant but, obviously, not quite elementary. Giorgi [38, 39] and Giorgi and Zuccotti [43] have presented a simple and self-contained proof of Berge and Ghouila-Houri's result, proof which requires only a classical separation theorem between convex sets. See also Fujimoto et al. [25].

Theorem 9. *Let be given the convex functions* $f_0(x)$, $f_1(x)$, $f_2(x)$, ..., $f_p(x)$, *defined on* $\mathbb{R}^n$ *and the linear affine functions* $h_1(x)$, $h_2(x)$, ..., $h_m(x)$, *where* $h_i(x) = A_i(x) - b_i$, $A_i \in \mathbb{R}^n$, $i = 1, \ldots, m$ (A_i *is the* i*-th row of matrix* A *of order* (m, n)). *If the system*

$$\begin{cases} f_0(x) < 0, \\ f_k(x) \leqq 0, \ k = 1, \ldots, p, \\ A_i x \leqq b_i, \ i = 1, \ldots, m, \end{cases}$$

admits no solution, but there exists $x^0 \in \mathbb{R}^n$ *such that it holds* ("*Slater constraint qualification*")

$$\begin{cases} f_k(x^0) < 0, \ k = 1, \ldots, p, \\ A_i x^0 \leqq b_i, \ i = 1, \ldots, m, \end{cases}$$

then there exist multipliers

$$y_1 \geqq 0, \ldots, y_p \geqq 0, u_1 \geqq 0, \ldots, u_m \geqq 0$$

such that

$$f_0(x) + \sum_{k=1}^{p} y_k f_k(x) + \sum_{i=1}^{m} u_i (A_i x - b_i) \geqq 0, \ \forall x \in \mathbb{R}^n.$$

Farkas' theorem is easily obtained from Theorem 9. It is immediate to see that systems

$$(S_1) : Ax = b, \quad x \geqq [0]$$

and

$$(S_2) : A^\top u \geqq [0], \quad b^\top u < 0$$

cannot admit both solutions. It remains to prove that if (S_2) does not admit solution, then (S_1) admits solution. Let us rewrite (S_2) putting $f_0(u) \equiv b^\top u < 0,$ and $-A^\top u \leqq [0].$ From Theorem 9 it will exist $x \in \mathbb{R}^n_+$ such that $b^\top u - x^\top A^\top u = u^\top (b - Ax) \geqq 0,$ for every $u \in \mathbb{R}^m,$ and therefore we have $b - Ax = [0],$ i.e., (S_1) admits solution.

We point out another general approach to build nonlinear theorems of the alternative, approach due to Giannessi [31-34]. The following result is a particular case of a more general theorem, proved by the said author.

Theorem 10. *Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and* $g : \mathbb{R}^n \to \mathbb{R}^m$. *Assume that* φ *and* g *be linear affine. Then the following system*

$$(S_3) : \begin{cases} \varphi(x) > 0, \\ g(x) \geqq 0 \end{cases}$$

is impossible if and only if there exist $\theta \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^m$ *such that*

$$S_3^* : \begin{cases} \theta_{\varphi(x)} + \lambda^\top g(x) \leqq 0, \ \forall x \in \mathbb{R}^n, \\ \theta \geqq 0, \ \lambda \geqq [0], \ (\theta, \lambda) \neq [0], \end{cases}$$

where the first inequality of (S_3^*) *must be verified in strict sense if* $\theta = 0$.

Let us rewrite Farkas' theorem in the form

$$(S_1) \equiv Ax \geqq [0], \quad a^\top x < 0$$

and

$$S_1^* \equiv z^\top A = a, \quad z \geqq [0].$$

Set $\varphi(x) = -a^\top x$ and $g(x) = Ax$. Theorem 10 can be applied. At $\theta = 0$, (S_3^*) becomes $\lambda \geqq [0]$, $\lambda^\top Ax < 0$, $\forall x \in \mathbb{R}^n$, which is obviously impossible. At $\theta = 1$, (S_3^*) becomes $\lambda \geqq [0]$, $-a^\top x + \lambda^\top Ax \leqq 0$, $\forall x \in \mathbb{R}^n$ which holds if and only if $\lambda \geqq [0]$, $-a + \lambda^\top A = [0]$, which is equivalent to (S_1^*).

(G) Finally, we point out the proof of Farkas' theorem given by Bonnans and Shapiro ([9], Lemma 5.43). These authors present a short and seemingly new proof of Farkas' theorem, based on properties of the solution of an appropriate optimization problem. In a previous result (Proposition 2.41) the same authors give an elegant proof that a finitely generated cone is closed.

References

[1] S. Achmanov, Programmation Linéaire, Editions MIR, Moscou, 1984.

[2] H. A. Antosiewicz, A theorem of the alternative for pairs of matrices, Pacific J. Math. 5 (1955), 641-642.

[3] D. Avis and B. Kaluzny, Solving inequalities and proving Farkas's lemma made easy, Amer. Math. Monthly 111 (2004), 152-157.

[4] D. Bartl, A short algebraic proof of the Farkas lemma, SIAM J. Optim. 19 (2008), 234-239.

[5] D. Bartl, A very short algebraic proof of the Farkas lemma, Math. Meth. Oper. Res. 75 (2012), 101-104.

[6] M. S. Bazaraa and C. M. Shetty, Foundations of Optimization, Springer Verlag, Berlin, New York, 1976.

[7] A. Ben-Israel, Linear equations and inequalities on finite dimensional, real or complex, vector spaces: a unified theory, J. Math. Anal. Appl. 27 (1969), 367-389.

[8] C. Berge and A. Ghouila-Houri, Programming, Games & Transportation Networks, Methuen & Co., London, 1965.

[9] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.

[10] J. M. Borwein, A note on the Farkas lemma, Utilitas Mathematica 24 (1983), 235-241.

[11] J. M. Borwein and A. S. Lewis, Convex Analysis and Nonlinear Optimization, Springer Verlag, New York, 2000.

[12] C. G. Broyden, A simple algebraic proof of Farkas's lemma and related theorems, Optimization Methods and Software 8 (1998), 185-199.

[13] A. Cambini and L. Martein, Generalized Convexity and Optimization. Theory and Applications, Springer Verlag, Berlin, 2009.

[14] W. B. Carver, Systems of linear inequalities, Ann. Math. 23 (1921), 212-220.

[15] B. D. Craven, Mathematical Programming and Control Theory, Chapman and Hall, London, 1978.

[16] A. Dax, An elementary proof of Farkas' lemma, SIAM Rev. 39 (1997), 503-507.

[17] M. E. De Giuli, G. Giorgi and U. Magnani, A general theorem of the alternative: how to get its special cases quickly, Pure Math. Appl. 8 (1997), 215-232.

[18] M. E. De Giuli, G. Giorgi, M. Maggi and U. Magnani, Matematica per l'Economia e la Finanza, Zanichelli, Bologna, 2008.

[19] Ky Fan, On systems of linear inequalities, Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Princeton Univ. Press, Princeton, 1956, pp. 99-156.

[20] J. Farkas, Über die Theorie der einfachen Ungleichungen, J. Reine Angew. Math. 124 (1902), 1-24.

[21] W. Fenchel, Convex cones, sets, and functions, Lecture Notes by D. W. Blackett, Department of Mathematics, Princeton University, Princeton, 1953.

[22] M. Fiedler and V. Ptak, Some generalizations of positive definiteness and monotonicity, Numerische Mathematik 9 (1966), 163-172.

[23] A. Forsgren, An elementary proof of optimality conditions for linear programming, Technical Report TRITA-MAT-2008, OS6, available on the WEB, 2008.

[24] J. B. J. Fourier, Solution d'une question particulière du calcul des inégalités, Nouveau Bulletin des Sciences par la Société Philomatique de Paris, 1826, pp. 99-101; Reprinted in G. Darboux, ed., Oeuvres de Jean-Baptiste-Joseph Fourier, Vol. 2, Gauthier-Villars, Paris, pp. 317-319.

[24] T. Fujimoto, A simple proof of the Tucker theorem, J. Economic Studies, Toyama University 21 (1976), 269-272.

[25] T. Fujimoto, B. B. Upeksha P. Perera, G. Giorgi and R. R. Ranade, Non-linear generalizations of Farkas-Minkowski's theorem, Kagawa University Economic Review 88 (2015), 1-17.

[26] T. Fujimoto, B. B. Upeksha P. Perera and G. Giorgi, A proof of the Farkas-Minkowski theorem by a tandem method, Metroeconomica 69 (2018), 142-150. See also www.economiaweb.unipv.it, DEM Working Paper Series N. 129, 2016.

[27] D. Gale, Convex polyhedral cones and linear inequalities, Activity Analysis of Production and Allocation, T. C. Koopmans, ed., J. Wiley, New York, 1951, pp. 287-297.

[28] D. Gale, The basic theorems of real linear equations, inequalities, linear programming and game theory, Naval Research Logistics Quarterly 3 (1956), 193-200.

[29] D. Gale, The Theory of Linear Economic Models, Mc Graw-Hill, New York, 1960.

[30] D. Gale, Comment, Econometrica 40 (1972), 391-392.

[31] F. Giannessi, Theorems of alternative, quadratic programs and complementarity

problems, Variational Inequalities and Complementarity Problems, R. W. Cottle, F. Giannessi and J. L. Lions, eds., J. Wiley, New York, 1980, pp. 151-186.

[32] F. Giannessi, Metodi Matematici della Programmazione. Problemi Lineari e Non Lineari, Pitagora Editrice, Bologna, 1982.

[33] F. Giannessi, Theorems of the alternative and optimality conditions, J. Optim. Theory Appl. 42 (1984), 331-365; Errata Corrige in J. Optim. Theory Appl. 44 (1984), 363-364.

[34] F. Giannessi, Constrained Optimization and Image Space Analysis, Volume 1, Separation of Sets and Optimality Conditions, Springer, New York, 2005.

[35] G. Giorgi, Alla ricerca di una dimostrazione elementare del teorema di Farkas-Minkowski, Atti XXIII Convegno A.M.A.S.E.S., Rende, 8-11 Settembre 1999, pp. 211-221.

[36] G. Giorgi, Sui teoremi di Gordan, Stiemke e Tucker, Atti XXIV Convegno A.M.A.S.E.S., Padenghe sul Garda, 6-9 Settembre 2000, Università di Bergamo e di Brescia, 2000, pp. 623-630.

[37] G. Giorgi, Osservazioni sui teoremi dell.alternativa non lineari implicanti relazioni di uguaglianza e vincolo insiemistico, Optimality in Economics, Finance and Industry, G. Crespi and others, eds., Datanova, Milan, 2002, pp. 171-183.

[38] G. Giorgi, Lemolte dimostrazioni del teorema (lemma) di Farkas-Minkowski, Quaderno di Ricerca N. 3, Dipartimento di Ricerche Aziendali, Riccardo Argenziano, Università di Pavia, Aracne Editrice, Roma, 2007, ISBN 978-88-548-1053-2.

[39] G. Giorgi, Again on the Farkas theorem and the Tucker key theorem proved easily, DEM Working Paper Series N. 94, Department of Economics and Management, University of Pavia, 2014a, www.economiaweb.unipv.it

[40] G. Giorgi, A mathematical note on all-productive, all-engaging and related systems, J. Math. Res. 6 (2014b), 1-13.

[41] G. Giorgi and M. R. Meriggi, Structural properties of von Neumann and related models; Proceedings of the XI A.M.A.S.E.S. Conference, Aosta, 9-11 Settembre 1987, Pitagora Editrice, Bologna, 1987, pp. 301-349.

[42] G. Giorgi and M. R. Meriggi, On the hierarchy of some qualitative structures in the von Neumann and related models, Metroeconomica XXXIX (1988), 121-140.

[43] G. Giorgi and C. Zuccotti, Teoremi di separazione e sistemi di disuguaglianze in $\mathbb{R}^n$,

Quaderno di Ricerca N. VII, Dipartimento di Ricerche Aziendali "Riccardo Argenziano", Università di Pavia, Aracne Editrice, Roma, 2007, ISBN 978-88-548-1346-5.

[44] P. Gordan, Über die Auflösungen linearer Gleichungen mit reelen Coefficienten, Math. Annalen 6 (1873), 23-28.

[45] O. Güler, Foundations of Optimization, Springer, New York, 2010.

[46] M. R. Hestenes, Calculus of Variations and Optimal Control Theory, J. Wiley & Sons, New York, 1966.

[47] M. R. Hestenes, Optimization Theory. The Finite Dimensional Case, J. Wiley & Sons, New York, 1975.

[48] J.-B. Hiriart-Urruty, A short proof of the variational principle for approximate solutions of a minimization problem, Amer. Math. Monthly 90 (1983), 206-207.

[49] C. W. Howe, An alternative proof of the existence of general equilibrium in a von Neumann model, Econometrica 28 (1960), 635-637.

[50] T. Iles and K. Meszeros, A new and constructive proof of two basic results of linear programming, Yugoslav J. Oper. Res. 11 (2001), 15-30.

[51] M. Jacimovic, Farkas' lemma of alternative, The Teaching of Mathematics, XIV, 2011, pp. 77-86.

[52] M. C. Kemp and Y. Kimura, Introduction to Mathematical Economics, Springer Verlag, Berlin, 1978.

[53] H. W. Kuhn, Solvability and consistency of linear equations and inequalities, Amer. Math. Monthly 63 (1956), 217-232.

[54] H. D. Kurz and N. Salvadori, Theory of Production. A Long-Period Analysis, Cambridge Univ. Press, Cambridge, 1995.

[55] J. Łos, A simple proof of the existence of equilibrium in a von Neumann model and some of its consequences, Bull. de l'Acad. Polonaise des Sciences, Série de Sciences Math. 1971, pp. 971-979.

[56] O. L. Mangasarian, Nonlinear Programming, Mc Graw-Hill, New York, 1969.

[57] W. H. Marlow, Mathematics for Operations Research, John Wiley & Sons, New York, 1978.

[58] M. Morishima, Theory of Economic Growth, Clarendon Press, Oxford, 1969; Reprinted with corrections, Clarendon Press, Oxford, 1970.

[59] T. S. Motzkin, Two consequences of the transposition theorem on linear inequalities, Econometrica 19 (1951), 184-185.

[60] Y. Murata, Mathematics for Stability and Optimization of Economic Systems, Academic Press, New York, 1977.

[61] J. Von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton Univ. Press, Princeton, 1944.

[62] H. Nikaido, Convex Structures and Economic Theory, Academic Press, New York, 1968.

[63] Y. Ozaki, Theorems of the alternative and linear programming, 2006, available on the WEB.

[64] M. H. Pearl, A matrix proof of Farkas's theorem, Quarterly J. Math. 18 (1967), 193-197.

[65] C. T. Perng, A note on D. Bartl's algebraic proof of Farkas's lemma, Int. Math. Forum 7 (2012), 1343-1349.

[66] C. T. Perng, A note on Gordans's theorem, British J. Math. Computer Sci. 10 (2015), 1-6.

[67] C. T. Perng, On a class of theorems equivalent to Farkas's lemma, Appl. Math. Sci. 11 (2017), 2175-2184.

[68] C. Roos and T. Terlaki, Note on a paper of Broyden, Oper. Res. Lett. 25 (1999), 183-186.

[69] C. Singh and I. Praton, Some variations, extensions and generalizations of Tucker's theorem, J. Inform. Optim. Sci. 17 (1996), 243-253.

[70] M. Slater, A note on Motzkin's transposition theorem, Econometrica 19 (1951), 185-187.

[71] E. Stiemke, Über positive Lösungen Homogener linearer Gleichungen, Math. Annalen 76 (1915), 340-342.

[72] J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions - I, Springer, Berlin, 1970.

[73] K. Svanberg, Farkas' lemma derived by elementary linear algebra, Technical Report TRITA-MAT-2008-OS7, 2008, available on the WEB.

[74] P. Szilagyi, Nonhomogeneous linear theorems of the alternative, Pure Math. Appl. 10 (1999a), 141-159.

[75] P. Szilagyi, A simple proof and short forms for the Kuhn-Fourier theorem, Pure Math. Appl. 10 (1999b), 381-384.

[76] J. Ville, Sur la théorie des jeux où intervient l'habileté des joueurs, Traité du Calcul des Probabilités et de ses Applications, E. Borel, ed., Vol. IV, Part 2, Gauthier-Villars, Paris, 1938, pp. 105-113.

[77] R. Webster, Convexity, Oxford Univ. Press, Oxford, 1994.

[78] R. J. B. Wets, Elementary constructive proofs of the theorems of Farkas, Minkowski and Weyl, Economic Decision-Making: Games, Econometrics and Optimisation, Contributions in Honour of Jacques H. Drèze, J. J. Gabszewicz, J.-F. Richard and L. A. Wolsey, eds., North Holland, Amsterdam, 1990, pp. 427-432.

[79] A. W. Tucker, Dual systems of homogeneous linear relations, Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Annals of Mathematics Studies, N. 38, Princeton University Press, Princeton, 1956, pp. 3-18.