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ADDITIONAL WEAKLY $P1$ PROPERTIES AND “NOT-(WEAKLY $P1$)” PROPERTIES

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Abstract

Within this paper, weakly $P1$ properties continue to be examined and “not-(weakly $P1$)” properties are investigated.

1. Introduction and Preliminaries

In 1975 [6], T_0-identification spaces, which were introduced in 1936 [7], were used to further characterize weakly Hausdorff spaces.

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [7].

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Within the 1936 paper [7], T_0-identification spaces were used to further characterize pseudometrizable spaces.

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff its* T_0*-identification space* $(X_0, Q(X, Q(X, T)))$ *is metrizable* [7].

Theorem 1.2. *A space* (X, T) *is weakly Hausdorff iff its* T_0*-identification space* $(X_0, (Q(X, T))$ *is Hausdorff* [6].

In the 1975 paper [6], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

Within the 1961 paper [1], A. Davis was interested in separation axioms R_i, which together with T_i are equivalent to T_{i+1}; $i = 0, 1,$ respectively, leading to the definition of R_1 and the rediscovery of the R_0 separation axiom.

Definition 1.3. A space (X, T) is R_0 iff for each $O \in T$ and each $x \in O$, $Cl(\{x\}) \subseteq O$ [1].

The separation axioms R_i; $i = 0, 1,$ satisfied Davis' expectations [1].

Within a 2015 paper [2], weakly Hausdorff was generalized to weakly Po properties.

Definition 1.4. Let P be a topological property for which $P\text{o} = (P \text{ and } T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

As a result of the role of T_0 in the weakly Po property process within the introductory paper [2], it was proven that for a topological property P for which weakly Po exists, a space is weakly Po iff its T_0-identification space has property Po.

Even though weakly Po properties were undefined at the time, since (pseudometrizable)o equals metrizable, metrizable was the first known weakly Po property and weakly (pseudometrizable)o = weakly (metrizable) = pseudometrizable. Within the paper [2], it was established that both T_2 and T_1 are weakly Po properties, with weakly (R_1)o = weakly T_2 = R_1 and weakly (R_0)o = weakly T_1 = R_0.

In the introductory weakly Po property paper [2], the search for a topological property which was not a weakly Po property led to a need and a use for the topological property "not-T_0", where "not-T_0" is the negation of T_0. In that paper [2], it was shown that both T_0 and "not-T_0" are not weakly Po properties. Also, it was shown that a space is weakly Po iff its T_0-identification space is weakly Po. The combination of this result with the fact that other topological properties are simultaneously shared by a space and its T_0-identification space led to the introduction and investigation of T_0-identification P properties [3].

Definition 1.5. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property S [3].

Within the paper [3], it was proven that property Q is a T_0-identification P property iff Qo exists and Q = weakly Qo.

As in the case of weakly Po properties, both T_0 and "not-T_0" fail to be T_0-identification P properties [3]. The knowledge and insights obtained from the investigations of weakly Po and T_0-identification P properties was used to define and investigate weakly P1 and to further investigate weakly Po and T_0-identification P properties [4]. In this paper, the study of weakly P1 properties continues and "not-(weakly P1)" properties are investigated.

2. Weakly $P1$

Definition 2.1. Let P be a topological property for which $P1 = (P \text{ and } T_1)$ exists. Then (X, T) is weakly $P1$ iff $(X_0, Q(X, T))$ is $P1$. A topological property $P1$ for which weakly $P1$ exists is called a weakly $P1$ property [4].

Within the paper [4], it was proven that for a weakly $P1$ property $Q1$, weakly $Q1 = ((\text{weakly } Qo) \text{ and } R_0)$. Since both weakly Qo and R_0 are topological properties and $((\text{weakly } Qo) \text{ and } R_0) = \text{weakly } Q_1$ exists, then weakly $Q1 = ((\text{weakly } Qo) \text{ and } R_0)$ is a topological property.

A natural question to ask at this point is whether there are topological properties P for which T_0-identification P, weakly Po, and weakly $P1$ are equal and, if so, is there a least topological property for which all three are equal?

Theorem 2.1. *The least topological property for which* T_0*-identification* P = *weakly* Po = *weakly* $P1$ *is* R_0.

Proof. Since R_0 = weakly (R_0)o and R_1 = weakly (R_1)o, then R_0 and R_1 are T_0-identification properties. Since weakly $(R_0)1 = R_0$ and weakly $(R_1)1 = R_1$ [4], then for $P = R_0$ or $P = R_1$, each of the three properties are equal.

Let Q be a topological property for which T_0-identification P = weakly Po = weakly $P1$. Since weakly $Q1 = ((\text{weakly } Qo) \text{ and } R_0)$, then Q = weakly $Q1$ = $((\text{weakly } Qo)$ and $R_0) = (Q \text{ and } R_0)$, which implies R_0. Thus, R_0 is the least topological property P for which each of T_0-identification P = weakly Po = weakly $P1$.

As in the case of weakly Po and T_0-identification P properties, neither T_0 nor "not-T_0" are weakly $P1$ properties [4]. Also, within the paper [4], it was proven that for a weakly $P1$ property, weakly $P1$ = ($P1$ or ((weakly $P1$) and "not-T_0")), where both $P1$ and ((weakly $P1$) and "not-T_0") exist and are distinct, and neither are weakly $P1$ properties. In the paper [2], it was proven that for a weakly Po property

Qo, weakly Qo = (Qo or ((weakly Qo) and "not-T_0")), where both Qo and ((weakly Qo) and "not-T_0") exist, are distinct, and neither are weakly Po properties. Thus, the question of whether "not-T_0" in the statement above for weakly Q1 could be replaced by "not-T_1" arises.

The use of "not-T_0" in the weakly Po paper [2] as an example of a topological property that is not a weakly Po property led to the investigation of "not-P" properties, where P is a topological property and "not-P" exists [5], which led to the discovery of $L = (T_0$ or "not-T_0"); the least of all topological properties [5]. In [5], it was shown that L is not a weakly Po property and, thus, by the results above, L is not a weakly P1 property. Within that paper [5], it was proven that L is also equal to (P or "not-P"), where P is a topological property for which "not-P" exists, which is used below.

Theorem 2.2. *Let Q be a topological property for which weakly Q1 exists and let (X, T) be a weakly Q1 space. Then (X, T) is "not-T_0" iff (X, T) is "not-T_1".*

Proof. Since (X, T) is (Q1 or ((weakly Q1) and "not-T_0")), where both Q1 and ((weakly Q1) and "not-T_0") exist and are distinct, then (X, T) is not Q1.

Since weakly Q1 = ((weakly Q1) and L) = ((weakly Q1) and (T_1 or "not-T_1")) = ((weakly Q1)1 or ((weakly Q1) and "not-T_1")) and (weakly Q1)1 = Q1 [4], then (X, T) is (Q1 or ((weakly Q1) and "not-T_1")) and, since (X, T) is not Q1, then both Q1 and ((weakly Q1) and "not-T_1") exist and are distinct. Thus ((weakly Q1) and "not-T_0") = ((weakly Q1) and "not-Q1") = ((weakly Q1) and "not-T_1") and in (X, T), "not-T_0" and "not-T_1" are equivalent.

Corollary 2.1. *Let Q be a topological property for which weakly Q1 exists. Then weakly Q1 = (Q1 or ((weakly Q1) and "not-T_1")), where both Q1 and ((weakly Q1) and "not-T_1") exist, are distinct, and neither of which are weakly P1 properties.*

The negation of Theorem 2.2 gives the next result.

Corollary 2.2. *Let Q be a topological property for which weakly $Q1$ exists and let (X, T) be a weakly $Q1$ space. Then (X, T) is T_0 iff (X, T) is T_1.*

Within the study of weakly Po properties, the introduction and investigation of "not-P" topological properties raised questions about "not-(weakly Po)" for a weakly Po property Po, which led to the following discoveries. For a topological property P for which weakly Po exists, "not-(weakly Po)" exists and is a topological property, both $(P$ and $T_0)$ and (P and "not-T_0") exist, ("not-P")o = ("not-Po")o, weakly (("not-P")o) exists, weakly (("not-P")o) = weakly (("not-(Po)")o) = "not-(weakly Po)" $\neq$ weakly Po, and ("not-P")o $\neq$ Po [5] raising similar questions for weakly $P1$ properties, which are addressed below.

3. "(Not-(Weakly $P1$)" Properties for Weakly $P1$ Properties

Since for a topological property Q for which $Q1$ is a weakly $P1$ property, Qo is a weakly Po property [4], the results below follow immediately from the results above.

Corollary 3.1. *Let Q be a topological property for which weakly $Q1$ exists. Then Qo is a weakly Po property and "not-(weakly Po)" exists and is a topological property, both (P and T_0) and (P and "not-T_0") exist, ("not-P")o = ("not-Po")o, weakly (("not-P")o) exists, weakly (("not-P")o) = weakly (("not-(Po)")o) = ("not-weakly Po)" $\neq$ weakly Po, and ("not-P)"o $\neq$ Po.*

Theorem 3.1. *("Not-R_0" and T_1) does not exist.*

Proof. Let (X, T) be a "not-R_0" space. Let $O \in T$ and let $x \in O$ such that $Cl(\{x\})$ is not a subset of O. Let $y \in Cl(\{x\}) \setminus O.$ Then every open set containing y contains x and (X, T) is not T_1. Hence ("not-R_0" and T_1) does not exist.

Theorem 3.2. *Let Q be a topological property for which weakly $Q1$ exists. Then ("not-(weakly $Q1$)" and T_1) does not exist and thus "not-(weakly $Q1$)" is not a weakly $P1$ property.*

Proof. Since weakly $Q1$ = ((weakly Qo) and R_0), then weakly $Q1$ implies R_0 and "not-R_0" implies "not-(weakly $Q1$)". Since "not-R_0" implies "not-(weakly $Q1$)"

is true, then (“not-R_0” and T_1) implies (“not-(weakly $Q1$)” and T_1) is true and, since (“not-R_0” and T_1) does not exist, then (“not-(weakly $Q1$)” and T_1) does not exist and “not-(weakly $Q1$)” is not a weakly $P1$ property.

Theorem 3.3. *Let Q be a topological property for which weakly $Q1$ exists. Then (“not-$Q1$)o exists, “not-(weakly $Q1$)” = weakly (“not-$Q1$”)o, and (“not-$Q1$”)o is a weakly Po property.*

Proof. Since a space is weakly $Q1$ iff its T_0-identification space is $Q1$, then a space is “not-(weakly $Q1$)” iff its T_0-identification space is “not-$Q1$”. Since all T_0-identification spaces are T_0 [7], then a space is “not-(weakly $Q1$)” iff its T_0-identification space is (“not-$Q1$”)o. Thus “not-(weakly $Q1$)” = weakly (“not-$Q1$”)o exists and (“not-$Q1$”)o is a weakly Po property.

References

[1] A. Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly 68 (1961), 886-893.

[2] C. Dorsett, Weakly *P* properties, Fundamental J. Math. Math. Sci. 3(1) (2015), 83-90.

[3] C. Dorsett, T_0-identification *P* and weakly *P* properties, Pioneer J. Math. Math. Sci. 15(1) (2015), 1-8.

[4] C. Dorsett, Weakly *P*1, weakly *P*o, and T_0-identification *P* properties, Fundamental J. Math. Math. Sci. 6(1) (2016), 33-43.

[5] C. Dorsett, Weakly *P* corrections and new, fundamental topological properties and facts, Fundamental J. Math. Math. Sci. 5(1) (2016), 11-20.

[6] W. Dunham, Weakly Hausdorff spaces, Kyungpook Math. J. 15(1) (1975), 41-50.

[7] M. Stone, Application of Boolean algebras to topology, Mat. Sb. 1 (1936), 765-771.