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ADDITIONAL NO BETWEEN TOPOLOGICAL PROPERTIES

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Abstract

In classical topology, the separation axioms T_0, T_1, T_2, Urysohn, T_3, completely Hausdorff, $T_{3\frac{1}{2}}$, and T_4 were introduced and investigated. Later the separation axioms perfectly normal and perfectly Hausdorff were added. A logical, natural question to pose is whether there are topological properties between two of the separation axioms P and Q, where P immediately precedes Q in the decreasing strength listing of the separation axioms. The question was unaddressed until a recent paper, where it was shown the answer is "no". In this paper, the contrapositive implications listing of the separation axioms above are investigated and it is shown that there are no topological properties between two consecutive

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properties in the decreasing contrapositive implications listing of the separation axioms, and the results are used to give another example of topological lattice.

1. Introduction and Preliminaries

Within a recent paper [3], it was thought that for two topological properties in the classical topological properties T_0, T_1, T_2, Urysohn, completely Hausdorff, T_3, $T_{3\frac{1}{2}}$, and T_4, where the first property immediately precedes the second, there are no topological properties between the two. In the paper [4], needed verification to justify the earlier claim was given by focusing on each pair of separation axioms, where the first immediately precedes the second, and showing that the first is a minimal strengthening of the second, and, thus there are no topological properties between the two. An example is given.

The T_0 separation axiom is credited to Kolmogorff [9]: A space (X, T) is T_0 iff for distinct elements x and y in X, there exists an open set containing only one of the two elements. Thus one element can be separated from the other by an open set, but without choice. A minimal strengthening of T_0 would be to separate each of the two elements from the other by an open set.

The T_1 separation axiom was introduced in 1906 [7] and [8], and is the separation axiom immediately preceding T_0: A space (X, T) is T_1 iff for distinct elements x and y in X, there exist an open set containing x and not y and an open set containing y and not x. Thus T_1 is a minimal strengthening of T_0 and there are no topological properties between T_0 and T_1.

From classical topology, it is known that T_0 implies T_1, which implies T_2, which implies Urysohn, which in one branch implies completely Hausdorff, which implies $T_{3\frac{1}{2}}$, and in another branch implies T_3, which implies $T_{3\frac{1}{2}}$, which implies T_4, with none of the implications reversible.

In 1950 [1], perfectly normal was added to the separation axioms and in 1977 [2], perfectly Hausdorff was added. A space (X, T) is perfectly normal iff it is T_1 and for disjoint closed sets C and D, there exists a continuous function $f : (X, T) \to (I, U)$, where $I = [0, 1]$ and U is the usual relative metric topology on I, such that $C = f^{-1}(0)$ and $D = f^{-1}(1)$. A space (X, T) is perfectly Hausdorff iff for distinct elements x and y in X, there exists a continuous function $f : (X, T) \to (I, U)$ such that $\{x\} = f^{-1}(0)$ and $\{y\} = f^{-1}(1)$. It is known that completely Hausdorff implies perfectly Hausdorff, which implies perfectly normal, and T_4 implies perfectly normal, with the implication non-reversible. The two new separation axioms along with their implications can be added at the end of the listing and ordering of the classical separation axioms above to obtain a complete listing and ordering of the separation axioms under consideration.

As given above, in the paper [4] for consecutive separation axioms Q and P in the extended ordered listing of separation axioms above, where Q implies P, it was shown that a minimal strengthening of Q gives P and, thus there are no topological properties between Q and P. In the paper [5], the contrapositive implications of the ordered list of separation axioms were considered and it was reported that there are no topological properties between two of those consecutive properties. However, the proof falls short of proving the claim and below, the results above are used to verify the claim is true.

2. Verification of Additional No Between Topological Properties

For each topological property in the extended order listing of separation axioms above, its negation exists. The contrapositive implications of the separation axioms in the extended ordered listing of the separation axioms above is, in decreasing order: “not-T_0”, which implies “not-T_1”, which implies “not-T_2”, which implies “not-Urysohn”, which in one branch implies “not-completely Hausdorff”, which in one branch implies “not-$T_{3\frac{1}{2}}$”, and in another branch implies “not-perfectly Hausdorff”, which implies “not-perfectly normal”; and in another branch from “not-Urysohn” implies “not-T_3”, which implies “not-$T_{3\frac{1}{2}}$”, which implies “not-T_4”, which implies “not-perfectly normal”, with none of the implications reversible.

As given above, within this paper, the results above concerning the ordered listing of separation axioms are used to show there are no topological properties between two consecutive topological properties in the ordered contrapositive implications listing given above. Thus to be able to transition between the two ordered listings above, for topological properties P, questions about the existence of “not-P” arise and one answer to the question is given below.

Theorem 2.1. *Let P and Q be topological properties for which P implies Q and “not-Q” exists. Then “not-P” exists.*

Proof. Since P implies Q and “not-Q” exists, then (P and “not-Q”) does not exist. Thus “not-Q” is not P and “not-P” exists.

Theorem 2.2. *Let U and V be consecutive topological properties in the ordered contrapositive implications listing in this section, with V implying U. Then there are no topological properties between Q and V.*

Proof. Since U and V are topological properties in the ordered contrapositive implications listing, then $P =$ “not-U” and $Q =$ “not-V” exist and are separation axioms in the ordered listing of separation axioms given in Section 1. Since V implies U, then P implies Q and, since U and V are consecutive topological properties in the ordered contrapositive implications listing, then P and Q are consecutive separation axioms in the ordered listing of the separation axioms above.

If Z is a topological property between U and V, then Z implies U, where “not-U” exists, and, by Theorem 2.1, “not-Z” exists, but, then “not-Z” is a topological property between P and Q, which is a contradiction. Thus there are no topological properties between U and V.

Theorem 2.3. *Let M be a topological property in the ordered contrapositive implication listing of topological properties above with immediate predecessor V and immediate successor U. Then M is "isolated" in that it is the only topological property between V and U.*

Proof. By hypotheses M is a topological property between V and U. If Z is a topological property between V and U other than M, then Z is between V and M or between Z and U, which is a contradiction. Thus M is "isolated" in that it is the only topological property between V and U.

In the paper [6], it was shown that $\mathcal{L} = \{P \mid P$ is weaker than or equal to perfectly normal and stronger than or equal to $T_0\}$, with the partial order $\leq$ defined by $P \leq Q$ iff P is weaker than or equal to Q, is a lattice. Below this result is used to show $\mathcal{NL} = \{Q \mid Q$ is stronger than or equal to "not-perfectly normal" and weaker than or equal to "not-T_0"$\}$, with the partial order $\leq^*$ defined by $P \leq^* Q$ iff P is weaker than or equal to Q, is another topological lattice.

3. Another Topological Lattice

In the paper [4], it was shown that $\mathcal{L}$ equals the set of 10 separation axioms given in the ordered listing of separation axioms in Section 1. Thus by the transition between $\mathcal{L}$ and $\mathcal{NL}$ given above, $\mathcal{NL}$ is the set of 10 topological properties given in the ordered contrapositive implications listing above. Below this result will be used to show $(\mathcal{NL}, \leq^*)$ is a lattice.

Definition 3.1. Let $(\mathcal{U}, \leq^+)$ and $(\mathcal{V}, \leq^{**})$ be partially ordered systems. Then a one-to-one function g from $(\mathcal{U}, \leq^+)$ onto $(\mathcal{V}, \leq^{**})$ is a reverse order preserving function iff for a and b in $\mathcal{U}$ such that $a \leq^+ b$, $g(b) \leq^{**} g(a)$ in $\mathcal{V}$.

Theorem 3.1. *Let $(\mathcal{U}, \leq^+)$ and $(\mathcal{V}, \leq^{**})$ be partially ordered systems and let g be a reverse order preserving function from $(\mathcal{U}, \leq^+)$ onto $(\mathcal{V}, \leq^{**})$. If*

$(\mathcal{V}, \leq^{**})$ *is a lattice*, *then* $(\mathcal{U}, \leq^{+})$ *is a lattice*.

Proof. Suppose $(\mathcal{V}, \leq^{**})$ is a lattice. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{U}$. Then $g(\mathcal{A})$ is a nonempty subset of $\mathcal{V}$ and, since $(\mathcal{V}, \leq^{**})$ is a lattice, $g(\mathcal{A})$ has a least upper bound (lub) a and a greatest lower bound (glb) b in $(\mathcal{V}, \leq^{**})$. If $c \in \mathcal{A}$, then $g(c) \in g(\mathcal{A})$, $g(c) \leq^{**} a$, and $g^{-1}(a) \leq^{+} c$. Thus $g^{-1}(a)$ is a lower bound of $\mathcal{A}$. Suppose $g^{-1}(a)$ is not the glb of $\mathcal{A}$. Let $c \in \mathcal{U}$ such that c is a lower bound of $\mathcal{A}$ bigger than $g^{-1}(a)$. Then for each $d \in \mathcal{A}$, $g(d) \leq^{**} g(c)$ and $g(c)$ is smaller than a. Hence $g(c)$ is an upper bound of $g(\mathcal{A})$, which is smaller than a, the lub of $g(\mathcal{A})$, which is a contradiction. Therefore, $g^{-1}(a)$ is the glb of $\mathcal{A}$. Similarly $g^{-1}(b)$ is the lub of $\mathcal{A}$.

Thus every nonempty subset of $\mathcal{U}$ has a lub and a glb and $(\mathcal{U}, \leq^{+})$ is a lattice.

Theorem 3.2. $(\mathcal{NL}, \leq^{*})$ *is a lattice*.

Proof. Let $g : (\mathcal{NL}, \leq^{*}) \to (\mathcal{L}, \leq)$ defined by $g(\text{“not-}P\text{”}) = P$ for all “not-P” in $\mathcal{NL}$. Then, by the results above, g is a one-to-one reverse order preserving function from $(\mathcal{NL}, \leq^{*})$ onto $(\mathcal{L}, \leq)$ and, by the result above, $(\mathcal{NL}, \leq^{*})$ is a lattice.

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