Fundamental Journal of Mathematics and Mathematical Sciences
Volume 12, Issue 2, 2019, Pages 43-49
This paper is available online at http://www.frdint.com/
Published online November 22, 2019

ADDITIONAL ANSWERS TO QUESTIONS IN TOPOLOGY

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Abstract

The mathematical ancestors in what is today called modern topology did an incredible job, but there remained many natural, unaddressed questions. Either the questions did not arise or there were not needed tools and properties available to resolve the questions successfully. In either case, the continued study of topology has produced new properties and tools that have been used to resolve some of those unaddressed questions, and in this paper, answers to those unaddressed questions, together with new properties and tools, are used to resolve related, unanswered questions.

1. Introduction and Preliminaries

In classical topology, the topological properties T_0, T_1, T_2, Urysohn, completely

Keywords and phrases: classical separation axioms, "not-*P*", where *P* is a topological property.

2010 Mathematics Subject Classification: 54D05, 54D10, 54B05, 54B15.

Received October 22, 2019; Accepted November 1, 2019

Hausdorff, T_3, $T_{3\frac{1}{2}}$, and T_4 were introduced and thoroughly investigated. Each of those properties is important, useful properties in mathematics, making topology useful and helpful in the continued study and advancement of mathematics. From classical topology, it is known T_4 implies $T_{3\frac{1}{2}}$, which in separate branches implies each of T_3 and completely Hausdorff, T_3 implies Urysohn and completely Hausdorff implies Urysohn, Urysohn implies T_2, T_2 implies T_1, and T_1 implies T_0, with none of the implications reversible. A logical, natural question to ask from classical topology is whether there are topological properties between two of the classical topological properties given above, where the first property immediately implies the second. In a recent paper [1], it was proven that the answer is "no". As a example, there are no topological properties between T_1 and T_0.

In the continued study of T_0-identification spaces [2], "not-P", where P is a topological property and the negation of P exists, proved to be useful properties and tools in the continued investigation and expansion of topology. The addition of "not-P", where P is a topological property and "not-P" exists, into topology as important, useful tools has opened a never before imagined, fertile topological territory within the study of topology, leading to new, never before imagined, extremely useful properties and insights in the study of topology. Two such examples are: (1) L, the least of all topological properties, is given by $L = (T_0$ or "not-T_0") $= (P$ or "not-P"), where P is a topological property for which "not-P" exists [3], and (2) let P and Q be topological properties for which each of "not-P" and (Q and "not-P") exist, and P implies Q. Then (P and "not-Q") does not exist, $Q = (P$ or (Q and "not-P")) $= (P$ or $Q)$, and $((P$ or $Q)$ or $Q) = (P$ or $Q) = Q$ [1]. Those two properties were used to establish the results given above and will continue to be used in this paper. Having the needed tools and properties, and the knowledge of those tools and properties, can make answers to previously challenging, if not unanswerable questions, answerable.

Below contrapositive equivalences of the implications and the results above are used to show there are no topological properties between "not-P" and "not-Q", where P and Q are classical topological properties given above and P immediately implies Q.

2. Contrapositive Equivalences of the Implications and Results above

Theorem 2.1. *Let P and Q be topological properties such that "not-P" exists and P is stronger than Q. Then (Q and "not-P") exists, (P and "not-Q") does not exist, $Q = (P$ or (Q and "not-P")$) = (P$ or $Q)$, and $((P$ or $Q)$ or $Q) = (P$ or $Q)$ $= Q$.*

Proof. Since P is stronger than Q, then (Q and "not-P") exists. Then the remainder of the Theorem follows immediately from result (2) above.

Applications of Theorem 2.1 include the following: ((compact and T_1) or (compact and T_0)) = (compact and T_0) and ((second countable) or (first countable)) = (first countable).

For each classical topological property P given above, "not-P" exists and the contrapositive equivalences of the implications above give the following: ("not-T_0") implies ("not-T_1"), which implies ("not-T_2"), which implies ("not-Urysohn"), which in one branch implies ("not-T_3") and in another branch implies ("not-completely Hausdorff"), each of ("not-T_3") and ("not-completely Hausdorff") implies ("not-$T_{3\frac{1}{2}}$"), which implies ("not-T_4"), where none of the implications are reversible.

Theorem 2.2. *Let P and Q be classical topological properties given above, where P is stronger than Q. Then (("not-Q") or ("not-P")) = ("not-P").*

Proof. Since P and Q are classical topological properties given above, then each of "not-P" and "not-Q" exist. Then by Theorem 2.1, (("not-Q") or ("not-P")) = ("not-P").

Applications of Theorem 2.2 include each of the following: (("not-T_0") or (“not-T_1”) = (“not-T_1”), ((“not-T_2”) or (“not-T_1”)) = (“not-T_2”), ((“not-T_3”) or (“not-$T_{3\frac{1}{2}}$”)) = (“not-$T_{3\frac{1}{2}}$”), and ((“not-T_1”) or (“not-T_4”)) = (“not-T_4”).

Theorem 2.3. *Let P and Q be classical topological properties given above, where P and Q are consecutive properties in the listing above, with P stronger than Q. Then there are no topological properties between* (“*not-Q*”) *and* (“*not-P*”).

Proof. Suppose there exists a topological property W between (“not-Q”) and (“not-P”). Then both (W and (“not-Q”)) and (W and (“not-P”)) exist, and W = ((W and (“not-Q”)) or (W and (“not-P”))) = (W and ((“not-Q”) or (“not-P”))) = (W and (“not-P”)), which is (“not-P”) and a contradiction. Thus there are no topological properties between (“not-Q”) and (“not-P”).

Applications of Theorem 2.3 include the following: There are no topological properties between (“not-T_0”) and (“not-T_1”) and there are no topological properties between (“not-completely Hausdorff”) and (“not-Urysohn”).

3. Additional Applications of the Results above

Theorem 3.1. *Let P, Q, and W be classical topological properties given above, where P and Q, and Q and W are consecutive properties in the listing above, P implies Q, which implies W. Then there is exactly one topological property between* (“*not-W*”) *and* (“*not-P*”), *namely* (“*not-Q*”).

Proof. Suppose there exists a topological property Z other than (“not-Q”) between (“not-W”) and (“not-P”). Then Z is a topological property between (“not-P”) and (“not-Q”) or between (“not-Q”) and (“not-W”), which, by Theorem 2.3, is a contradiction. Thus there is exactly one topological property between (“not-W”) and (“not-P”), namely (“not-Q”).

Applications of Theorem 3.1 include the following: There is exactly one topological property between (“not-T_0”) and (“not-T_2”), namely, (“not-T_1”), and there is exactly one topological property between (“not-T_3”) and (“not-T_4”), namely (“not-$T_{3\frac{1}{2}}$”).

With appropriate choices of topological properties, Theorem 3.1 can be extended to include more than one topological property.

Theorem 3.2. *Let P, Q, W, and Z be consecutive classical topological properties given above, listed in the given chronological order. Then there are exactly two topological properties between* (“*not-Z*”) *and* (“*not-P*”), *namely* (“*not-Q*”) *and* (“*not-W*”).

The proof is straightforward and omitted.

Theorem 3.2 could be extended to three or more between topological properties.

The properties and proofs above can be used to extend the results above to compound properties connected by “and” or “or” with classical topological properties given above or their negations.

Theorem 3.3. *Let Z be a topological property for which* “*not-Z*” *exists, and let P, Q, and W be consecutive classical topological properties as listed above such that each of* (*Z and W*), (*Z and Q*), (*Z and P*), *and all their negations exist. Then* ((*Z and W*) *or* (*Z and Q*)) = (*Z and W*), *there are no topological properties between* (*Z and W*) *and* (*Z and Q*), *there is exactly one topological property between* (*Z and W*) *and* (*Z and P*), *namely* (*Z and Q*), ((“*not*-(*Z and W*)”) *or* (“*not*-(*Z and Q*)”)) = (“*not*-(*Z and Q*)”), *there are no topological properties between* (“*not*-(*Z and W*)”) *and* (“*not*-(*Z and Q*)”), *and there is exactly one topological property between* (“*not*-(*Z and W*)”) *and* (“*not*-(*Z and P*”)), *namely* (“*not*-(*Z and Q*)”).

Proof. By Theorem 2.1, ((*Z* and *W*) or (*Z* and *Q*)) = (*Z* and *W*).

Suppose there is a topological property *U* between (*Z* and *W*) and (*Z* and *Q*). Then both (*U* and (*Z* and *W*)) and (*U* and (*Z* and *Q*)) exist, and *U* = ((*U* and (*Z* and *W*)) or (*U* and (*Z* and *Q*))) = (*U* and ((*Z* and *W*) or (*Z* and *Q*))) = (*U* and (*Z* and *W*)), which is (*Z* and *W*) and a contradiction. Thus there are no topological properties between (*Z* and *W*) and (*Z* and *Q*).

Since (*Z* and *Q*) is a topological property between (*Z* and *W*) and (*Z* and *P*), then there is at least one topological property between (*Z* and *W*) and (*Z* and *P*). Suppose there are additional topological properties between (*Z* and *W*) and (*Z* and *P*). Let *U* be a topological property different from (*Z* and *Q*) between (*Z* and *W*) and (*Z* and *P*). Then *U* is between (*Z* and *W*) and (*Z* and *Q*) or between (*Z* and *Q*) and (*Z* and *P*), which is a contradiction. Thus there is exactly one topological property between (*Z* and *W*) and (*Z* and *P*), namely (*Z* and *Q*).

By Theorem 2.1, ((“not-(*Z* and *W*)”) or (“not-(*Z* and *Q*)”)) = (“not-(*Z* and *Q*)”), or equivalently, (((“not-*Z*”) or (“not-*W*”)) or ((“not-*Z*”) or (“not-*Q*”))) = ((“not-*Z*”) or (“not-*Q*”)).

The remainder of the proof is similar to that above and is omitted.

Applications of Theorem 3.3 include the following: ((normal and T_0) or (normal and T_1)) = (normal and T_0), there are no topological properties between (normal and T_0) and (normal and T_1), there is exactly one topological property between (normal and T_0) and (normal and T_2), namely (normal and T_1), ((“not-((second countable) and T_1)”) or (({not-((second countable) and T_2))”) = (“not-((second countable) and T_2)”), there is no topological property between ((“not-((second countable) and T_1)”) and ((“not-((second countable) and T_2)”)), and there is exactly one topological property between (“not-((second countable) and T_1)”) and (“not-((second countable) and Urysohn)”), namely (“not-((second countable) and T_2)”).

With proper selection of the topological properties, there could be more than one topological property between the two given topological properties.

Theorem 3.4. *Let Z be a topological property for which* “*not-Z*” *exists, and let P*, *Q*, *and W be consecutive classical topological properties as listed above such that each of* (“*not*-(*Z or W*)”), (“*not*-(*Z or Q*)”), and (“*not*-(*Z and P*)”) *exist*. *Then* ((*Z or W*) *or* (*Z or Q*)) = (*Z or W*), *there are no topological properties between* (*Z or W*) *and* (*Z or Q*), *there is exactly one topological property between* (*Z or W*) *and* (*Z or P*), *namely* (*Z or Q*), ((“*not*-(*Z or W*)”) *or* (“*not*-(*Z or Q*))) = (“*not*-(*Z or Q*)”), *there are no topological properties between* (“*not*-(*Z or W*)”) *and* (“*not*-(*Z or Q*)”), *and there is* *exactly one topological property between* (“*not*-(*Z or W*)”) *and* (“*not*-(*Z or P*)”), *namely* (“*not*-(*Z or Q*)”).

The proof is similar to the proof of Theorem 3.3 and is omitted.

To get quick applications of Theorem 3.4, replace the “and” in the applications of Theorem 3.3 by “or”. For all the theorems above with given applications, there are many more applications.

References

[1] C. Dorsett, Properties and applications for (*P* or *Q*) for topological properties *P* and *Q*, Pioneer J. Math. Math. Sci. 27(1) (2019), 25-36.

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