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A VIABLE WORMHOLE MODEL IN A FIVE-DIMENSIONAL SPACETIME

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Abstract

This paper generalizes two of the author's earlier wormhole solutions that are characterized by an extra spatial dimension. This paper adds the assumption that the components of the line element are functions not only of the radial coordinate r but of the extra coordinate l as well, resulting in a significant generalization. It is shown that the throat of the wormhole can be threaded with ordinary matter and that the unavoidable violation of the null energy condition can be attributed to the extra dimension. To be consistent with string theory, we also assume that this extra dimension has a small magnitude.

1. Introduction

While wormholes are as good a prediction of Einstein's theory as

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black holes, they are subject to severe restrictions from quantum field theory [1, 2, 3, 4]. In particular, to hold a wormhole open requires the need to violate the null energy condition, calling for the existence of “exotic matter”. It has been shown, however, that this requirement can be met by assuming the existence of an extra spatial dimension. Some of these issues were discussed previously by the author [5, 6]: Ref. [5] assumes that the line element has an extra term of the form $e^{2\mu(r,l)}dl^2$, where r is the usual radial coordinate, while l is the extra coordinate. Ref. [6] assumes an extra time-dependent term of the form $e^{2\mu(r,l,t)}dl^2$. In this paper, the extra term has the simpler form $[\mu(r, l)]^2 dl^2$; in addition, the redshift and shape functions are also functions of r and l. It was shown once again that the throat of the wormhole could be lined with ordinary matter, while the extra dimension is then responsible for the unavoidable energy violation.

To that end, let us first recall the static and spherically symmetric form of a wormhole in Schwarzschild coordinates [1]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where

$$e^{2\lambda(r)} = 1 - \frac{b(r)}{r}. \tag{2}$$

Here $\Phi = \Phi(r)$ is called the *redshift function* and $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. A strong advocate of retaining an extra spatial dimension had been Paul Wesson [7]. The main reason is that the field equations for a five-dimensional totally flat spacetime yield the Einstein field equations in four dimensions containing matter, also called the *induced-matter theory*. It can be argued that our understanding of four-dimensional gravity is greatly enhanced by assuming a fifth dimension. We therefore need to construct a model that is consistent with our knowledge of general relativity. To that end, we are going to let l denote the extra coordinate and assume that there is an additional term having the form $[\mu(r, l)]^2 dl^2$, thereby retaining the dependence on the r-coordinate. To make the model as general as possible, we also assume that the first two terms in line element (1) are functions of r and l, as well. So the line element becomes

$$ds^2 = -e^{2\Phi(r, l)}dt^2 + e^{2\lambda(r, l)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + [\mu(r, l)]^2 dl^2, \quad (3)$$

where $e^{2\lambda(r, l)} = 1 - b(r, l)$. This line element is more general than those discussed in Refs. [5, 6]. To be consistent with string theory, we also assume that the coefficient $\mu(r, l)$ has a small magnitude.

2. Basic Calculations

To study the effects of our extra spatial dimension, we start with line element (3) and choose an orthonormal basis $\{e_{\hat{\alpha}}\}$ which is dual to the following 1-form basis:

$$\theta^0 = e^{\Phi(r, l)}dt, \qquad \theta^1 = \left[1 - \frac{b(r, l)}{r}\right]^{-1/2} dr, \qquad \theta^2 = r d\theta,$$

$$\theta^3 = r \sin\theta d\phi, \qquad \theta^4 = \mu(r, l) dl. \quad (4)$$

These forms yield

$$dt = e^{-\Phi(r, l)}\theta^0, \qquad dr = \left[1 - \frac{b(r, l)}{r}\right]^{1/2} \theta^1, \qquad d\theta = \frac{1}{r}\theta^2,$$

$$d\phi = \frac{1}{r\sin\theta}\theta^3, \qquad dl = \frac{1}{\mu(r, l)}\theta^4. \quad (5)$$

To obtain the components of the Riemann curvature tensor, we need to determine the curvature 2-forms. Here we use the method of differential forms, following Ref. [8]. To that end, we calculate the exterior derivatives in terms of θ^i, starting with

$$d\theta^0 = \frac{\partial\phi(r,\, l)}{\partial r}\left(1 - \frac{b(r,\, l)}{r}\right)^{1/2}\theta^1 \wedge \theta^0 + \frac{\partial\phi(r,\, l)}{\partial l}\frac{1}{\mu(r,\, l)}\theta^4 \wedge \theta^0. \tag{6}$$

As noted in the Introduction, to be consistent with string theory, we need to assume that $\mu(r,\, l)$ is extremely small. So $d\theta^0$ in Eq. (6) becomes physically unacceptable unless

$$\frac{\partial\phi(r,\, l)}{\partial l} \equiv 0. \tag{7}$$

A related problem in Eq. (6) is the shape function. Momentarily returning to Eq. (1), let us recall that in a Morris-Thorne wormhole, the spherical surface $r = r_0$ is called the *throat* of the wormhole, where $b(r_0) = r_0$ and $b(r) < r$ for $r > r_0$, while $b'(r_0) < 1$. In line element (3), the first of these conditions becomes $b(r_0,\, l) = r_0$ for every l, which does not appear to be a natural requirement. We are thereby forced to assume that the shape function is a function of r alone. The form of $d\theta^0$ therefore becomes [in view of Eq. (7)]

$$d\theta^0 = \frac{\partial\phi(r,\, l)}{\partial r}\left(1 - \frac{b(r)}{r}\right)^{1/2}\theta^1 \wedge \theta^0. \tag{8}$$

The remaining exterior derivatives are listed next:

$$d\theta^1 = 0, \tag{9}$$

$$d\theta^2 = \frac{1}{r}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^1 \wedge \theta^2, \tag{10}$$

$$d\theta^3 = \frac{1}{r}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^1 \wedge \theta^3 + \frac{1}{r}\cot\theta\; \theta^2 \wedge \theta^3, \tag{11}$$

$$d\theta^4 = \frac{\partial\mu(r,\, l)}{\partial r}\frac{1}{\mu(r,\, l)}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^1 \wedge \theta^4. \tag{12}$$

The connection 1-forms $\omega^i{}_k$ have the symmetry $\omega^0{}_i = \omega^i{}_0$ $(i = 1, 2, 3, 4)$ and $\omega^i{}_j = -\omega^j{}_i$ $(i, j = 1, 2, 3, 4, i \neq j)$ and are related to the basis $\{\theta^i\}$ by

$$d\theta^i = -\omega^i{}_k \wedge \theta^k. \tag{13}$$

The solution of this system is

$$\omega^0{}_1 = \frac{d\Phi(r,\, l)}{dr}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^0, \tag{14}$$

$$\omega^4{}_1 = \frac{\partial\mu(r,\, l)}{\partial r}\left[1 - \frac{b(r)}{r}\right]^{1/2}\frac{1}{\mu(r,\, l)}\theta^4, \tag{15}$$

$$\omega^3{}_2 = \frac{1}{r}\cot\theta\ \theta^3, \tag{16}$$

$$\omega^3{}_1 = \frac{1}{r}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^3, \tag{17}$$

$$\omega^2{}_1 = \frac{1}{r}\left[1 - \frac{b(r)}{r}\right]^{1/2}\theta^2, \tag{18}$$

$$\omega^0{}_4 = \frac{\partial\Phi(r,\, l)}{\partial l}\frac{1}{\mu(r,\, l)}\theta^0 = 0; \tag{19}$$

$\omega^0{}_4 = 0$ thanks to Eq. (7).) Finally,

$$\omega^0{}_2 = \omega^0{}_3 = \omega^2{}_4 = \omega^3{}_4 = 0. \tag{20}$$

The curvature 2-forms $\Omega^i{}_j$ are calculated directly from the Cartan structural equations

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{21}$$

The results for $\Omega^i{}_j$ are given in Appendix A.

The components of the Riemann curvature tensor can be read off directly from the form

$$\Omega^i{}_j = -\frac{1}{2} R_{mnj}{}^i \theta^m \wedge \theta^n \tag{22}$$

and are listed next:

$$R_{011}{}^0 = -\frac{1}{2}\frac{\partial\Phi(r,\, l)}{\partial r}\frac{rb'(r) - b(r)}{r^2} + \frac{\partial^2\Phi(r,\, l)}{\partial r^2}\left(1 - \frac{b(r)}{r}\right)$$

$$+ \left(\frac{\partial\Phi(r,\, l)}{\partial r}\right)^2\left(1 - \frac{b(r)}{r}\right), \tag{23}$$

$$R_{022}{}^0 = R_{033}{}^0 = \frac{1}{r}\frac{\partial\Phi(r,\, l)}{\partial r}\left(1 - \frac{b(r)}{r}\right), \tag{24}$$

$$R_{044}{}^0 = \frac{\partial\Phi(r,\, l)}{\partial r}\frac{\partial\mu(r,\, l)}{\partial r}\left(1 - \frac{b(r)}{r}\right)\frac{1}{\mu(r,\, l)}, \tag{25}$$

$$R_{122}{}^1 = R_{133}{}^1 = -\frac{1}{2}\frac{rb'(r) - b(r)}{r^3}, \tag{26}$$

$$R_{144}{}^1 = \left[\frac{\partial^2\mu(r,\, l)}{\partial r^2}\left(1 - \frac{b(r)}{r}\right) - \frac{1}{2}\frac{\partial\mu(r,\, l)}{\partial r}\frac{rb'(r) - b(r)}{r^2}\right]\frac{1}{\mu(r,\, l)}, \tag{27}$$

$$R_{233}{}^2 = -\frac{b(r)}{r^3}, \tag{28}$$

$$R_{244}{}^2 = R_{344}{}^3 = \frac{1}{r}\frac{\partial\mu(r,\, l)}{\partial r}\left(1 - \frac{b(r)}{r}\right)\frac{1}{\mu(r,\, l)}. \tag{29}$$

The last form to be derived in this section is the Ricci tensor, which is obtained by a trace on the Riemann curvature tensor:

$$R_{ab} = R_{acb}{}^{c}. \tag{30}$$

The components are listed in Appendix B. These forms will come into play in text section.

3. The Main Result

It was noted in Section 1 that a Morris-Thorne wormhole can only be kept open by violating the null energy condition (NEC). This condition states that for the energy-momentum tensor $T_{\alpha\beta}$, $T_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0$ for all null vectors $T_{\alpha\beta}$. In this section, we are going to show that thanks to the extra spatial dimension, the wormhole throat can be lined with ordinary matter, while the violation of the NEC can be attributed to the existence of the fifth dimension.

To that end, we start with the four-dimensional null vector $(1, 1, 0, 0)$, leaving the other null vectors for later. The Einstein field equations in the orthonormal frame are

$$G_{\hat{\alpha}\hat{\beta}} = R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2} R g_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}, \tag{31}$$

where

$$g_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

To simplify the notation, we now omit the hats. So $T_{00} = \rho$ is the energy density and $T_{11} = p_r$ is the radial pressure. (The NEC is therefore violated whenever $\rho + p_r < 0$). We now have

$$G_{00} + G_{11} = 8\pi(\rho + p_r) = \left[R_{00} - \frac{1}{2}R(-1)\right] + \left[R_{11} - \frac{1}{2}R(1)\right] = R_{00} + R_{11}. \tag{33}$$

Since we are primarily interested in the vicinity of the throat, we assume that $1 - b(r_0)/r_0 = 0$. We can now go directly to Appendix B and deduce that

$$8\pi(\rho + p_r)\big|_{r=r_0} = R_{00} + R_{11}\big|_{r=r_0} = \frac{b'(r_0) - 1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial\mu(r_0, l)}{\partial r}\frac{1}{\mu(r_0, l)}\right]. \tag{34}$$

Recalling that $b'(r_0) < 1$, it follows that for the four-dimensional case, $8\pi(\rho + p_r)\big|_{r=r_0} > 0$ whenever

$$\frac{\partial\mu(r_0, l)}{\partial r} < 0 \tag{35}$$

since $\mu(r_0, l)$ is assumed to be extremely small.

For the five-dimensional case, consider the null vector $(1, 0, 0, 0, 1)$. Then

$$8\pi T_{\alpha\beta}k^{\alpha}k^{\beta} = G_{00} + G_{44} = \left[R_{00} - \frac{1}{2}R(-1)\right] + \left[R_{44} - \frac{1}{2}R(1)\right] = R_{00} + R_{44}. \tag{36}$$

Given that $1 - b(r_0)/r_0 = 0$ again, we get

$$R_{00} + R_{44}\Big|_{r=r_0} = \frac{1}{2}\frac{b'(r_0)-1}{r_0}\left[-\frac{\partial\Phi(r_0,\, l)}{\partial r} + \frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)}\right]. \tag{37}$$

With inequality (35) in mind, suppose we also have

$$\frac{\partial\Phi(r_0,\, l)}{\partial r} < 0. \tag{38}$$

Then $R_{00} + R_{44}\Big|_{r=r_0} < 0$ whenever

$$\left|\frac{\partial\Phi(r_0,\, l)}{\partial r}\right| > \left|\frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)}\right|. \tag{39}$$

More precisely, from Eq. (34), if

$$\frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)} < -\frac{2}{r_0} \tag{40}$$

[which is consistent with (35)], then $R_{00} + R_{44}\Big|_{r=r_0} < 0$ provided that

$$\frac{\partial\Phi(r_0,\, l)}{\partial r} = -A < \frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)}. \tag{41}$$

So the NEC is indeed violated at the throat in the five-dimensional case, even though it is met in the four-dimensional case.

Summarizing, we have shown that the quantity

$$\left|\frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)}\right| \tag{42}$$

is large enough to satisfy condition (34) (four-dimensional case) and small enough to satisfy condition (37) (five-dimensional case).

4. The Remaining Conditions

It was noted in Section 3 that for the four-dimensional case, the NEC must also be satisfied for the null vectors $(1,\, 0,\, 1,\, 0)$ and $(1,\, 0,\, 0,\, 1)$. To that end, we require that

$$R_{00} + R_{22} \big|_{r=r_0} = R_{00} + R_{33} \big|_{r=r_0} = -\frac{1}{2}\frac{\partial\Phi(r_0,\, l)}{\partial r}\frac{r_0 b'(r_0) - b(r_0)}{r_0^2}$$

$$+\frac{1}{2}\frac{r_0 b'(r_0) - b(r_0)}{r_0^2} + \frac{b(r_0)}{r_0^3} = \frac{1}{2}\frac{b'(r_0) - 1}{r_0}\left(1 - \frac{\partial\Phi(r_0,\, l)}{\partial r}\right) + \frac{1}{r_0^2} > 0. \quad (43)$$

This condition is met provided that $b'(r_0)$ is sufficiently close to unity.

5. Conclusion

It is well known that a Morris-Thorne wormhole can only be sustained by violating the null energy condition, requiring the existence of "exotic matter". It has been shown that this requirement can be met by assuming the existence of an extra spatial dimension: the throat of the wormhole can be lined with ordinary matter, while the extra spatial dimension is responsible for the unavoidable energy violation.

The proposed line element is

$$ds^2 = -e^{2\Phi(r,\, l)}dt^2 + e^{2\lambda(r,\, l)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + [\mu(r,\, l)]^2 dl^2,$$

where l is the extra coordinate. It is subsequently shown that to be consistent with string theory, we must have $\partial\Phi(r,\, l)/\partial l = 0,$ while the shape function must have the usual form for a Morris-Thorne wormhole.

The requirements regarding the energy conditions can be summarized as follows: in the four-dimensional case [null vector $(1,\, 1,\, 0,\, 0)$],

$$8\pi(\rho + p_r)\big|_{r=r_0} = R_{00} + R_{11} \big|_{r=r_0}$$

$$= \frac{b'(r_0) - 1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial\mu(r_0,\, l)}{\partial r}\frac{1}{\mu(r_0,\, l)}\right] > 0 \quad (44)$$

whenever

$$\frac{\partial\mu(r_0, l)}{\partial r} < 0 \tag{45}$$

since $\mu(r_0, l)$ is assumed to be extremely small.

In the five-dimensional case [null vector (1, 0, 0, 0, 1)] suppose we have

$$\frac{\partial\Phi(r_0, l)}{\partial r} < 0. \tag{46}$$

Then

$$R_{00} + R_{44}\Big|_{r=r_0} = \frac{1}{2}\frac{b'(r_0) - 1}{r_0}\left[-\frac{\partial\Phi(r_0, l)}{\partial r} + \frac{\partial\mu(r_0, l)}{\partial r}\frac{1}{\mu(r_0, l)}\right] < 0 \tag{47}$$

whenever

$$\frac{\partial\mu(r_0, l)}{\partial r}\frac{1}{\mu(r_0, l)} < -\frac{2}{r_0} \tag{48}$$

and

$$\frac{\partial\Phi(r_0, l)}{\partial r} = -A < \frac{\partial\mu(r_0, l)}{\partial r}\frac{1}{\mu(r_0, l)}. \tag{49}$$

In summary, conditions (46), (48), and (49) imply that the NEC is met in the four-dimensional case and that the unavoidable violation of the NEC can be attributed to the fifth dimension.

To finish our discussion, we need to recall that for the shape function $b = b(r)$ at the throat $r = r_0$, $b'(r_0)$ must be sufficiently close to unity.

It should also be emphasized that the requirement that $\mu(r, l)$ has a small magnitude is consistent with string theory.

Appendix A. The curvature 2-forms

$$\Omega^0{}_1 = \left[-\frac{\partial^2 \Phi(r,\,l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) - \left(\frac{\partial \Phi(r,\,l)}{\partial r}\right)^2\left(1-\frac{b(r)}{r}\right) + \frac{1}{2}\frac{\partial \Phi(r,\,l)}{\partial r}\frac{rb'(r)-b(r)}{r^2}\right]\theta^0 \wedge \theta^1, \tag{50}$$

$$\Omega^0{}_2 = -\frac{1}{r}\frac{\partial \Phi(r,\,l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\theta^0 \wedge \theta^2, \tag{51}$$

$$\Omega^0{}_3 = -\frac{1}{r}\frac{\partial \Phi(r,\,l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\theta^0 \wedge \theta^3, \tag{52}$$

$$\Omega^0{}_4 = -\frac{\partial \Phi(r,\,l)}{\partial r}\frac{\partial \mu(r,\,l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\frac{1}{\mu(r,\,l)}\theta^0 \wedge \theta^4, \tag{53}$$

$$\Omega^1{}_2 = \frac{1}{2}\frac{rb'(r)-b(r)}{r^3}\theta^1 \wedge \theta^2, \tag{54}$$

$$\Omega^1{}_3 = \frac{1}{2}\frac{rb'(r)-b(r)}{r^3}\theta^1 \wedge \theta^3, \tag{55}$$

$$\Omega^1{}_4 = \left[-\frac{\partial^2 \mu(r,\,l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) + \frac{1}{2}\frac{\partial \mu(r,\,l)}{\partial r}\frac{rb'(r)-b(r)}{r^2}\right] \times \frac{1}{\mu(r,\,l)}\theta^1 \wedge \theta^4, \tag{56}$$

$$\Omega^2{}_3 = \frac{b(r)}{r^3}\theta^2 \wedge \theta^3, \tag{57}$$

$$\Omega^2{}_4 = -\frac{1}{r}\frac{\partial \mu(r,\,l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\frac{1}{\mu(r,\,l)}\theta^2 \wedge \theta^4, \tag{58}$$

$$\Omega^3{}_4 = -\frac{1}{r}\frac{\partial \mu(r,\,l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\frac{1}{\mu(r,\,l)}\theta^3 \wedge \theta^4. \tag{59}$$

Appendix B. The components of the Ricci tensor

$$\begin{aligned}R_{00} &= \frac{1}{2}\frac{\partial\Phi(r,\, l)}{\partial r}\frac{rb'(r)-b(r)}{r^2} + \frac{\partial^2\Phi(r,\, l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) \\ &+ \left(\frac{\partial\Phi(r,\, l)}{\partial r}\right)^2\left(1-\frac{b(r)}{r}\right) + \frac{2}{r}\frac{\partial\Phi(r,\, l)}{\partial r}\left(1-\frac{b(r)}{r}\right) \\ &+ \frac{\partial\Phi(r,\, l)}{\partial r}\frac{\partial\mu(r,\, l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\frac{1}{\mu(r,\, l)}, \end{aligned} \tag{60}$$

$$\begin{aligned}R_{11} &= \frac{1}{2}\frac{\partial\Phi(r,\, l)}{\partial r}\frac{rb'(r)-b(r)}{r^2} - \frac{\partial^2\Phi(r,\, l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) \\ &- \left(\frac{\partial\Phi(r,\, l)}{\partial r}\right)^2\left(1-\frac{b(r)}{r}\right) + \frac{rb'(r)-b(r)}{r^3} \\ &- \left[\frac{\partial^2\mu(r,\, l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) - \frac{1}{2}\frac{\partial\mu(r,\, l)}{\partial r}\frac{rb'(r)-b(r)}{r^2}\right]\frac{1}{\mu(r,\, l)}, \end{aligned} \tag{61}$$

$$\begin{aligned}R_{22} = R_{33} &= -\frac{1}{r}\frac{\partial\Phi(r,\, l)}{\partial r}\left(1-\frac{b(r)}{r}\right) + \frac{1}{2}\frac{rb'(r)-b(r)}{r^2} + \frac{b(r)}{r^3} \\ &- \frac{1}{r}\left(1-\frac{b(r)}{r}\right)\frac{\partial\mu(r,\, l)}{\partial r}\frac{1}{\mu(r,\, l)}, \end{aligned} \tag{62}$$

$$\begin{aligned}R_{44} &= -\frac{\partial\Phi(r,\, l)}{\partial r}\frac{\partial\mu(r,\, l)}{\partial r}\left(1-\frac{b(r)}{r}\right)\frac{1}{\mu(r,\, l)} \\ &- \left[\frac{\partial^2\mu(r,\, l)}{\partial r^2}\left(1-\frac{b(r)}{r}\right) - \frac{1}{2}\frac{\partial\mu(r,\, l)}{\partial r}\frac{rb'(r)-b(r)}{r^2}\right]\frac{1}{\mu(r,\, l)} \\ &- \frac{2}{r}\left(1-\frac{b(r)}{r}\right)\frac{\partial\mu(r,\, l)}{\partial r}\frac{1}{\mu(r,\, l)}. \end{aligned} \tag{63}$$

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