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# A SHARP APPROXIMATION FOR THE GAMMA FUNCTION AS CONTINUED FRACTION

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### Abstract

We introduce a new sharp approximation for the Gamma function as a continued fraction, which is faster than the classical Stirling series.

## 1. Introduction

There are many situations in pure mathematics or in other branches of science when we are forced to manipulate large factorials, e.g., see [1, 2, 5, 6]. Maybe one of

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the most well-known formulas for approximation the factorial function is the so-called Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{1.1}$$

Up to now, many researchers made great efforts in the area of establishing more precise inequalities and more accurate approximations for the factorial function and its extension gamma function, and had a lot of inspiring results. Recently, some authors also paid attention to giving increasing better approximations for the gamma function using continued fractions. For example, Mortici [5] found Stieltjes' continued fraction

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\cfrac{a_0}{x+\cfrac{a_1}{x+\cfrac{a_2}{x+\cdots}}}\right) \tag{1.2}$$

$$\underline{\underline{\Delta}}\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{a_0}{x+}\,\frac{a_1}{x+}\,\frac{a_2}{x+\cdots}\right), \tag{1.3}$$

where

$$a_0 = \frac{1}{12},\ a_1 = \frac{1}{30},\ a_2 = \frac{53}{210},\ \cdots.$$

In addition, Mortici [6] also provided a new continued fraction approximation starting from Nemes' formula as following:

$$\Gamma(x+1) \approx \sqrt{2\pi x}e^{-x} \exp\left(x + \frac{1}{12x-}\,\frac{1}{10x+}\,\frac{a}{x+}\,\frac{b}{x+}\,\frac{c}{x+}\,\frac{d}{x+\cdots}\right)^x, \tag{1.4}$$

where

$$a = -\frac{2369}{252},\ b = \frac{2117009}{1193976},\ c = \frac{393032191511}{1324011300744},$$

$$d = \frac{33265896164277124002451}{14278024104089641878840}\cdots.$$

Recently, Lu [9] provided a new continued fraction approximation starting from

Burnside's formula as following:

$$\Gamma(x+1) \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{a_1}{n+}\,\frac{a_2 n}{n+}\,\frac{a_3 n}{n+}\,\frac{a_4 n}{n+\cdots}\right), \tag{1.5}$$

where

$$a_1 = -\frac{1}{24},\ a_2 = \frac{1}{2},\ a_3 = -\frac{7}{60},\ a_4 = -\frac{37}{60},\ \cdots.$$

It is their works that motivate our study. In this paper, based on the early works of Mortici [5, 6, 8], we provide a similar continued fraction approximation for the fractorial function as follows:

**Theorem 1.** *For the factorial function*, *we have*

$$\Gamma(x+1) \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp(\eta_9(n)), \tag{1.6}$$

*where*

$$\eta_9(n) = \frac{a_1}{n+}\,\frac{a_2 n}{n+}\,\frac{a_3 n}{n+}\,\frac{a_4 n}{n+}\,\frac{a_5 n}{n+}\,\frac{a_6 n}{n+}\,\frac{a_7 n}{n+}\,\frac{a_8 n}{n+}\,\frac{a_9 n}{n+\cdots},$$

$$a_1 = -\frac{1}{24},\ a_2 = \frac{1}{2},\ a_3 = -\frac{7}{60},\ a_4 = \frac{37}{60},\ a_5 = -\frac{41}{98},$$

$$a_6 = \frac{45}{49},\ a_7 = -\frac{10981}{18204},\ a_8 = \frac{20083}{18204},\ a_9 = -\frac{317364295}{346549379},\ \cdots.$$

**Remark 1.** We get the coefficients $a_n,\ n = 1, 2, ..., 9$ by *Mathematica*. When calculating $a_9$, the computer has to spend more than one minute. So we do not continue to calculate $a_n,\ n \geq 10$ because of the computational difficulties.

It is easy to see that Burnside's formula is only a particular case of Theorem 1 for all $a_i = 0,\ i \geq 1$. Next, using Theorem 1, we provide some inequalities for the gamma function.

**Theorem 2.** *There exists an m, such that for every* $x > m$, *it holds*:

$$\exp(\eta_8(n)) < \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x} < \exp(\eta_9(n)), \tag{1.7}$$

*where*

$$\eta_8(n) = \frac{a_1}{n+}\,\frac{a_2 n}{n+}\,\frac{a_3 n}{n+}\,\frac{a_4 n}{n+}\,\frac{a_5 n}{n+}\,\frac{a_6 n}{n+}\,\frac{a_7 n}{n+}\,\frac{a_8 n}{n+\cdots}.$$

**Remark 2.** This is not only a more sharp inequalities than these similar inequalities in [5, 9], but also it has fast rate of convergence as a continued fraction than others. We give more tight upper and lower bounds for the approximation of gamma function. To obtain Theorem 1, we need the following lemma which was used in [5, 6] and is very useful for constructing asymptotic expansions.

**Lemma 1.** *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{1.8}$$

*with* $s > 1$, *then*

$$\lim_{n\to+\infty} n^{s-1} x_n = \frac{l}{s-1}. \tag{1.9}$$

Lemma 1 was first proved by Mortici. From Lemma 1, we can see that the rate of convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ increases together with the value $s$ satisfying (1.9).

## 2. The Proof of Theorem 1

Based on the argument of Theorem 2.1 in [5] or Theorem 5 in [6], we need to find the value $c_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{a_1}{n}\right). \tag{2.1}$$

To measure the accuracy of this approximation, a method is to define a sequence $(u_n)_{n\in\mathbb{N}}$ by the relations

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{a_1}{n}\right)\exp(u_n) \tag{2.2}$$

and to say that an approximation (2.1) is better if $u_n$ converges to zero faster. From (2.2), we have

$$u_n = \ln n! - \frac{1}{2}\ln 2\pi - \left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right) + n + \frac{1}{2} - \frac{a_1}{n}. \tag{2.3}$$

Thus

$$u_n - u_{n+1} = -1 + \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n+\frac{1}{2}}\right)$$

$$+ \ln\left(1+\frac{\frac{1}{2}}{n+1}\right) - \frac{a_1}{n} + \frac{a_1}{n+1}. \tag{2.4}$$

Developing the power series in $\frac{1}{2}$, we have

$$u_n - u_{n+1} = -\left(\frac{1}{24}+a_1\right)\frac{1}{n^2} + \left(\frac{1}{12}+a_1\right)\frac{1}{n^3}$$

$$-\left(\frac{41}{320}+a_1\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{2.5}$$

From Lemma 1, we know that the rate of convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ is even higher as the value $s$ satisfying (1.9). Thus using Lemma 1, we have:

1. If $a_1 \neq -\frac{1}{24}$, then the rate of convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ is $\frac{1}{n}$ since

$$\lim_{n\to+\infty} nu_n = -\left(\frac{1}{24} + a_1\right) \neq 0. \tag{2.6}$$

2. If $a_1 = -\frac{1}{24}$, then from (2.5), we have

$$u_n - u_{n+1} = \frac{1}{24}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right) \tag{2.7}$$

and the rate of convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ is $\frac{1}{n^2}$ since

$$\lim_{n\to+\infty} n^2 u_n = \frac{1}{48}. \tag{2.8}$$

We know that the fastest possible sequence $(u_n)_{n\in\mathbb{N}}$ is obtained only for $a_1 = -\frac{1}{24}$.

Next, we define the sequence $(v_n)_{n\in\mathbb{N}}$ by the relation

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{-\frac{1}{24}}{n+a_2}\right)\exp(v_n). \tag{2.9}$$

Using the same method, we have that the fastest possible sequence $(v_n)_{n\in\mathbb{N}}$ is obtained only for $a_2 = \frac{1}{2}$.

Then, we define the sequence $(w_n)_{n\in\mathbb{N}}$ by the relation

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{-\frac{1}{24}}{n+\frac{\frac{1}{2}n}{n+a_3}}\right)\exp(w_n). \tag{2.10}$$

Using the same method, we have that the fastest possible sequence $(w_n)_{n\in\mathbb{N}}$ is obtained only for $a_3 = -\frac{7}{60}$.

By induction, we have $a_4 = \frac{37}{64}$, $a_5 = -\frac{41}{98}$, $a_6 = \frac{45}{49}$, $a_7 = -\frac{10981}{18204}$, $a_8 = \frac{20083}{18204}$, $a_9 = -\frac{317364295}{346549379}$, ..., and the new asymptotic expansion (1.5) is obtained.

This completes the proof of Theorem 1.

## 3. The Proof of Theorem 2

We need the following basic result of Alzer [2], who defined the function $R_n(x)$ by

$$\ln \Gamma(x+1) = \left(x + \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + (-1)^n R_n(x), \tag{3.1}$$

where $R_n(x)$ is completely monotonic on $(0, +\infty)$, $B_j$ is the $j$th Bernoulli number defined by the power series expansion

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!}$$

$$= 1 - \frac{t}{2} + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \frac{1}{30240}t^6 - \frac{1}{1209600}t^8 + \frac{1}{47900160}t^{10} + O(t^{11}),$$

where $B_{2i+1} = 0$, for all $i \geq 1$, and the Bernoulli number $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}$, $B_{16} = -\frac{3617}{510}$, $B_{18} = \frac{43867}{798}$, $B_{20} = -\frac{174611}{330}$, ... .

In particular, $R_n(x)$ is positive, and for $n = 9$ and $n = 10$, we get the following inequalities, for $x > 0$,

$$\exp(H_{15}(x)) < \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x} < \exp\left(H_{15}(x) + \frac{43867}{244188x^{17}}\right), \tag{3.2}$$

where

$$\begin{aligned} H_{15}(x) &= \frac{B_2}{2 \cdot 1x^1} - \frac{B_4}{4 \cdot 3x^3} + \frac{B_6}{6 \cdot 5x^5} - \frac{B_8}{8 \cdot 7x^7} + \frac{B_{10}}{10 \cdot 9x^9} \\ &\quad - \frac{B_{12}}{12 \cdot 11x^{11}} + \frac{B_{14}}{14 \cdot 13x^{13}} - \frac{B_{16}}{16 \cdot 15x^{15}} \\ &= \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} \\ &\quad - \frac{691}{360360x^{11}} + \frac{1}{156x^{13}} - \frac{3617}{122400x^{15}}. \end{aligned}$$

Now the conclusion follows the inequalities

$$H_{15}(x) > \eta_8(n), \tag{3.3}$$

$$H_{15}(x) + \frac{43867}{244188x^{17}} < \eta_9(n). \tag{3.4}$$

This completes the proof of Theorem 2.

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