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A SECOND SOLUTION FOR THE RHIND PAPYRUS UNIT FRACTION DECOMPOSITIONS

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Abstract

Within this paper, a second, greatly-improved systematic solution, consistent with the mathematics used in the Rhind Papyrus, is given for the long unsolved mystery of a method for construction of the unit fraction decomposition table in the Rhind Papyrus.

1. Introduction

Most of our knowledge of ancient Egyptian mathematics is derived from two sizable papyri: the Rhind Papyrus and the Golenischev Papyrus. In 1858, A. Henry Rhind purchased a partial papyrus in Luxor, Egypt. The papyrus was reportedly found in Thebes, in the ruins of a building near the Ramesseum. The Rhind Papyrus was written in hieratic script about 1650 B.C. by a scibe named Ahmes. Since much of what we know about ancient Egyptian mathematics comes from the Rhind Papyrus, Ahmes' work is both mathematically and historically significant.

Readers of the papyrus are assured that its content is a likeness of earlier work

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dating back to the Twelfth Dynasty: 1849-1801 B.C. Early Egyptian mathematics with fractions, dating back to the Twelfth Dynasty, was made difficult by the computational practice of allowing only unit fractions, i.e., fractions of the form $\frac{1}{n}$, where n is a natural number. To overcome this difficulty, ancient Egyptian mathematicians constructed unit fraction decomposition tables for quick reference and use. The table at the beginning of the Rhind Papyrus gives unit fraction decompositions of fractions of the form $\frac{2}{n}$, where n is an odd natural number between 1 and 103, and is the most extensive of the unit fraction decomposition tables to be found and preserved among the ancient Egyptian papyri.

As stated within the Rhind Papyrus, the content of the papyrus was to give "a thorough study of all things, insight into all that exists, and knowledge of all obscure secrets." However, no insight or knowledge of the construction of the unit fraction decomposition table was given within the papyrus, leaving its method of construction an obscure secret. The lack of information about the construction of the table did, almost 3700 years after its creation, stimulate interest in its method of construction motivating many to try to end the mystery, all of which has made the table even more historically and mathematically important.

As indicated, through the years, many interested people have determinedly tried to solve the mystery of the table's construction. Because of the insights and systematic processes included within the Rhind Papyrus, most interested people searched for a systematic process that would give the unit fraction decompositions, but, up to the fall 2006, only patterns giving some of the entries within the table were known. For example, it was known that fractions within the table of the form $\frac{2}{3k}$ follow the general pattern $\frac{2}{3k} = \frac{1}{2k} + \frac{1}{6k}$. Thus some progress was made. However, many different observed patterns and unexpected entries within the table raised questions about the existence of a systematic method making the search for a systematic solution very much like looking for a needle in a haystack, not knowing if, in fact, there is a needle in the haystack. In addition, prior to the fall 2006, there were unit fraction decompositions that followed none of the known patterns continuing to leaving the construction of the table an unknown mystery.

The 2006 solution, which was published in 2008 [2], provided a single systematic method that could be used to generate all the entries in the table. The 2008 solution [2] gave a process that could be used, but fell short of the ideal solution since additional work was required to determine each entry. However, the discoveries, insights, and processes in the 2008 solution [2] proved to be mathematically significant since their use helped produce the more precise algorithm in this paper.

Additional information concerning the Rhind Papyrus unit fraction decompositions and the difficulties in searching for a systematic solution to the mystery can be found in David Burton's book: The History of Mathematics: An Introduction [1], and would serve as an excellent resource. As recognized in Burton's book [1], there are two totally different types of unit fraction decompositions in the table, the second one for $\frac{2}{101}$, which unlike the previous entries, abandoned the mission for the creation of the table giving a one mold fit all entry that was, with almost total certainty, a later addition to the table by someone other than the creator of the above entries.

2. The Motivation and Goal for the Construction of the Table

The motivation for constructing a mathematics table is need. As indicated above, there was a need among ancient Egyptian mathematicians for a unit fraction decomposition table. The goal would be to construct an easily used table that best served the need, without extraneous entries. The creator of the unit fraction decompositions in the table for n less than or equal to 99 would be highly motivated since he or she would be the first to use the table and would not want to further complicate an already difficult notation restriction with anything less than the best possible unit fraction decompositions.

3. Development of a Strategy

The strategy for the construction of a table is determined by the need, known information, known strategies, the difficulties of the construction, and the cleverness of the creator, together, in this case, with a determined resolve to find that best choice. As witnessed by the absence of fractions of the form $\frac{2}{n}$, where n is even, the creator of the table was aware of the multiplication and cancellation properties for fractions. Thus inclusion of fractions $\frac{2}{n}$, where n is even and 4 or bigger, would be extraneous information not to be included in the table.

The objective was to determine the best unit fraction decompositions for further calculations that would work best with the required notation. To accomplish these objectives, the number of terms in the decomposition would be an important consideration as would be the size of the denominators, particularly the denominators of the first and last fraction in the decomposition. Thus the ideal choice for an entry would be a two term decomposition with smallest possible first and second fraction denominators that meets the need. However, the creator of the entries realized that a three term or four term unit fraction decomposition could possibly better serve the needs and added those possibilities in the search for the best unit fraction decomposition. In addition, even though the focus was on the individual entries, and decisions were made at the entry level, uniformity within the entries would make use of the table easier and more productive.

As was the case for the Rhind Papyrus unit fraction decompositions table, for an entry $\frac{2}{n}$ in the table, obtaining unit fraction decomposition for multiples of $\frac{1}{n}$ between $\frac{1}{n}$ and 1 would be an important related calculation. With the best unit fraction decomposition for an entry $\frac{2}{n}$, this objective could be best accomplished: for an even integer $k = 2j$ between 1 and n, $k\left(\frac{1}{n}\right) = j\left(\frac{2}{n}\right)$ and for an odd integer $k = 2j+1$ between 1 and n, $k\left(\frac{1}{n}\right) = j\left(\frac{2}{n}\right) + \frac{1}{n}$. Thus a key in the search for a table entry is the unit fraction decomposition that best gives integer multiples of $\frac{1}{n}$. For example consider the table entry for $\frac{2}{15} : \frac{1}{10} + \frac{1}{30}$. Using that entry, $\frac{4}{15} = 2\left(\frac{2}{15}\right) = \frac{1}{5} + \frac{1}{15}$, $\frac{6}{15} = \frac{2}{5} = \frac{1}{3} + \frac{1}{15}$, $\frac{8}{15} = 2\left(\frac{4}{15}\right) = \frac{2}{5} + \frac{2}{15} = \frac{1}{3} + \frac{1}{15} + \frac{1}{10} + \frac{1}{30}$, $\frac{10}{15} = \frac{2}{3} = \frac{1}{2} + \frac{1}{6}$, and $\frac{12}{15} = \frac{4}{5} = 2\left(\frac{2}{5}\right) = \frac{2}{3} + \frac{2}{15} = \frac{1}{2} + \frac{1}{6} + \frac{1}{5} + \frac{1}{15}$ making the given unit fraction decomposition for $\frac{2}{15}$ the ideal unit fraction decomposition for $\frac{2}{15}$ based on need. Of all the many possibilities, how did the creator of the table come up with the 10 and 30? The creator would have to develop a strategy that identified the 10 and 30 for prime consideration and then establish that choice as the best choice.

For consistency and better comparison and use, each entry should be written using the same format. The entries in the table are written in increasing orders of the denominators. To maximize the use of the table, the denominator of the first fraction in the decomposition of $\frac{2}{n}$ should be less than *n*, allowing better and greater use of table. This could be, and, was done, by considering only values of *p* starting at $\frac{(n+1)}{2}$ and going to $n-1$. For each *p*, let $o = 2p - n$. The values for *o* will be increasing odd integers starting with 1 up to and including $n-2$ giving $\frac{2}{n} = \frac{2p}{np} = \frac{(n+o)}{np} = \frac{n}{np} + \frac{o}{np} = \frac{1}{p} + \frac{o}{np}$. The objective is to find the *o* that has a unit fraction decomposition with smallest last term denominator that best simplifies related calculations.

If $\frac{4}{n} = \frac{2}{n} + \frac{2}{n} < 1$, then *p* having a factor of 2 would immediately reduce $\frac{4}{n}$ to a sum of unit fractions. If $\frac{6}{n} = \frac{2}{n} + \frac{4}{n} < 1$, then *p* having another factor of 2 would immediately reduce $\frac{6}{n}$ to a sum of unit fractions. If $\frac{8}{n} = \frac{4}{n} + \frac{4}{n} < 1$, *p* having a third factor of 2 would immediately reduce $\frac{8}{n}$ to a sum of unit fractions. The process could be continued making values of *p* with factors of 2 highly attractive for use in constructing the unit fraction decompositions. However, selecting values of *p* with many factors of 2 makes the denominators big, defeating a goal in the creation of the table. The best entry would contain one factor of 2 and, quickly and easily, regenerate the desired factor of 2 within the calculations. Having a factor of 3 with the 2 would be helpful in calculating $\frac{6}{2n}$.

If *o* divides *np*, canceling *o* in the numerator and denominator of the second fraction gives an easily-calculated, highly desirable two unit fraction decomposition candidate for $\frac{2}{n}$. Whether or not *o* divides *np*, if *o* can be written as a decreasing sum of two or three factors of *np*, then $\frac{2}{n}$ can be reduced to a sum of unit fractions.

For a fixed such p, given a choice of decreasing sums, the sum with largest last term would make the denominator of the last two fractions as small as possible giving a candidate for the unit fraction decomposition.

For each n, either there exists an odd integer o greater than 1 that divides np or no such o exists. Consider the first case, where n is composite and o divides np.

If o divides n, then one natural place to look for the best entry would be the largest factor of n and, in many cases, gives the desired two term unit fraction decomposition. If o divides np and not n, then, as known by the creator of the table based on use of the property in the construction of the table and proven in the 2008 paper [2], each factor of o is a factor of n. Thus, to obtain the desired unit fraction decomposition, the focus falls on the factors of n greater than or equal to 1 and on the question of whether or not the focus could be extended to additional numbers.

If replacing a prime factor f of n of a factor $o(n)$ of n by the square of the smallest prime factor m of n gives a product less than n and $m^2 > f$, let k be the larger such value off, and let $o(n, k) = \frac{m^2(o(n))}{k}$. Let $q(n, k)$ be the integer such that $2(q(n, k)) = n + o(n, k)$. Then $q(n, k) = \frac{n + o(n, k)}{2}$ and $\frac{n + o(n, k(n))}{n(q(n, k))}$ reduces to a sum of two unit fractions giving a candidate for the used unit fraction decomposition. If $o(n, k)$ can be written as a sum of factors of $q(n, k)$, then the sum with the largest last term would give an additional candidate for the unit fraction decomposition. The smallest prime factor was selected in order to get a smaller value of p.

In this case, to allow the possibility of a two term unit fraction decomposition, only those values of o greater than 1 for which o divides np, where the p is the corresponding value for o, as discussed above, will be considered. Because of the desire for a small denominator, for each o, the focus is on the denominator of the last fraction in the corresponding decomposition or decompositions. If one of the denominators for the last unit fraction is much smaller than and contains factors located earlier in the table, then the unit fraction decomposition for that choice would be the entry in the table. Given two possible choices, if one gives an even value for p, because of the preference for even numbers, the unit fraction decomposition with even p would be used in the table. Given two choices, for ease of calculations within the entry itself, a factor of 2 and 3 would be an important consideration and if the prime factors of n are also included, then to make calculations for multiples of $\frac{2}{n}$ easier, that choice would be a serious consideration. If the p value is a multiple of a prime factor of n, calculations of multiples of $\frac{2}{n}$ would be simplified and thus a consideration when choosing the unit fraction decomposition.

The focus is on n and o, but the corresponding p can be calculated using $p = \frac{(n+o)}{2}$. Below information is given to help make the best choice. In some cases there is a clear choice. In the cases where there is no clear choice amongst the two best choices, the creator of the table, determined to obtain the best choice, would have used each to do calculations and then select the one that worked best.

Below the calculations needed in the search for the best entry is given for each composite value of n using the format (the value of o, the corresponding value for p, denominator of last term) or (the value of o considered, the corresponding value for p, the sum used, denominator of last term).

For $n = 9 = 3^2$, there is only one choice: $(3,\ 6,\ 2(3^2))$ giving the unit fraction decomposition $\frac{2}{9} = \frac{1}{6} + \frac{1}{18}$.

For $n = 15 = 3(5)$, the choices are $1 = (3,\ 9,\ (3^2)(5))$, $2 = (5,\ 10,\ 2(3)(5))$, $3 = (9,\ 12,\ 2^2(5))$, and $4 = (9,\ 12,\ 5+4,\ 3^2(5))$ giving a choice between 2 and 3. Use the criteria above, 2 would be the one selected giving the unit fraction decomposition $\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$.

For $n = 21 = 3(7)$, the choices are $1 = (3,\ 12,\ 2^2(3)(7))$, $2 = (7,\ 14,\ 2(3)(7))$, $3 = (9,\ 15,\ 5(7))$, and $4 = (9,\ 15,\ 5+3+1,\ 3^2(5)(7))$. Choice 3 would require use of an entry in the table following the entry for 21 and would be excluded. Using the criteria above, 2 would be selected giving the unit fraction decomposition $\frac{2}{21} = \frac{1}{14} + \frac{1}{42}$.

For $n = 25 = 5^2$, there is only one choice $(5,\ 15,\ 3(5^2))$ giving $\frac{2}{15} = \frac{1}{15} + \frac{1}{75}$.

For $n = 27 = 3^2$, the choices are $1 = (3, 15, 3^2(5))$, $2 = (9, 18, 2(3^3))$, and $3 = (9, 18, 9 + 6, 2(3^4))$. Of the choices, 2 is selected giving the table entry $\frac{2}{27} = \frac{1}{18} + \frac{1}{54}$.

For $n = 33 = 3(11)$, the choices are $1 = (3, 18, 2(3^2)(11))$ and $2 = (11, 22, 2(3)(11))$. The clear choice is 2 giving the decomposition $\frac{2}{33} = \frac{1}{22} + \frac{1}{66}$.

For $n = 35 = 5(7)$, the choices are $1 = (5, 20, 2^2(5)(7))$, $2 = (7, 21, 3(5)(7))$, $3 = (25, 30, 2(3)(7))$, and $4 = (25, 30, 15 + 10, 3(5)(7))$. Using the criteria above, 3 would be selected giving $\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$.

For $n = 39 = 3(13)$, the choices are $1 = (3, 21, 3(7)(13))$, $2 = (13, 26, 2(3)(13))$. Of the two, 2 is the choice giving $\frac{2}{39} = \frac{1}{26} + \frac{1}{78}$.

For $n = 45 = 3^2(5)$, the choices are $1 = (2, 24, 2^3(3^2)(5))$, $2 = (5, 25, 3^2(5^2))$, $3 = (9, 27, 3^3(5))$, and $4 = (15, 30, 2(3^2)(5))$. Of those choices, 4 is selected giving $\frac{2}{45} = \frac{1}{30} + \frac{1}{90}$.

For $n = 49 = 7^2$, there is only one choice $(7, 28, 2^2(7^2))$ giving $\frac{2}{49} = \frac{1}{28} + \frac{1}{196}$.

For $n = 51 = 3(17)$, the choices are $1 = (3, 27, 3^3(17))$ and $2 = (17, 34, 2(3)(17))$, Choice 2 would be selected giving $\frac{2}{51} = \frac{1}{34} + \frac{1}{102}$.

For $n = 55 = 5(11)$, the choices are $1 = (5, 30, 2(3)(5)(11))$, $2 = (11, 33, 3(5)(11))$, and $3 = (25, 35, 7(11))$. Since 1 gives an even p, 1 is the selection giving $\frac{2}{55} = \frac{1}{30} + \frac{1}{330}$.

For $n = 57 = 3(19)$, the choices are $1 = (3, 30, 2(3^2)(5)(19))$ and $2 = (19, 38, 2(3)(19))$ with 2 the best choice for table entry $\frac{2}{57} = \frac{1}{38} + \frac{1}{114}$.

For $n = 63 = 3^2(7)$, the choices are $1 = (3, 33, 3^2(7)(11))$, $2 = (7, 35, 3^2(5)(7))$, $3 = (9, 36, 2^2(3)(7))$, $4 = (21, 42, 2(3^2)(7))$, $5 = (27, 45, 3(5)(7))$, and $6 = (27, 45, 15 + 9 + 3, 3^3(5)(7))$ and the one selected is $4: \frac{2}{63} = \frac{1}{42} + \frac{1}{126}$.

For $n = 65 = 5(13)$, the choices are $1 = (5, 35, 5(7)(13))$, $2 = (13, 39, 3(5)(7))$, $3 = (25, 45, 3^2(13))$, and $4 = (25, 45, 13 + 9 + 3, 3(5^2)(13))$. There are no even values for p and 2 has the smallest last term making 2 the selection giving $\frac{2}{65} = \frac{2}{39} + \frac{1}{105}$.

For $n = 69 = 3(23)$, the choices are $1 = (3, 36, 2^2(3^2)(23))$ and $2 = (23, 46, 2(3)(23))$ giving selection 2 and decomposition $\frac{2}{69} = \frac{1}{46} + \frac{1}{138}$.

For $n = 75 = 3(5^2)$, the choices are $1 = (3, 39, 3(5^2)(13))$, $2 = (5, 40, 2^3(3)(5))$, $3 = (15, 45, 3^2(5^2)$, and $4 = (25, 50, 2(3)(5^2))$ giving selection 4 and decomposition $\frac{2}{75} = \frac{1}{50} + \frac{1}{150}$.

For $n = 77 = 7(11)$, the choices are $1 = (7, 42, 2(3)(7)(11))$, $2 = (11, 44, 2^2(7)(11))$, $3 = (49, 63, 3^2(11))$, and $4 = (49, 63, 33 + 9 + 7, 3^2(7)(11))$. This is a tough call between 1 and 2, but for 2, $p = 44$, is a multiple of 11 making calculations easier and giving the decomposition $\frac{2}{77} = \frac{1}{44} + \frac{1}{308}$.

For $n = 81 = 3^4$, the choices are $1 = (3, 42, 2(3^4)(7))$, $2 = (9, 45, 3^4)$, and $3 = (27, 54, 2(3^4))$. Following the criteria, the selection is 3 giving the decomposition $\frac{2}{81} = \frac{1}{54} + \frac{1}{162}$.

For $n = 85 = 5(17)$, the choices are $1 = (5, 45, 3^2(5)(17))$, $2 = (17, 51, 3(5)(17))$, and $3 = (25, 55, 11(17))$. Since 51 is a multiple of 17, 2 would be selected giving decomposition $\frac{2}{85} = \frac{1}{51} + \frac{1}{255}$.

For $n = 87 = 3(29)$, the choices are $1 = (3, 45, 3^3(5)(29))$ and $2 = (29, 58, 2(3)(29))$ giving selection 2 and table entry $\frac{2}{87} = \frac{1}{58} + \frac{1}{174}$.

For $n = 91 = 7(13)$, the choices are $1 = (7, 49, 7^2(13))$, $2 = (13, 52, 2^2(7)(13))$, $3 = (49, 70, 2(5)(13))$, and $4 = (49, 70, 35 + 14, 5(7)(13))$. Selected would be 3 giving decomposition $\frac{2}{91} = \frac{1}{70} + \frac{1}{130}$.

For $n = 93 = 3(31)$, the choices are $1 = (3, 48, 2^4(3)(31))$ and $2 = (31, 62, 2(3)(31))$. Clearly 2 is the choice giving table entry $\frac{2}{93} = \frac{1}{62} + \frac{1}{186}$.

For $n = 95 = 5(19)$, the choices are $1 = (5, 50, 2(5^2)(19))$, $2 = (19, 57, 3(5)(7))$, $3 = (25, 60, 2^2(3)(19))$, and $4 = (25, 60, 15 + 10, 2(3)(5)(19))$. Using the criteria, 4 would be selected giving decomposition $\frac{2}{95} = \frac{1}{95} + \frac{1}{380} + \frac{1}{570}$.

For $n = 99 = 3^2(11)$, the choices are $1 = (3, 51, 3^2(11)(17))$, $2 = (9, 54, 2(3^3)(11))$, $3 = (11, 55, 3^2(5)(11))$, $4 = (33, 66, 2(3^2)(11))$, and $5 = (33, 66, 22 + 11, 2(3^3)(11))$. Selected would be 4 giving decomposition $\frac{2}{99} = \frac{1}{66} + \frac{1}{198}$.

Thus consider $\frac{2}{n}$, where n is prime and not 101. As above, in the case of a composite number, the focus is on each n with objective obtaining the best unit fraction decomposition for performing needed calculations. For each n there are $\frac{n-3}{2}$ values for p and for each of those p values, there could be many possible unit fraction decompositions for consideration, all of which must be known and considered to determine the best choice. Once the unit fraction decompositions are known for each p, from those the one that is best suited for the needed calculations is the unit fraction decomposition for $\frac{2}{n}$ in the table. To do all this for one n, without a plan, is difficult, time consuming, and filled with uncertainty about the required completeness. To have to repeat the entire process for the next value of n perhaps requiring even more determinations can be overwhelming. Thus a strategy to make the selection process more easily managed and to ensure completeness would be in order.

In this case, because n is prime, only values of o, with corresponding value p, are considered for which o divides p or o can be written as a decreasing sum of two or three factors of p. Below a partial strategy is given intentionally leaving its completion for the interested reader. The strategy is applied to the values of p for a fixed, but arbitrary, prime number n. Thus, all the cases below are within the same odd prime n.

Consider the case that p is the product of two or more distinct odd primes. Then p is the product of exactly 2 odd primes since, otherwise, $p \geq 3(5)(7) = 105$, which is too large for consideration in the table. Let a and b be two distinct odd primes such that $3 \leq b \leq a$ and $p = ab$. Then the only possible way to form a unit fraction decomposition from p would be use of 1 or b or a or the sum $a + b + 1$, where obviously the larger possibility is the $a + b + 1$. Knowing the largest possibility m is useful since, if the corresponding o value for p is greater than m, p can be immediately removed from consideration. If $o = m$, then the resulting unit fraction decomposition would be saved for further consideration and all other possibilities can be discarded. For this p, and all others, there is mathematically at most one two term unit fractions saved for further consideration, which is easily determined by seeing if one of the possible factors is the value of o. All other possibilities must be checked saving only those whose value is o.

If p is prime and o is not 1, for p to be further considered, the corresponding o would have to be p, $2p = n + 0 = n + p$, and $p = n$, which is impossible. Thus all prime values for p, with o not 1, can be immediately discarded.

Consider the case where p is a power of an odd prime with exponent 2 or more, given by $p = a^k$. As above, the only possible two term unit fraction decomposition from p is the possible case where $o = 1$ or, because of the restrictions on o, $o = a^m$, where $1 \leq m < k$. If $a \geq 5$, then, because of the limitations in the table, $k = 2$ and there are no possible three or four term unit fraction decompositions. For $a = 3$, $2 \leq k \leq 4$. For 3^2, there would be no two term or three term decreasing sums for consideration. For 3^3, if $o = 1$ or $o = 3$ or $o = 3^2$ or if $n \geq 53$ and

$o = 3^2 + 3 + 1,$ there would be a two or a four term unit fraction decomposition from p for further consideration. For $3^4,$ if o is 1 or 3 or 9 or 27 or $3^3 + 3^2 + 3$ or $3^3 + 3 + 1$ or $3^2 + 3 + 1,$ there would two or four term unit decompositions to consider. As above, if $o > 3^3 + 3^2 + 3,$ then $p = 3^4$ can be discarded from consideration.

For $p = 2^k,$ $k \geq 2,$ the maximum value of k is 6 and there would be the possibility of two or three or four term unit fraction decompositions for further consideration all of which the last term is $\frac{1}{np}$. For example, if $k = 3$ and o is 1 or $2^3 + 1$ or $2^2 + 1$ or $2 + 1$ or $2^3 + 2^2 + 1$ or $2^3 + 2 + 1$ or $2^2 + 2 + 1,$ there would be a two or four term unit fraction decomposition for further consideration.

If $p = 2a,$ where a is an odd prime, for further consideration of p, the corresponding value for o, would have to be a or $a + 2$.

Obviously, having a strategy makes the process more manageable and also gives not before observed, important insights into the problem. As indicated above, the remaining cases include values of p with many factors that possibly can be used to create many sums for consideration in the search for unit fraction decompositions and are left as an exercise for the interested reader.

As in the case above where n is composite, the process is labor intensive, but producing the best in almost all instances requires a greater effort. For example, for $\frac{2}{61}$, there are 13 possible unit fraction decompositions saved from use of the above completed algorithm. The 13 candidates are given using the format (p, factor or sum, denominator of last term): $1 = (31, 1, 61(31)),$ $2 = (32, 2 + 1, 61(32)),$ $3 = (36, 9 + 2, 61(18)),$ $4 = (36, 6 + 3 + 2, 61(18)),$ $5 = (36, 6 + 4 + 1, 61(36)),$ $6 = (39, 13 + 3 + 1, 61(39)),$ $7 = (40, 10 + 5 + 4, 61(10)),$ $8 = (40, 10 + 8 + 1, 61(40)),$ $9 = (42, 21 + 2, 61(24)),$ $10 = (42, 14 + 6 + 3, 61(14)),$ $11 = (45, 15 + 9 + 5, 61(9)),$ $12 = (48, 24 + 8 + 3, 61(16)),$ and $13 = (54, 27 + 18 + 2, 61(27)).$ Choosing the two with the smallest last denominators give the unit fraction decompositions $a = \frac{1}{40} + \frac{1}{61(10)} + \frac{1}{61(8)} + \frac{1}{61(10)}$ and $b = \frac{1}{45} + \frac{1}{61(3)} + \frac{1}{61(5)} + \frac{1}{61(9)}$. Both candidates have positives to consider, but because of the power of 2 as discussed above, decomposition *a* is used for the table entry.

After the developed algorithm has been applied to an odd prime *n* and unit fraction decompositions have been saved, then, as in the composite case, and the example above, size matters. Also, even numbers continue to be good choices making calculations easier. However, each entry presents its own special circumstances that impact the selection process making a one fit all selection process impossible when searching for the best choice.

As a general rule, from the choices pick the two with smallest last term denominators. This was the case for $\frac{2}{61}$. If one of the selections is a two term unit fraction decomposition and the denominator of the last term of the two term unit fraction is the smaller, then, as a general rule, that two term unit fraction decomposition is the table entry. Otherwise the other unit fraction decomposition is the table entry. For example, for $\frac{2}{5}$, the two selected are $\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$ and $\frac{2}{5}=\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$ and the table entry is the first one. For $\frac{2}{17}$, the choices are $\frac{2}{17}=\frac{1}{9}+\frac{1}{153}$ and $\frac{2}{17}=\frac{1}{12}+\frac{1}{51}+\frac{1}{68}$, the last of which is the entry in the table. An exception occurs when the compared denominator is smaller, but odd, and the denominators of the two term unit fraction decomposition are even, in which case the two term unit fraction decomposition is the table entry. For example, for $\frac{2}{7}$, the two choices are $\frac{2}{7}=\frac{1}{4}+\frac{1}{24}$ and $\frac{2}{7}=\frac{1}{6}+\frac{1}{14}+\frac{1}{21}$ with the first one the table entry.

When the two selected are three term unit fraction decompositions, to ensure easiest use for calculations, the focus is on the denominators of the last two terms for each decomposition. If the denominator of the last term with smaller denominator is even and, for the other, is odd, then the one with the smaller last denominator is selected. For example, for $\frac{2}{19}$, the choices are $\frac{1}{12}+\frac{1}{19(4)}+\frac{1}{19(6)}$ and $\frac{1}{14}+\frac{1}{19(2)}+\frac{1}{19(7)}$ and the first decomposition is the entry in the table. In a consistent manner, the same thing happens when the two selected decompositions are a three term and a four term decomposition as seen for $\frac{2}{41}$. The two selected decompositions are $\frac{1}{24}+\frac{1}{41(6)}+\frac{1}{41(8)}$ and $\frac{1}{36}+\frac{1}{41(2)}+\frac{1}{41(4)}+\frac{1}{41(9)}$ and the first is the entry in the table. When comparing a three and four term decomposition both having even last denominator, the smallest second from last denominator becomes a factor for consideration. For example, for $\frac{2}{47}$, the two choices are $\frac{1}{30}+\frac{1}{47(3)}+\frac{1}{47(10)}$ and $\frac{1}{40}+\frac{1}{47(2)}+\frac{1}{47(5)}+\frac{1}{47(8)}$ with the first the table entry. A similar situation occurs when comparing two four term unit fraction decompositions.

The remaining entries are left as an exercise to give the interested reader the opportunity to retrace the process used by the creator of the table. The remaining entries in the table are given below for the interested readers to check their answers:

$\frac{2}{3}=\frac{1}{2}+\frac{1}{6}$, $\frac{2}{11}=\frac{1}{6}+\frac{1}{66}$, $\frac{2}{13}=\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$, $\frac{2}{23}=\frac{1}{12}+\frac{1}{176}$, $\frac{2}{29}=\frac{1}{24}+\frac{1}{58}+\frac{1}{174}+\frac{1}{232}$, $\frac{2}{31}=\frac{1}{20}+\frac{1}{124}+\frac{1}{155}$ $\frac{2}{37}=\frac{1}{24}+\frac{1}{111}+\frac{1}{296}$, $\frac{2}{43}=\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$, $\frac{2}{53}=\frac{1}{30}+\frac{1}{318}+\frac{1}{795}$, $\frac{2}{59}=\frac{1}{36}+\frac{1}{236}+\frac{1}{531}$, $\frac{2}{67}=\frac{1}{40}+\frac{1}{335}+\frac{1}{536}$, $\frac{2}{71}=\frac{1}{40}=\frac{1}{568}+\frac{1}{710}$, $\frac{2}{73}=\frac{1}{60}+\frac{1}{219}+\frac{1}{292}+\frac{1}{365}$, $\frac{2}{79}=\frac{1}{60}+\frac{1}{237}+\frac{1}{316}+\frac{1}{790}$, $\frac{2}{83}=\frac{1}{60}+\frac{1}{332}+\frac{1}{415}+\frac{1}{498}$, $\frac{2}{89}=\frac{1}{60}+\frac{1}{356}+\frac{1}{534}+\frac{1}{890}$, and $\frac{2}{97}=\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$.

Because the method of construction of the table was so time consuming and labor intensive, the creator recorded his or her creation in written form to avoid having to recreate it. As witnessed above, the creator's focus was to obtain the best unit fraction decomposition for each fraction in the table based on the need at that time. Under such circumstances, to try to reverse the process by using the table to explain the individual entries in the table, as many have tried, is self defeating resulting only with patterns for some of the entries within the table.

Unlike many modern day creators that require all to fit one mold quickly and easily, whether or not it is a good fit, the creator of the table within the Rhind Papyrus made sure each part was the best fit for the need, making the whole a masterpiece not only for his or her time, but for all time.

For the creator of the table to make a written record of the method of construction of the table would have been extremely difficult: the notation of the period was totally inadequate for such a major undertaking and much space would be required for the explanation of how the candidates for each entry were determined and then, even more space for an explanation of how the candidates were narrowed to the best choice, making even the availability of writing material an obstacle. As a result, most likely, a very small number of people knew through verbal discussions the method of construction of the table and, once those people were gone, the method of construction of the table was a mystery to even the faithful users of the table.

As is true for all man-made creations, with of without explanation, the true test of value for the creation is efficient, dependable, lasting utility of the creation for as long as there is a need for the creation. Almost 200 years after its creation, the creator's unit fraction decomposition table continued to be the best of the unit fraction decomposition tables. To recognize the table's importance and to ensure its continued availability, Ahmes included it in his papyrus and, as a result, preserved the work of a mathematical master for all time.

References

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