Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 13, Issue 2, 2020, Pages 79-102
This paper is available online at http://www.frdint.com/
Published online March 31, 2020

A ROBUST VOTING PROCEDURE USING GEOMETRIC MEAN AS AGGREGATION FUNCTION

RUFFIN-BENOÎT M. NGOIE and ZOÏNABO SAVADOGO*

Département de Mathématiques
Institut Supérieur Pédagogique de Mbanza-Ngungu
R. D. Congo
e-mail: benoitmpoy@hotmail.com

UFR/SEA LANIBIO
Université Joseph Ki-Zerbo
Burkina Faso
e-mail: serezenab@yahoo.fr

Abstract

Geometric mean is a statistical parameter with fair properties. Because of its robustness, it is used for aggregating cardinal preferences, especially to guarantee strategy-proofness. The Geometric Voting Function (GVF) is a simplified formulation of a voting function based on this parameter. It is presented in this article as an innovative voting function with many democracy-desired properties. In addition, it is supplied with a tie-

Keywords and phrases: cardinal preference, electoral system, geometric mean, geometric voting function, standard deviation.

2020 Mathematics Subject Classification: 00A06, 91A80, 91B12, 91B14, 91-02, 91A35.

*Corresponding author

Received February 20, 2020; Accepted March 6, 2020

breaking mechanism involving standard deviation to guarantee its resoluteness. A characterization of the method was carried out as well as a study of its time complexity.

1. Introduction

Voting is a vital activity in any modern democratic society. It consists on collecting individuals' preferences in order to find the social preference for the community. In this hypothesis, it can be seen as an effective way for a community to make collective decisions. To find the collective preference, one uses a mathematical function called "aggregation function". A voting function is a method that is designed to aggregate individuals' preferences into a collective one. In social choice theory, there are many kinds of such functions but all of them are vulnerable to electoral paradoxes and are struck by impossibility theorems. Nevertheless, some voting methods are likely more democratic than others. This is the case for expressive voting methods. In group decision theory, several procedures have been proposed to determine, from individual preferences, a collective preference.

However, it is not easy to always decide which is the best function among all of existing ones. Baujard and Igersheim [4] argue that the legitimacy of a voting function lies on the confidence which the voters have in its successive restrictions and its capacity to accurately represent their preferences. In addition, Norris [15] has proposed four normative criteria to evaluate an electoral system: government effectiveness, government responsiveness and accountability, fairness to minor parties and social representation.

Several criticisms are therefore made towards voting methods by social choice theoreticians and especially in relation to the controversial results they generate. Most of these controversial results are due to voters' strategic manipulation, the algorithmic complexity and the methods calculation procedures.

Many authors have questioned the majority system, which seems to give sometimes surprising results. It appears that the ordinal approach of individual preferences, widely used in the beginning of the social choice theory (see Borda [6] and Condorcet [8]), is now disputed by authors such as Hillinger [12], Balinski and

Laraki [2]. These authors reject ranking voting systems and propose aggregation methods based on the evaluation principle.

Smaoui and Lepelley [19] agree with the fact that ranking voting systems use voters' ordinal preferences and do not make it possible to judge (or appreciate) independently the different options. Moreover, the one or two-round majority voting system has a large number of widely known and proven defects such as failure in Condorcet winner and loser conditions, majority requirement, participation, reinforcement, monotonicity, etc. (see Condorcet [8] and Felsenthal [9]).

In this paper, we suggest a simple voting mechanism based on evaluation principle where geometric mean is used to determine final score for a candidate. As we will show that further, herein presented method meets many democracy-desired properties and needs only but a polynomial time to converge. Therefore, our paper is organized as follows: Section 2 is devoted to important concepts and preliminaries. Section 3 deals with voting functions. A novel voting rule called "Geometric Voting Function" (GVF) is presented in Section 4. Sections 5, 6, 7 and 8 are, respectively, devoted to the characterization of GVF, its robustness and time complexity, desirable properties it fills and paradoxical results it yields when used in real life. Concluding remarks are given in Section 9.

2. Basic Terminology and Preliminaries

Definition 2.1 (Average)**.** Let S be an arbitrary set and let $\mathbb{R}$ denote the system of real numbers. For each function f from S to $\mathbb{R}$, we define an "average" or "mean" as a function $M(f)$ that satisfies the following axioms:

- Axiom 1: $M(f)$ is a real number;

- Axiom 2: $M(cf) = c.M(f)$ for any real constant c;

- Axiom 3: $M(f) \geq 0$ for all functions f what assume only nonnegative values;

- Axiom 4: $M(1) = 1,$ where 1 denotes the function on S that everywhere assumes the value 1;

• Axiom 5: $M(f) \leq M(g)\ \forall f \leq g$, i.e., $f(s) \leq g(s)\ \forall s \in S$.

Axiom 2 is called *homogeneity condition*. It assumes that multiplying the argument f by a constant c is worth to multiply M by c. It is a consequence of Axioms 2 and 4 that $M(a) = a, \forall a \in \mathbb{R}$.

Definition 2.2 (Arithmetic mean)**.** Let a function denoted A be an average. A is called "arithmetic mean" if and only if it satisfies the following supplementary axiom:

• Axiom 6: $A(f + g) = A(f) + A(g)$ for any two functions f and g.

Definition 2.3 (Geometric mean)**.** Let f be a positive function on S and $\log f$ its natural logarithm. We define the "geometric mean" $G(f)$ by

$$G(f) = e^{A(\log f)},$$

where e is the base of natural logarithms.

Definition 2.4 (Extended mean function)**.** Stolarsky [20] has generalized mean by an expression as follows:

$$E(r, s; x, y) = \left[\frac{r(y^s - x^s)}{s(y^r - x^r)}\right]^{\frac{1}{s-r}},\ rs(r-s)(x-y) \neq 0.$$

Then

$$E(1, 2; x, y) = \frac{x + y}{2},$$

and

$$E(0, 0; x, y) = \sqrt{xy},$$

are, respectively, artithmetic and geometric means.

Definition 2.5 (Completely monotonic function)**.** A function f is said to be completely monotonic on an interval $I \subseteq R$ if f has derivatives of all orders on I and

$$(-1)^n f^n(t) \geq 0,$$

for all $t \in I$ and $n \in \mathbb{N}$.

Definition 2.6 (Bernstein function)**.** A function $f : I \subseteq \mathbb{R} \to [0, +\infty[$ is called a "Bernstein function" on I if $f(t)$ has derivatives of all orders and $f'(t)$ is completely monotonic on I.

A theorem due to Schilling et al. [17] characterizes Bernstein functions. It states that a function $f :]0, 1[\to \mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x) = a + bx + \int_0^{+\infty} (1 - e^{-xt}) d\mu(t),$$

where $a, b \geq 0$ and μ is a measure on $]0, +\infty[$ satisfying

$$\int_0^{+\infty} \min(1, t) d\mu(t) < +\infty.$$

Theorem 2.1. *Arithmetic and geometric means are Bernstein functions.*

Proof. It is easy to see that the arithmetic mean

$$A_{x, y}(t) = A(x + t, y + t) = A(x, y) + t,$$

is a trivial Bernstein function of $t \in \{-\min(x, y), +\infty\}$ for $x, y > 0$.

To establish the second part of the theorem, it is clear that geometric mean $G_{x, y}(t)$ satisfies

$$G'_{x, y}(t) = \frac{1}{2}\left(\sqrt{\frac{x + t}{y + t}} + \sqrt{\frac{y + t}{x + t}}\right) = \frac{1}{2}\left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) = f(u)$$

and

$$[\log G_{x, y}(t)]' = \frac{1}{2}\left(\frac{1}{x + t} + \frac{1}{y + t}\right),$$

where $u = u_{x,y}(t) = \frac{x+t}{y+t} = 1 + \frac{y-x}{y-t}$ if $0 < x < y$, then $0 < u'_{x,y}(t) < 1$, $\forall t \in \,]-x, +\infty[$. On the other hand, the function $f(u)$ is positive and

$$f^{(i)}(u) = \frac{1}{2}\left[(-1)^{i-1}\frac{(2i-3)!!}{2^i}u^{-(2i-1)/2} + (-1)^i\frac{(2i-1)!!}{2^i}u^{-(2i+1)/2}\right]$$

$$= (-1)^i\frac{(2i-3)!!}{2^i+1}\frac{1}{u^{(2i-1)/2}}\left(\frac{2i-1}{u} - 1\right),$$

for $i \in \mathbb{N}$, which implies that the function $f(u)$ is completely monotonic on $]0, 1[$.

So, when $y > x > 0$, the derivative $G'_{x,y}(t)$ is completely monotonic and the geometric mean $G_{x,y}(t)$ is a Bernstein function. □

2.1. Additional properties

Let $M = f(a_1, a_2, \ldots, a_n)$. In other words, M is some unknown function, *f*, of $a_1, \ldots, a_n$. We assume $a_i > 0$. Since M is an unweighted expected value or mean, the function *f* must satisfy the following three properties:

Property 1 (Reflexive property):

$$f(a, a, \ldots, a) = a.$$

Property 2 (Symmetric property):

$$f(a_1, a_2, \ldots, a_n) = f(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}),$$

for all permutations σ of the numbers $1, \ldots, n$. This second property maintains that the order of the arguments of *f* does not affect M.

Property 3 (Multiplicative property):

$$f(a_1b_l, a_2b_2, \ldots, a_nb_n) = f(a_1, a_2, \ldots, a_n)f(b_1, b_2, \ldots, b_n).$$

2.2. Characterization of the geometric mean

Fleming and Wallace [10] characterize the geometric mean as follows:

Theorem 2.2 (Fleming and Wallace [10]). *The unique function that fills Properties* 1 *through* 3 *is the geometric mean*.

Proof. See related paper from Fleming and Wallace [10].

3. Voting Functions

In practice, according to Balinski and Laraki [3], voting invokes issues that go well beyond the problem of how to elect one candidate among several or how to determine their order of finish. Candidates may be elected as the representatives of regions to legislatures, or as the representatives of political parties, or of both regions and parties, to legislatures. A multitude of different systems are used; they raise different problems, invoke different information, ask for different inputs, and are resolved with different mechanisms. Nevertheless, several central problems are common to many electoral systems.

There are mainly two frameworks for preference expression: ranking and grading. The first model (ranking) also called traditional or Arrow's framework (see Arrow [1]) is consecrated by some seven centuries of use wherein individuals' opinions as to preferences are often given in a pairwise binary choice format. While traditional model is dedicated to ordinal preferences, the second model is concerned by cardinal preferences. Voting procedures are more and more designed in this framework to answer the social choice theory central question: how can we amalgamate the appreciations or evaluations of many individuals into one collective appreciation or evaluation?

Most of these voting functions use statistical parameters to aggregate individuals' grades into a collective grade. So, approval voting (AV), range voting (RV) and Borda Majority Count (BMC) are arithmetic mean-based voting functions. Majority Judgment (MJ) is a median-based voting function. Mean-Median Compromise Method (MMCM) hybridizes mean and median to aggregate cardinal preferences[1].

---

[1]For more details, we refer the interested reader to Brams and Fishburn [7] for AV, Balinski and Laraki [3] for MJ, Zahid and De Swart [22] for BMC and RV, and Ngoie et al. [14] for MMCM.

3.1. Social grading functions

Let $\Lambda = \{g_1, g_2, \ldots, g_p\}$ be a finite set called a *common language* or a well-defined grading system. Voters have to grade candidates using Λ. The grading system can be made by a range of positive integers, a set of letters, words or phrases denoting the opinion or how the voter finds (judges) the candidates. Following Balinski and Laraki [3], a common language is a set of strictly ordered grades. A profile $\Phi(A, J)$ is an $m \times n$ matrix of the grades $\Phi(a_i, j) \in \Lambda$ assigned by each voter $j \in J$ to each of the candidates $a_i \in A$.

Definition 3.1 (Method of grading)**.** A method of grading is a function F defined as follows:

$$F : \Lambda^{m \times n} \to \Lambda^m$$

$$\Phi(A, J) = \begin{pmatrix} g_{11} & \cdots & g_{1j} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{i1} & \cdots & g_{ij} & \cdots & g_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{m1} & \cdots & g_{mi} & \cdots & g_{mn} \end{pmatrix} \mapsto (g_1^*, g_2^*, \ldots, g_m^*)$$

with g_{ij} the grade given by judge j to candidate a_i, and $g_i^* \in \Lambda$ the final grade assigned to a_i by function F.

MJ and AV are social grading functions as they assign to each candidate a unique grade taken in the adopted common language. Notice that for AV common language is the binary set $\Lambda = \{0, 1\}$ where 0 = “approved” and 1 = “disapproved”. We recommend interested reader to Balinski and Laraki [3] for more details on MJ common language.

3.2. Social ranking functions

Given a finite language Λ, judges assign grades to any number of competitors that are the inputs or profile

$$\Phi(A, J) = \begin{pmatrix} g_{11} & \cdots & g_{1j} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{i1} & \cdots & g_{ij} & \cdots & g_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{m1} & \cdots & g_{mi} & \cdots & g_{mn} \end{pmatrix}.$$

Let us consider $S = \{1, 2, \dots, n\}$ a set of n voters and two candidates A and B. By definition,

- $A \succcurlyeq_S B$ indicates that A is collectively preferred to B according to the chosen decision rule;
- $A \approx_S B$ if $A \succcurlyeq_S B$ and $B \succcurlyeq_S A$;
- $A \succ_S B$ if $A \succcurlyeq_S B$ but it is not true that $A \approx_S B$.

Definition 3.2 (Method of ranking)**.** A method of ranking is a nonsymmetric binary relation $\succcurlyeq_S$ that compares any two competitors, A and B, whose grades belong to some profile Φ.

Readers can easily check that AV, MJ, BMC, and MMCM are social ranking functions.

3.3. Desirable properties for voting functions

There are many democracy-desired properties for voting functions. In the social choice literature, many authors such as Arrow [1], Balinski and Laraki [3], Felsenthal [9] and Taylor [21], agree with the following fair properties for a voting functions:

- Universality $(\mathbf{U})$: A fair voting function must guarantee a certain freedom of the individuals taking part in the decision-making process. In addition, each of their preferences must be taken into account in the process of deciding the election winner.

- Unanimity or Pareto $(\mathbf{P})$: If all the individuals involved in the collective decision making process prefer an alternative a to another b, this same decision must be transcribed to the collective decision.

• Resoluteness $(\mathbf{R})$: Given m candidates $c_1, c_2, \ldots, c_m$, the function must always decide who is the winner among them.

• Independence of irrelevant alternatives $(\mathbf{I})$: The collective decision between two options a and b must depend only on individual preferences on both options even when individuals modify their preferences among other options while leaving unchanged their preferences on a and b.

• Neutrality $(\mathbf{N})$: Permuting the names of the alternatives will result in the same permutation in the final outcome.

• Non-dictatorship $(\mathbf{D})$: A fair voting function must exclude any situation in which an individual would be able to impose her or his choice to all the individuals involved in the decision process.

• Monotonicity $(\mathbf{M})$: A winning alternative is never harmed whenever some voters decide to lift up this alternative in their grades or rankings without changing anything else.

• Anonimity $(\mathbf{A})$: Permuting the names of the voters do not have any impact on the final outcome.

• Clone-resistance $(\mathbf{C})$: Whenever a clone of a losing alternative is introduced, this does not alter the original outcome.

• Homogeneity $(\mathbf{H})$: Given a preference profile and the corresponding outcome, replicating this profile λ times $(\lambda > 1, \lambda \in \mathbb{N})$ does not change the outcome. In our framework, we will say that a voting function is homogeneous if whenever we replicate λ times the original profile, the outcome remains unchanged.

• Reinforcement condition $(\mathbf{RC})$: The reinforcement condition requires that when an electorate is divided into two groups of voters and the voting outcome is the same for both groups, this outcome will remain unchanged when both groups of voters are merged.

• Participation condition $(\mathbf{PC})$: A decision rule is said to meet the Participation condition if when some voters supporting a winning candidate are added, this will not turn this candidate into a losing one.

• Majority condition $(\mathbf{MC})$: A decision rule is said to meet the Majority condition if when an absolute majority prefers a candidate *a* to another candidate *b*, it always returns candidate *a* as winner.

• Condorcet Winner condition $(\mathbf{CW})$: A decision rule is said to meet the Condorcet Winner condition if it always selects the Condorcet winner when it exists. A Condorcet winner is a candidate who is preferred by an absolute majority to any other candidate.

• Condorcet Loser condition $(\mathbf{CL})$: A decision rule is said to meet the Condorcet Loser condition if it never selects the Condorcet loser when it exists. A Condorcet loser is a candidate who is defeated by any other candidate in one-to-one match.

• Transitivity condition $(\mathbf{TC})$: A decision rule is said transitive if $A \succcurlyeq_S B$ and $B \succcurlyeq_S C$ implies $A \succcurlyeq_S C$. This condition demands that the Condorcet paradox be avoided.

• Simpson's condition $(\mathbf{SC})$: A decision rule is said to meet Simpson's condition when it does yield the Simpson's paradox: a phenomenon where an apparent relationship in different groups seems to be reversed when the groups are combined.

• Liberalism $(\mathbf{L})$: A decision rule is said to guarantee liberalism if for each voter $i \in S$ there is at least two candidates A and B such that $A \succcurlyeq_i B$ (resp., $B \succcurlyeq_i A$) implies $A \succcurlyeq_S B$ (resp., $B \succcurlyeq_S A$). This condition means that each voter must be decisive in both ways on at least a pair of candidates.

• Minimal liberalism (L^*): A decision rule is said to guarantee liberalism if there are at least two decisive voters in both ways on at least two distinct candidates.

• Non-manipulability condition $(\mathbf{NM})$: A decision rule is said to be manipulable if a voter can modify final result (social candidates' ranking) by modifying his individual preference.

3.4. Impossibility theorems

3.4.1. General possibility theorem

The General Possibility Theorem stated by Arrow [1] shows that there is no aggregation method that simultaneously satisfies the four conditions $\mathbf{I}$, $\mathbf{U}$, $\mathbf{P}$, and $\mathbf{D}$ when at least two individuals have to decide on at least three options.

3.4.2. Paretian liberal impossibility theorems

Sen [18] stated two impossibility theorems after introducing the concept of liberalism (condition $\boldsymbol{L}$) and its associate weak formulation of minimal liberalism (condition $\boldsymbol{L}^*$).

Theorem 3.1 (Sen [18])**.** *There is no aggregation method that simultaneously satisfies the three conditions* $\boldsymbol{U}$**,** $\boldsymbol{P}$ *and* $\boldsymbol{L}$.

Theorem 3.2 (Sen [18])**.** *There is no aggregation method that simultaneously satisfies the three conditions* $\boldsymbol{U}$**,** $\boldsymbol{P}$ *and* $\boldsymbol{L}^*$.

3.4.3. Gibbard-Satterthwaite theorem

Gibbard [11] and Satterthwaite [16] state the following impossibility theorem:

Theorem 3.3. *If there is at least three candidates*, *no aggregation function can simultaneously satisfy conditions* $\boldsymbol{U}$**,** $\boldsymbol{D}$ *and* $\boldsymbol{NM}$**.**

4. Geometric Voting Function

In this section, we propose a voting function based on the geometric mean of grades allotted by voters to candidates. The proposed function is called *Geometric Voting Function* (GVF). Let us recall that for n strictly positive real numbers $g_1, g_2, \ldots, g_n$ the geometric mean of these numbers denoted G is defined as follows:

$$G = \sqrt[n]{\prod_{i=1}^{n} g_i}.$$

The above-defined mean is indicated in case of long-queueing distributions. In fact, when one deals with a normal distribution, arithmetic mean, mode and median are the same. But most of cases, other means are preferred to the arithmetic mean as the latter is very sensitive to data individual variations.

For example, to evaluate a political candidate *c*, his partisans would allot to him the highest mark. So, they tend to multiply the candidate's real value by 2, 3 or more. In addition, *c*'s opponents will tend to divide his real value by 2, 3 or more. The correct central tendency parameter is therefore the geometric mean, not the arithmetic one nor other means.

As we will show that later in this article, the geometric mean is more robust than the arithmetic one. While arithmetic mean is very sensitive to extreme values and increases with the standard deviation, geometric mean is as higher as the standard deviation is smaller. For example, consider three candidates A, B, and C with respective list of grades $\{4, 4, 5, 6, 6\}$, $\{1, 4, 5, 7, 8\}$, and $\{5, 5, 5, 5, 5\}$. As it is shown in Table 1, all the three candidates are equal according to arithmetic mean. However, we can notice that there is a consensus for candidate C who is the best candidate according to geometric mean. So, while we have $A \approx B \approx C$ with arithmetic mean, we easily check that for geometric mean there is no tie with the three candidates as $C \succ A \succ B$.

Table 1. Statistics for candidates A, B and C

Candidate	Arithmetic mean	Geometric mean	Standard deviation
A	5.00000	4.91902	1.00000
B	5.00000	4.07234	2.73861
C	5.00000	5.00000	0.00000

In case the distribution is very large, we use logarithms to compute the geometric mean. Indeed,

$$Log(G) = \frac{1}{n}\sum_{i=1}^{n} Log(x_i),$$

where $Log(x_i)$ is the decimal logarithm of x_i.

4.1. Geometric score

Consider a set E of m candidates for an election, with $m \geq 2$ and a set of s voters with $s \geq 2$. The Geometric Voting Function (GVF) is defined as follows:

Each voter allots to each candidate a unique grade as defined in Balinski and Laraki [3] common language. For more accuracy, the literal marks correspond to a 1-10 scale of numerical data (see Table 2).

For example, if a voter judges that a candidate deserves “Very Good” as final grade, she or he may allot to him 7 or 8 points.

The method consists in considering the geometric mean of each candidate’s numerical grades. As it is not easy to compute such a value for large electorate (presidential election for example), we suggest the use of corresponding geometric grades which are the decimal logarithms of numerical grades.

We notice from Table 2 that the grading variation is not linear in the geometric scale. This is one of advantages of the geometric mean. In real life, it is easier for a voter to increase (resp., decrease) a candidate’s grade from 2 to 3 (resp., from 3 to 2) than she or he could do the same from 8 to 9 (resp., from 9 to 8).

Table 2. Common language and corresponding geometric scale

Literal grade	Numerical grade	Geometric grade (log)	Variation
To reject	1	0.00	–
Poor	2	0.30	–
	3	0.48	60.00%
	4	0.60	25.00%
Acceptable	5	0.70	16.67%
Good	6	0.78	11.43%
	7	0.85	8.97%
Very Good	8	0.90	5.88%
	9	0.95	5.56%
Excellent	10	1.00	5.26%

Definition 4.1 (Geometric score)**.** The geometric score for a candidate *j* denoted $G(j)$ is the sum of all obtained geometric grades from *n* voters, i.e.,

$$G(j) = \frac{1}{n}\sum_{i=1}^{n} g_{ij}^{*},$$

where $g_{ij}^{*} = Log(g_{ij})(1 \le i \le n)$ indicates the geometric grades allotted by voters to candidate *j*. Finally, the candidate with the highest rating is the one who represents the consensus.

Notice that the geometric score for a candidate is always less or equal to 1. It can be expressed as percentage.

For example, a candidate with grades 10, 5, 4 and 1 receives as geometric score

$$\frac{1.00 + 0.70 + 0.60 + 0.00}{4} = 57.5\%.$$

The geometric mean is a robust statistical parameter with desirable properties. It does not compensate for weak grades by high ones like the arithmetic mean which is widely used in many voting systems. It favors candidates who do not have very weak scores and whose grades are not very dispersed. For that reason, we suggest a tie-breaking mechanism taking into account data dispersion.

4.2. Tie-breaking mechanism

As the probability of *ex aequo* is high when one deals with small electorates, we suggest a tie-breaking mechanism in this subsection. When the geometric score of two candidates are different, the one with the higher geometric score naturally ranks ahead of the other. The geometric ranking $\succ_g$ between two candidates *a* and *b* evaluated by the same jury is determined as follows:

- If $G(a) > G(b)$, then $a \succ_g b$;

- If $G(a) = G(b)$, then the one with the lower Standard Deviation (SD) ranks ahead of the other.

4.3. Numerical examples

Example 1. Small electorate.

Five judges evaluate the performance of two skaters. The results are presented as follows:

$$a : 10 \quad 5 \quad 6 \quad 4 \quad 2$$

$$b : 4 \quad 7 \quad 7 \quad 5 \quad 3$$

Using the corresponding geometric grades, we have:

$$G(a) = \frac{1.00 + 0.70 + 0.78 + 0.60 + 0.30}{5} = 67.6\%,$$

$$G(b) = \frac{0.60 + 0.85 + 0.85 + 0.70 + 0.48}{5} = 69.6\%.$$

We then finally have $b \succ_g a$.

Example 2. Small electorate with ex aequo.

Four judges evaluate the performances of three skaters. The results are presented as follows:

$$a : 10 \quad 5 \quad 4 \quad 1$$

$$b : 5 \quad 4 \quad 2 \quad 5$$

$$c : 1 \quad 10 \quad 2 \quad 10$$

Using the corresponding geometric grades, we have:

$$G(a) = \frac{1.00 + 0.70 + 0.60 + 0.00}{4} = 57.5\%,$$

$$G(b) = \frac{0.70 + 0.60 + 0.30 + 0.70}{4} = 57.5\%,$$

$$G(c) = \frac{0.00 + 1.00 + 0.30 + 1.00}{4} = 57.5\%.$$

We then have $G(a) = G(b) = G(c)$ and must use the appropriate tiebreaking mechanism. The SD for candidates a, b, and c are, respectively, 3.74, 1.41, and 4.92. The candidate with the lowest SD ranks ahead of the others. So, we finally have $b \succ_g a \succ_g c$.

Example 3. Large electorate.

For large electorates, data can be represented as percentages and the geometric score is then the balanced arithmetic mean of geometric grades. Let us consider the following profile in Balinski-Laraki's framework:

		10	9	8	7	6	5	4	3	2	1
a	:	12%	8%	5%	3%	3%	30%	15%	8%	10%	6%

We can compute the geometric score as follows:

$$G(a) = (1.00 \times 12 + 0.95 \times 8 + 0.90 \times 5 + 0.85 \times 3 + 0.78 \times 3$$

$$+ 0.70 \times 30 + 0.60 \times 15 + 0.48 \times 8 + 0.30 \times 10 + 0.00 \times 6) / 100 = 65.83\%.$$

5. Characterization of GVF

In this section, we study fair properties that only GVF and no other function can meet. Let us recall that GVF is equivalent to the decimal logarithm of the above-defined geometric mean. And thus, what Fleming and Wallace [10] stated for geometric mean in Theorem 2.2 above is also true for GVF. In addition, the following theorem is true:

Theorem 5.1 (Characterization of GVF)**.** *The GVF is the unique voting rule that meets simultaneously the reflexive property, the symmetric property, the multiplicative property and the additive property.*

Proof. Observe that for any $k > 0$

$$\begin{aligned} k &= f(k, k, \dots, k) \\ &= f(k, 1, \dots, 1) f(1, k, \dots, 1) \dots f(1, 1, \dots, k) \\ &= f(k, 1, \dots, 1)^n. \end{aligned}$$

The first equality follows from reflexive property, the second is arrived at by repeated applications of multiplicative property, and the last is symmetric property. Hence, $f(k, 1, \ldots, 1) = k^{\frac{1}{n}}$ for any $k > 0$. Then we note that symmetric and multiplicative properties, together with the above calculation, imply that

$$f(a_1, a_2, \ldots, a_n) = f(a_1, 1, \ldots, 1) f(1, a_2, \ldots, 1) \ldots f(1, 1, \ldots, a_n)$$

$$= \prod_{i=1}^{n} f(1, \ldots, a_i, \ldots, 1)$$

$$= \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}.$$

To finally establish our theorem, we use decimal logarithm

$$Log(f(a_1, a_2, \ldots, a_n)) = Log\left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}$$

$$= \frac{1}{n} \sum_{i=1}^{n} Log(a_i).$$

The last equality (which reflects the additive property) is the definition of the geometric score. □

6. Computational Issues

6.1. Robustness of GVF

In this subsection, we present the results of an *in silico* experiment. We have generated 10 000 000 random numbers on the 1-10 discrete scale and calculated their arithmetic and geometric means as well as the median and the standard deviation. The following stage was to select in a random way 5%, 10%, 25%, and 50% of these data which will be replaced by extreme values (1 if the number is lower than the average and 10 otherwise). The results are given in Table 3 below. The last column indicates a simulation where all the data were modified. It is about the case where all the voters are strategic.

Table 3. Simulation of strategic voting

Strategic voters	0%	5%	10%	25%	50%	100%
Median	6.00000	6.00000	6.00000	6.00000	6.0000	10.00000
GVF	0.67272	0.66920	0.68173	0.69550	0.71849	0.61065
GM	4.70678	4.66873	4.80542	4.96024	5.22989	4.07990
AM	5.49339	5.53935	5.65799	5.90982	6.33016	6.49585
SD	2.62916	2.75169	2.78644	2.99388	3.26821	4.38845

GVF: Geometric Voting Function GM: Geometric Mean AM: Arithmetic Mean SD: Standard Deviation

Table 4 indicates, for each proportion of strategic voters, the relative variation of the parameter compared to the initial data (column of 0% in Table 3). It follows that GVF is the parameter which is the most strategy-proof after the median. If one considers an electorate where all the voters are strategic, GVF is shown even more robust than the median.

Table 4. Parameters relative variation

Strategic voters	0%	5%	10%	25%	50%	100%
Median	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	66.6666%
GVF	0.0000%	− 0.5232%	1.3393%	3.3862%	6.8027%	− 9.2267%
GM	0.0000%	− 0.8084%	2.0957%	5.3850%	11.1139%	− 13.3186%
AM	0.0000%	0.8366%	2.9963%	7.5805%	15.2323%	18.2484%
SD	0.0000%	4.6604%	5.9821%	13.8721%	24.3062%	66.9145%

GVF: Geometric Voting Function GM: Geometric Mean AM: Arithmetic Mean SD: Standard Deviation

6.2. Complexity study

It is obvious that GVF can be computed in a polynomial time. For an n-data set, if data are not grouped, one needs $n - 1$ additions and a final division. If data are already grouped in 10 classes (1 to 10) the number of operations is constant (10 multiplications, 9 additions and 1 division). In general, we claim that time complexity for GVF is $\mathcal{O}(n)$.

7. Desirable Properties Filled by GVF

Let us consider a set of m candidates $\mathcal{C} = \{c_1, \ldots, c_i, \ldots, c_m\}$ and a finite set of n voters $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$. A problem is completely specified by a profile $\Phi = \Phi(\mathcal{C}, \mathcal{J})$: an $m \times n$ matrix of the grades $\Phi(I, j) \in \Lambda$ assigned by each of the voters $j \in \mathcal{J}$ to each of the candidates $c_i \in \mathcal{C}$.

A method of grading is a function F that assigns to any profile Φ one final grade in the same language for every candidate $F : \Lambda^{m \times n} \to \Lambda^m$. Designed to assign grades, it must satisfy certain basic properties.

Balinski and Laraki [2] notice that the mean value function (may it be arithmetic or geometric) is the universally used aggregation function in practice, though sometimes highest and lowest grades are dropped. This means that the output language is almost always richer than the input grades (inputs are usually restricted to discrete levels).

In conformity with most practical applications, the common language is parametrized as a subset of real numbers and whatever aggregation is used, small changes in the parametrization or the input grades should imply small changes in the final grades. Hence, with Balinski and Laraki [2], we agree that if the initial language is finite, all possible prametrizations must be considered. We will naturally take the output common language to be [1, 10] and impose the continuity condition to GVF. By doing so we can then consider GVF as a method of grading and establish the following theorem:

Theorem 7.1. *GVF is a social grading function* (*SGF*).

Proof. We know that the aggregation function f for GVF is the geometric mean. This means that for every $c_i \in \mathcal{C}$, we have $GVF(\Phi)(c_i) = f(\Phi(c_i))$, where f is the geometric mean. According to possibility theorem stated by Balinski and Laraki [2], the latter equality means that GVF satisfies neutrality, anonymity, unanimity, monotonicity and independence of irrelevant alternatives. As the considered aggregation function f (geometric mean) is a Bernstein function, we can easily check that GVF is continuous. A social grading function (SGF) is a method that satisfies the six previous conditions. □

GVF defines, and is defined by a unique continuous aggregation function f that is geometric mean. In the sequel, because a SGF and its aggregation function go hand in hand, properties are defined in terms of aggregation functions, theorems stated in terms of SGFs.

Theorem 7.2. *GVF meets universality, resoluteness, cloneresistance, homogeneity, reinforcement, and participation conditions.*

Proof. The resoluteness is guaranteed by the tie-breaking mechanism. The clone-resistance property is corollary to the independence of irrelevant alternatives. Since geometric mean is homogeneous, we can easily check that GVF is homogeneous. Universality, reinforcement and participation conditions are evident since the geometric mean (even simple addition and multiplication) fills them. □

As Balinski and Laraki [2] argued, enriching a language by embedding it into a real interval opens the door to many more methods of grading, but it will turn out that the aggregation functions that emerge as those that must be used are directly applicable in the seemingly more restrictive finite language as well.

8. Paradoxical Results with GVF

In this section, we present weaknesses of GVF in a controversial example. We prove that GVF fails the majority requirement, the Condorcet winner criterion and the Condorcet loser criterion.

In fact, consider three candidates a, b, c and five voters v_1, v_2, v_3, v_4, v_5 with preferences as indicated in Table 5.

Table 5. Voters' preferences and GVF results

		v_1	v_2	v_3	v_4	v_5	Geometric score	SD
a	:	5	8	2	5	5	66.0%	2.44949
b	:	10	1	2	10	10	66.0%	4.92443
c	:	1	10	5	8	5	66.0%	3.91578

The reader can easily check that candidate *b* is the Condorcet winner while candidate *a* is the Condorcet loser. However, GVF with its tie-breaking mechanism selects candidate *a* as the best one. It follows that the GVF fails the majority requirement, the Condorcet winner criterion and the Condorcet loser criterion. Notice also that MJ, BMC, and MMCM select *b* as the best candidate and returns *a* as the worst.

9. Concluding Remarks

The aggregation of individual preferences into a collective preference is performed by means of aggregation functions, which are generally mathematical formulas or social choice functions. These formulas make it possible to rank candidates to an election on the basis of voters' appreciation or evaluation. Determining a social choice function with desirable properties allows the transformation of individuals' choice into a choice representing the general interest.

Especially, in this article, we have presented a geometric mean-based function to circumvent electoral paradoxes often observed with common voting rules such as one and two-round majority systems. We have therefore brought to the literature a good voting system in the sense that it fills many democracy-desired properties even though, according to impossibility theorems, it still remains vulnerable to a restricted number of electoral paradoxes such as failure to majority requirement, the Condorcet winner and the Condorcet loser criteria.

To circumvent computing complexity, the herein presented voting system proposes to voters a logarithmic scale of evaluation. Final scores for candidates are then expressed in percent. The resoluteness is guaranteed in all cases since the function is supplied with a tie-breaking mechanism. In addition, its time complexity is linear. That is to say that it does not take much time, even at human level, to converge to the result.

In electoral field, it is known as stated by Baujard et al. [5] that the choice of a voting method shapes the democracy in which we live. In accordance with Igersheim et al. [13], it appears that the experimental results confirm that, for given preferences, changing the voting system is likely to change the outcome of the election. Studies highlight then the fact that aggregate results differ from one voting system to another. In this paper, we advocate for GVF a fair voting rule since it yields results such that at least two normative criteria for a "good" electoral system as stated by Norris [15] are filled: Government effectiveness and fairness to minor parties. Research into the handling of this function to linear and continuous scales and finding other paradoxical results can be an important contribution to the field.

Data Availability

Results on GVF robustness are computed from a spreadsheet. Interested reader can request the file by mailing to the corresponding author (Ruffin-Benoit M. Ngoie: benoitmpoy@hotmail.com).

Acknowledgements

The authors are grateful to Eric Kamwa and Berthold Ulungu for their helpful suggestions and comments.

References

[1] K. J. Arrow, Social Choice and Individual Values, Yale University Press, New-York, 1963.

[2] M. Balinski and R. Laraki, A theory of measuring electing and ranking, Proceedings of the National Academy of Sciences of the United States of America 104(21) (2007), 8720-8725.

[3] M. Balinski and R. Laraki, Majority Judgement: Measuring ranking and electing, The MIT Press Massachusett, 2010.

[4] A. Baujard and H. Igersheim, Experimentation du vote par note et du vote par approbation lors de l'élection présidentielle française du 22 avril 2007, Rapport Final Centre d'Analyse Stratégique, 2007.

[5] A. Baujard, F. Gavrel, H. Igersheim, J.-F. Laslier and I. Lebon, Vote par approbation, vote par note: Une expérimentation lors de l'élection présidentielle du 22 avril 2012, Revue Économique 64(2) (2013), 345-356.

[6] J.-C. Borda, Mémoire sur les élections au scrutin, Histoire de l'Académie Royale des Sciences, 1784.

[7] S. J. Brams and P. C. Fishburn, Approval Voting, 2nd Edition, Springer Science, New-York, 2007.

[8] J.-M. Condorcet, Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, Imprimerie Royale, 1785.

[9] D. S. Felsenthal, Electoral Systems, Paradoxes, Assumptions, and Procedures, Chapter 3: Review of Paradoxes Afflicting Procedures for Electing Single Candidate, Springer-Verlag, Berlin, Heidelberg, 2012.

[10] P. J. Fleming and J. J. Wallace, How not to lie with statistics: The correct way to summarize benchmark results, Communications of the ACM 29(3) (1986), 218-221.

[11] A. Gibbard, Manipulation of voting schemes: A general result, Econometrica 41 (1973), 587-601.

[12] C. Hillinger, Voting and the Cardinal Aggregation of Judgements, Discussion Paper 2004-25, University of Munich, 2004.

[13] H. Igersheim, A. Baujard and J.-F. Laslier, La question du vote, Expérimentation en laboratoire et In Situ, L'actualité Économique 92(1-2) (2016), 151-189.

[14] R.-B. M. Ngoie, Z. Savadogo and B. E.-L. Ulungu, New prospects in Social Choice Theory: Median and average as tools for measuring, electing and ranking, Advanced Studies in Contemporary Mathematics 25(1) (2015), 19-38.

[15] P. Norris, Choosing electoral systems: Proportional, majoritarian and mixed systems, International Political Science Review 18(3) (1997), 297-312.

[16] M. A. Satterthwaite, Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory 10 (1973), 187-217.

[17] R. L. Schilling, R. Song and Z. Vondracek, Bernstein Functions, de Gruyter Studies in Mathematics 37, De Gruyter, Berlin, Germany, 2010.

[18] A. K. Sen, The impossibility of a Paretian liberal, Journal of Political Economy 78 (1970), 152-157.

[19] H. Smaoui and D. Lepelley, Le système de vote par note à trois niveaux: Étude d'un nouveau mode de scrutin, Revue d'Écononomie Politique 123(6) (2013), 827-848.

[20] K. B. Stolarsky, Generalizations of the logarithmic mean, Math. Mag. 48 (1975), 87-92.

[21] A. Taylor, Social Choice and the Mathematics of Manipulation, Cambridge University Press, New York, 2005.

[22] M. A. Zahid and H. De Swart, The Borda majority count, Information Sciences 295 (2015), 429-440.