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A POSTULATE-FREE TREATMENT OF LORENTZ BOOSTS IN MINKOWSKI SPACE

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Abstract

Fundamental results of special relativity, such as the linear transformation for Lorentz boosts, and the invariance of the spacetime interval, are derived from a system of differential equations. The method so used dispenses with the need to make any physical assumption about the nature of spacetime.

1. Introduction

In his original work, A. Einstein [1] derived the Lorentz transformation based on the postulates of the principle of relativity and the invariance of the speed of light c. The transformation [2] in $(1+1)$-dimensional Minkowski space (in units where $c = 1$) is given by

$$\bar{t} = t \cosh \phi - x \sinh \phi, \tag{1a}$$

$$\bar{x} = x \cosh \phi - t \sinh \phi, \tag{1b}$$

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where ϕ is the rapidity. A central assumption in special relativity is the invariance of the spacetime interval under any transformation between inertial frames. This is instated (for the interval from the origin) by the relation

$$\bar{t}^2 - \bar{x}^2 = t^2 - x^2. \tag{2}$$

In $(3+1)$-dimensional Minkowski space, the transformation for a boost along the direction specified by the unit vector (n_1, n_2, n_3) is

$$\begin{bmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cosh\phi & -n_1\sinh\phi & -n_2\sinh\phi & -n_3\sinh\phi \\ -n_1\sinh\phi & 1+n_1^2(\cosh\phi-1) & n_1n_2(\cosh\phi-1) & n_1n_3(\cosh\phi-1) \\ -n_2\sinh\phi & n_1n_2(\cosh\phi-1) & 1+n_2^2(\cosh\phi-1) & n_2n_3(\cosh\phi-1) \\ -n_3\sinh\phi & n_1n_3(\cosh\phi-1) & n_2n_3(\cosh\phi-1) & 1+n_3^2(\cosh\phi-1) \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}, \tag{3}$$

and the preservation of the spacetime interval is expressed by stating that

$$\bar{t}^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2 = t^2 - x^2 - y^2 - z^2. \tag{4}$$

In more recent approaches [3, 4, 5], only Einstein's first postulate is used, and in addition, spacetime is assumed to be homogeneous and isotropic in nature. In this work, these assumptions are relaxed, and the four spacetime coordinates t, x, y and z are taken to be functions of the rapidity ϕ. In this process, the physical assumptions regarding spacetime are replaced by mathematical conditions. An event in an inertial frame (in $(3+1)$-dimensional Minkowski space) is then given by $(t(\phi), x(\phi), y(\phi), z(\phi))$. The choice of ϕ for the initial frame is arbitrary (in this work, it is chosen to be zero), since there exists no preferred frame.

Furthermore, since distinct values of ϕ result in boosted frames with distinct rapidities, ϕ may be said to "label" an inertial frame.

2. Boosts in $(1+1)$-dimensional Minkowski Space

2.1. The fundamental equations

Consider the system of equations (with primes denoting differentiation with respect to ϕ)

$$t' = -x, \tag{5a}$$

$$x' = -t. \tag{5b}$$

Successive differentiation yields

$$t'' = t, \tag{6a}$$

$$x'' = x. \tag{6b}$$

The solutions for t and x are

$$t(\phi) = t(0)\cosh\phi + t'(0)\sinh\phi, \tag{7a}$$

$$x(\phi) = x(0)\cosh(\phi) + x'(0)\sinh\phi. \tag{7b}$$

Noting that $t'(0) = -x(0)$ and $x'(0) = -t(0)$, we get

$$t(\phi) = t(0)\cosh\phi - x(0)\sinh\phi, \tag{8a}$$

$$x(\phi) = x(0)\cosh(\phi) - t(0)\sinh\phi. \tag{8b}$$

Equations (8) are strikingly similar to (1), and describe a boost of rapidity ϕ, relative to the frame $\phi = 0$.

We also obtain from (5)

$$tt' - xx' = 0, \tag{9}$$

which on integrating gives

$$t(\phi)^2 - x(\phi)^2 = \text{constant } (= s^2). \tag{10}$$

This proves the invariance of the (squared) Minkowski norm under boosts.

From (10), we have

$$t(\phi)^2 - x(\phi)^2 = t(0)^2 - x(0)^2. \tag{11}$$

Setting $x(\phi) = 0,$ we get

$$t(\phi)^2 = t(0)^2 - x(0)^2. \tag{12}$$

Defining $\beta(\phi) = \tanh \phi,$ we have from (8b)

$$x(0) = \beta(\phi) t(0). \tag{13}$$

$\beta(\phi)$ is thus identified as the velocity of the frame ϕ with respect to the frame $\phi = 0.$ Substituting (13) in (12) and taking the positive square root, we obtain

$$t(\phi) = t(0)\sqrt{1 - \beta(\phi)^2}. \tag{14}$$

Equation (14) describes the time dilation of the frame ϕ with respect to the frame $\phi = 0.$ The quantity $t(\phi)$ is identified as the proper time of the frame ϕ.

Now, on setting $x(0) = 0,$ we get from (8)

$$t(\phi) = t(0) \cosh \phi, \tag{15a}$$

$$x(\phi) = -t(0) \sinh \phi, \tag{15b}$$

which results in

$$x(\phi) = -\beta(\phi)t(\phi). \tag{16}$$

Here, $-\beta(\phi)$ is the velocity of the frame $\phi = 0$ with respect to the frame ϕ.

Now, let $t(\phi) = 0$. Returning to (11), we obtain

$$-x(\phi)^2 = t(0)^2 - x(0)^2, \tag{17}$$

where from (8a),

$$t(0) = \beta(\phi)x(0). \tag{18}$$

Substituting in (17) and taking the positive square root again, we have

$$x(\phi) = x(0)\sqrt{1 - \beta(\phi)^2}. \tag{19}$$

Equation (19) describes the spatial contraction of the frame ϕ with respect to the frame $\phi = 0$. The quantity $x(0)$ is identified as the proper length of the frame $\phi = 0$.

Now, let the event $(t(0), x(0))$ be simultaneous with the origin. In that case, $t(0) = 0$ and we get from (8)

$$t(\phi) = -x(0)\sinh\phi, \tag{20a}$$

$$x(\phi) = x(0)\cosh\phi, \tag{20b}$$

from which

$$t(\phi) = -\beta(\phi)x(\phi). \tag{21}$$

Equation (21) expresses the relativity of simultaneity of the two events.

2.2. Pair of events and the spacetime interval

Consider the events $(t_1(\phi), x_1(\phi))$ and $(t_2(\phi), x_2(\phi))$. Defining t_{12} =

$t_2 - t_1$ and $x_{12} = x_2 - x_1$, we get from (8) (by virtue of the linear nature of the transformation)

$$t_{12}(\phi) = t_{12}(0)\cosh\phi - x_{12}(0)\sinh\phi, \tag{22a}$$

$$x_{12}(\phi) = x_{12}(0)\cosh\phi - t_{12}(0)\sinh\phi, \tag{22b}$$

and hence from (10),

$$t_{12}(\phi)^2 - x_{12}(\phi)^2 = \text{constant } (= s_{12}^2). \tag{23}$$

Here, s_{12} is the (invariant) spacetime interval between the two events.

3. Boosts in $(3+1)$-dimensional Minkowski Space

Let n_1, n_2 and n_3 be real numbers such that $n_1^2 + n_2^2 + n_3^2 = 1$. Consider now the system of equations

$$t' = -n_1 x - n_2 y - n_3 z, \tag{24a}$$

$$x' = -n_1 t, \tag{24b}$$

$$y' = -n_2 t, \tag{24c}$$

$$z' = -n_3 t. \tag{24d}$$

Successive differentiation yields

$$t'' = t, \quad x''' = x', \quad y''' = y', \quad z''' = z'. \tag{25}$$

The solution for t is

$$t(\phi) = A_1 \cosh\phi + A_2 \sinh\phi, \tag{26}$$

where

$$A_1 = t(0), \tag{27a}$$

$$A_2 = t'(0) = -n_1 x(0) - n_2 y(0) - n_3 z(0). \tag{27b}$$

The solution for x is

$$x(\phi) = B_1 \cosh\phi + B_2 \sinh\phi + B_3, \tag{28}$$

where

$$B_1 + B_3 = x(0), \tag{29a}$$

$$B_2 = x'(0) = -n_1 t(0), \tag{29b}$$

$$B_1 = x''(0) = n_1^2 x(0) + n_1 n_2 y(0) + n_1 n_3 z(0). \tag{29c}$$

The solutions for y and z may be obtained in a way similar to that of x. The solution to (24) is

$$\begin{bmatrix} t(\phi) \\ x(\phi) \\ y(\phi) \\ z(\phi) \end{bmatrix} = \begin{bmatrix} \cosh\phi & -n_1 \sinh\phi & -n_2 \sinh\phi & -n_3 \sinh\phi \\ -n_1 \sinh\phi & 1 + n_1^2(\cosh\phi - 1) & n_1 n_2(\cosh\phi - 1) & n_1 n_3(\cosh\phi - 1) \\ -n_2 \sinh\phi & n_1 n_2(\cosh\phi - 1) & 1 + n_2^2(\cosh\phi - 1) & n_2 n_3(\cosh\phi - 1) \\ -n_3 \sinh\phi & n_1 n_3(\cosh\phi - 1) & n_2 n_3(\cosh\phi - 1) & 1 + n_3^2(\cosh\phi - 1) \end{bmatrix} \begin{bmatrix} t(0) \\ x(0) \\ y(0) \\ z(0) \end{bmatrix}. \tag{30}$$

Equation (30) describes a boost of rapidity ϕ in the direction specified by the unit vector (n_1, n_2, n_3) (note the similarity with (3)). It may be checked that (24) reduces to (5) for $(n_1, n_2, n_3) = (1, 0, 0)$. Eliminating n_1, n_2 and n_3 from (24) results in

$$tt' - xx' - yy' - zz' = 0, \tag{31}$$

which on integrating gives

$$t(\phi)^2 - x(\phi)^2 - y(\phi)^2 - z(\phi)^2 = \text{constant}. \tag{32}$$

This proves the invariance of the (squared) Minkowski norm under boosts.

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