A NOTE ON WORMHOLES AS COMPACT STELLAR OBJECTS

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Abstract

A Morris-Thorne wormhole can only be sustained by violating the null energy condition, thereby requiring the existence of "exotic matter", a requirement that many researchers consider unphysical. Using a mostly qualitative approach, it is proposed in this note that the basic problems can be overcome by purely relativistic considerations. The implication is that a Morris-Thorne wormhole is likely to be a compact stellar object.

1. Introduction

It is well known that a Morris-Thorne wormhole can only be held open by violating the null energy condition (NEC), calling for the existence of exotic matter at or near the throat. While the need for exotic mater is rather problematical, it is not a conceptual problem, as we know

from the Casimir effect [1]. In other words, exotic matter can be made in the laboratory. An open question is whether enough could be produced locally to sustain a macroscopic traversable wormhole. Many researchers consider any wormhole solution unphysical if exotic matter cannot be avoided, as would be the case, for example, in some modified gravitational theories. The purpose of this note is to reexamine some of these requirements. We are primarily interested in qualitative results.

2. Wormhole Structure

We start this section with the following line element, using units in which $c = G = 1$ [2]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r \leq R$$

$$= -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r > R. \quad (1)$$

Here $m(r)$ is the effective mass inside radius r and M is the mass of a star of radius R as seen by a distant observer. If $\rho(r)$ is the energy density, then the total mass-energy inside radius r is given by

$$m(r) = \int_0^r 4\pi r^2 \rho(r)dr, \quad m(0) = 0. \quad (2)$$

The line element of a Morris-Thorne wormhole is [1]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

using units in which $c = G = 1$. Here $\Phi = \Phi(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon.

The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [1]. The spherical surface $r = r_0$ is called the *throat* of the wormhole, where $b(r_0) = r_0$. The shape function must also meet the requirement $b'(r_0) < 1$, called the *flare-out condition*, while $b(r) < r$ for $r > r_0$. We also require that $b'(r_0) > 0$. A final requirement is asymptotic flatness: $\lim_{r \to \infty} \Phi(r) = 0$ and $\lim_{r \to \infty} b(r)/r = 0$.

The flare-out condition can only be met by violating the null energy condition (NEC) which states that

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0 \tag{4}$$

for all null vectors k^α, where $T_{\alpha\beta}$ is the energy-momentum tensor. Matter that violates the NEC is called "exotic" in Ref. [1]. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation has the form

$$T_{\alpha\beta} k^\alpha k^\beta = \rho + p_r < 0. \tag{5}$$

Here $T^t{}_t = -\rho$ is the energy density, $T^r{}_r = p_r$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t$ is the lateral (transverse) pressure. Before continuing, let us list the Einstein field equations:

$$\rho(r) = \frac{b'}{8\pi r^2}, \tag{6}$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{7}$$

and

$$p_t(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r - b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)}\right]. \tag{8}$$

To see the connection to the flare-out condition at the throat, observe that from Eqs. (5), (6), and (7), we have

$$8\pi[\rho(r_0) + p_r(r_0)] = \frac{r_0 b'(r_0) - b(r_0)}{r_0^3}. \tag{9}$$

Given that the radial tension $\tau(r)$ is the negative of $p_r(r)$, Eq. (5) can be written

$$\tau - \rho c^2 > 0, \tag{10}$$

temporarily reintroducing c. The last inequality has given rise to the designation "exotic matter" since $\tau > \rho c^2$ implies that there is an enormous radial tension at the throat. This problem has been discussed extensively in Refs. [3, 4, 5, 6, 7].

3. The Flare-out Condition

It follows from Eq. (6) that

$$b(r) = r_0 + \int_{r_0}^{r} 8\pi r^2 \rho(r) dr, \tag{11}$$

confirming that $b(r_0) = r_0$, now viewed as an initial condition. The physical significance of this condition will be examined more closely in this note. Observe also that

$$b(r) = 2m(r). \tag{12}$$

As noted in the Introduction, we are primarily interested in qualitative results. To that end, we first recall that $\rho(r)$ is likely to be very small in geometrized units. So it follows that

$$b'(r_0) = 8\pi r_0^2 \rho(r_0) < 1; \tag{13}$$

so the flare-out condition has been met. To show that the assumption

regarding ρ is realistic, suppose we try $\rho(r_0) = 10^{-2} \mathrm{m}^{-2}$. Then

$$\rho(r_0) = 10^{-2} \frac{c^2}{G} = 1.35 \times 10^{25} \frac{\mathrm{kg}}{\mathrm{m}^3} .$$

Given that nuclear matter has a density of 10^{18} kg/ m^3, our choice of ρ could be even smaller than $10^{-2} \mathrm{m}^{-2}$.

4. The Condition $b(r_0) = r_0$

We know from Eq. (12) that $\frac{1}{2} b(r)$ is the effective mass inside radius r. Since $r = r_0$ is the throat of the wormhole, it follows from the definition of throat that the interior $r < r_0$ is outside the wormhole spacetime. So $\frac{1}{2} b(r_0) = m(r_0)$ must be the mass of the interior $r < r_0$. This can be compared to a thin-shell wormhole from a Schwarzschild black hole [8]. The radius of the throat necessarily exceeds the radius of the event horizon. So while the black hole is not part of the wormhole spacetime, it helps produce the necessary gravitational field. Similarly, according to Ref. [3], the existence of a massive core of quark matter at the center of a neutron star could give rise to a wormhole. (See also Ref. [9].)

In general, then, the mass of the interior is $\frac{1}{2} b(r_0) = \frac{1}{2} r_0$. At first glance this appears to be impossible: for example, in geometrized units, the mass of the Earth is $0.44 \, \mathrm{cm}$ and the mass of the sun is $1.5 \, \mathrm{km}$; both are very much less than the corresponding radii. Although seemingly absurd, we can accept the condition $b(r_0) = r_0$ by appealing to exotic matter, not because this solves the problem but because we can now claim that a sufficiently far advanced civilization may be able to handle the resulting technical difficulties. An obvious alternative is to take into

account certain relativistic effects in order to satisfy the condition
$m(r_0) = \dfrac{1}{2} r_0$.

Since we are primarily interested in qualitative results, let us assume that ρ is constant, as in the original Schwarzschild interior solution. It follows from line element (1) that the element of volume is

$$dV = 4\pi r^2 \frac{1}{\sqrt{1 - \dfrac{2m(r)}{r}}} \, dr \tag{14}$$

and the rest mass is

$$m_0 = \int_0^r 4\pi r^2 \frac{1}{\sqrt{1 - \dfrac{2m(r)}{r}}} \rho \, dr. \tag{15}$$

Since $2m(r)/r = \dfrac{8}{3}\pi r^2 \rho$, we get for the total rest mass inside radius r

$$m_0 = \int_0^r 4\pi r^2 \frac{1}{\sqrt{1 - \dfrac{8}{3}\pi r^2 \rho}} \, dr = 4\pi \left[\frac{\sin^{-1}\left(r\sqrt{\dfrac{8}{3}\pi\rho}\right)}{2\left(\dfrac{8}{3}\pi\rho\right)^{3/2}} - \frac{r\sqrt{1 - \dfrac{8}{3}\pi r^2 \rho}}{2\left(\dfrac{8}{3}\pi\rho\right)} \right]. \tag{16}$$

Next, we observe that

$$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{\dfrac{1}{\sqrt{1 - x^2}}}{1} = 1$$

by L'Hospital's rule. So for small values of x, $\sin^{-1} x \approx x$ and for ρ sufficiently small, it then follows that

$$m_0 \approx 4\pi \left[\frac{r\sqrt{\dfrac{8}{3}\pi\rho}}{2\left(\dfrac{8}{3}\pi\rho\right)^{3/2}} - \frac{r\sqrt{1 - \dfrac{8}{3}\pi r^2 \rho}}{2\left(\dfrac{8}{3}\pi\rho\right)} \right]$$

$$= 4\pi\left[\frac{r}{\frac{16}{3}\pi\rho} - \frac{r\sqrt{1-\frac{8}{3}\pi r^2 \rho}}{\frac{16}{3}\pi\rho}\right]$$

$$= 4\pi \frac{r\left(1-\sqrt{1-\frac{8}{3}\pi r^2 \rho}\right)}{\frac{16}{3}\pi\rho}. \tag{17}$$

We conclude that for ρ sufficiently small,

$$m_0 \approx 4\pi\frac{r}{\frac{16}{3}\pi\rho} = \frac{r}{\frac{4}{3}\rho}. \tag{18}$$

We now have

$$\frac{r}{\frac{4}{3}\rho} \gg \frac{4}{3}\pi r^3 \rho \quad \text{for} \quad \rho \ll 1. \tag{19}$$

So purely qualitatively, the total rest mass m_0 of the region inside $r = r_0$ can be very much larger than $\frac{4}{3}\pi r^3 \rho$. The relativistic mass m_0 could therefore be large enough to meet the condition $m_0 = \frac{1}{2}r_0$ and hence $b(r_0) = r_0$. Condition (9), which is usually associated with exotic matter, also follows from relativistic considerations. The conclusion depends entirely on having a sufficiently large mass inside $r < r_0$ to yield desired relativistic effects.

5. Summary

Our purely qualitative approach has led to the conclusion that the wormhole structure would have to be extremely massive to begin with. With the neutron-star example in mind, it seems quite likely that a traversable Morris-Thorne wormhole would be a compact stellar object.

This conclusion is consistent with the proposed search for wormholes by means of gravitational lensing. This technique lends itself to a direct detection of photon spheres, but these would only exist for compact stellar objects.

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