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A NOTE ON THE HARMONIC OSCILLATOR PATH INTEGRALS

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Abstract

The harmonic oscillator as treated by the path integral formalism in quantum mechanics originally due to Feynman is critically examined. This is a sequel to earlier published works by the author on the method applied to free particles.

1. Introduction

Recently, we have demonstrated [1] that the path integral formulation of the probability amplitude of transition as calculated by Feynman [2] by dividing the time interval into infinitesimally small pieces in which a particle travels from an eigenstate $|x_0\rangle$ of position to another state $|x\rangle$ is mathematically incorrect. We have also given a better formula for the same quantity [3] for the free particle. In this present note, we deal with the harmonic oscillator problem whose

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solution using the method of Feynman also appears in textbooks [4] along with that of the free particle. It will be demonstrated that the transition probability amplitude expression of the harmonic oscillator which is supposed to yield the same eigenstates and eigenvalues of the energy as those obtained by other quantum mechanical methods is mathematically incorrect. A correct formulation of the problem as it was done for the free particle in reference [3] is also attempted and the values of the transition probability amplitude are numerically calculated using this formulation.

2. The Existing Transition Probability Amplitude Formula in Textbooks for the Harmonic Oscillator

Using Feynman's method the transition probability amplitude of a particle executing harmonic oscillations initially in an eigenstate of position $|x_0\rangle$ at time t_0 and finally in an eigenstate $|x\rangle$ at time t is (see Eqs. (12.61) and (12.7) of reference [4])

$$\langle x|e^{-iH(t-t_0)}|x_0\rangle = \left[\frac{m\omega}{2\pi i \sin\omega(t-t_0)}\right]^{\frac{1}{2}}$$

$$\times \exp\left[\frac{im\omega}{2\sin\omega(t-t_0)}\left\{\left(x^2+x_0^2\right)\cos\omega(t-t_0)-2xx_0\right\}\right]. \quad (1)$$

Putting $T = t - t_0$ and through a series expansion, it is shown that the leading term of this expression for $iT \to \infty$ limit is (see Eq. (12.63) of reference [4])

$$\langle x|e^{-iHT}|x_0\rangle = e^{-\frac{i\omega T}{2}}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}} \exp\left[-\frac{m\omega}{2}\left(x^2+x_0^2\right)\right]. \quad (2)$$

The first objection to this limiting procedure is that $T = t - t_0$ is a real quantity that is all motion in physics be it classical or quantum is with respect to time which is nothing but a real parameter. Whenever we state

$iT \to \infty$, we introduce an imaginary component into time which is physically unacceptable. In fact the time, actually, is made complex since $\underset{iT\to\infty}{\text{Lt}}\left[\frac{m\omega}{2\pi i \sin \omega T}\right]^{\frac{1}{2}}$ will yield $e^{-\frac{i\omega T}{2}}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}}$ only if T is complex having a negative imaginary part. If we keep t and t_0 real as is true for any physical problem, $\underset{iT\to\infty}{\text{Lt}}\left[\frac{m\omega}{2\pi i \sin \omega T}\right]^{\frac{1}{2}}$ will be oscillatory in nature yielding no limit at all. The second objection to expressions of the type of Eq. (1) and this, in my opinion, is a much more serious problem is that the transition probability amplitude starting from $t = t_0$ periodically becomes infinite for all positions x simultaneously. Given this fact, one can ask a very relevant question as to what is the meaning of putting the system into an eigenstate $|x_0\rangle$ of position at $t = t_0$ if it exists with equal and infinite probability density at all $|x\rangle$ specifically at this very point of time. The transition probability amplitude being infinite cannot also be normalized although it is an essential requirement of quantum mechanics (see ref. [3]). Thus this again is a wrong result and as pointed out in reference [1] is a consequence of the incorrect use of the value of the integral $\int_{-\infty}^{\infty} e^{i\eta^2} d\eta = \sqrt{\pi i}$.

3. The Correct Path Integral Expression for the Quantum Harmonic Oscillator

Path integrals can be expressed in terms of the eigenstates of energy (see Eq. (8) of reference [1]) and this fact has been successfully applied by the author to the free particle case. We have

$$\langle x | e^{-iH(t-t_0)} | x_0 \rangle = \sum_n e^{-iE_n(t-t_0)} \phi_n^*(x_0)\phi_n(x) \tag{6}$$

into which the eigenstates (see reference [5], p. 134 with the system of units $\hbar = 1$)

$$\phi_n(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega x^2}{2}} H_n\left(\sqrt{m\omega}x\right) \tag{7}$$

and the eigenvalues of energy

$$E_n = \left(n + \frac{1}{2}\right)\omega \tag{8}$$

can be substituted to evaluate $\langle x|e^{-iH(t-t_0)}|x_0\rangle$ numerically. Here $H_n\left(\sqrt{m\omega}x\right)$ are Hermite polynomials of degree n. Instead of trying out a direct numerical summation of the series in Eq. (6) which will anyway be difficult especially because the factorial function for large n necessitates special methods for its evaluation, we just see some of the properties of $\langle x|e^{-iH(t-t_0)}|x_0\rangle$ as given by Eqs. (6) to (8) and also see how it differs from Eq. (1). The first point to note is the fact that Eq. (7) shows that $\phi_n(x)$ has a $e^{-\frac{m\omega x^2}{2}}$ factor which means that $\langle x|e^{-iH(t-t_0)}|x_0\rangle$ will fall off to zero exponentially at large x according to Eq. (6). On the other hand in Eq. (1) the dependence on the coordinate x is of an oscillatory nature since the exponential has an imaginary argument and cannot fall off to zero at large x. We conclude that Eq. (6) in conjunction with Eqs. (7) and (8) is physically more appealing than Eq. (1). Secondly, we give a somewhat crude calculation based on Eqs. (6), (7) and (8), where we truncate the series arbitrarily at $n = 149$. We take $m = 1$, $\omega = 1$, $x_0 = \sqrt{11}$ and $x = x_0/2$ and then calculate for a time interval $t - t_0 = 0$ to 4π by dividing that interval into 200 parts. After this, we plot $\left|\langle x|e^{-iH(t-t_0)}|x_0\rangle\right|^2 = |\psi(x, t)|^2$ (see ref. [3]) for each value of $t - t_0$ (from

0 to 4π) that is as a function of $t - t_0$. Clearly there is an indication of movement consistent with the motion of a classical harmonic oscillator. At $t = t_0$, the classical particle is at $x = x_0$ and the corresponding quantum mechanical one is in an eigenstate of position $|x_0\rangle$. So at $x = x_0/2$, $\left|\langle x|e^{-iH(t-t_0)}|x_0\rangle\right|^2$ will be expected to be zero as the plot (see Figure 1 below) confirms (at $t = t_0$). As time passes the classical particle passes through $x = x_0/2$ and moves to $x = -x_0$ at $t - t_0 = \pi$. Our quantum mechanical oscillator when observed at $x = x_0/2$ shows a non zero value of $\left|\langle x|e^{-iH(t-t_0)}|x_0\rangle\right|^2$ at these times before it reduces to zero at $t - t_0 = \pi$. One should not fail to observe the sharp increase in the value of probability density to a peak immediately after $t = t_0$ which in our opinion is not very inconsistent with classical motion. The classical oscillator then turns back towards $x = x_0$ while passing through $x = x_0/2$ at some time before $t - t_0 = 2\pi$. Our quantum mechanical oscillator shows a symmetric $\left|\langle x|e^{-iH(t-t_0)}|x_0\rangle\right|^2$ about $t - t_0 = \pi$ consistent with the behavior of its classical counterpart. The periodic nature of the plot is demonstrated by the second cycle.

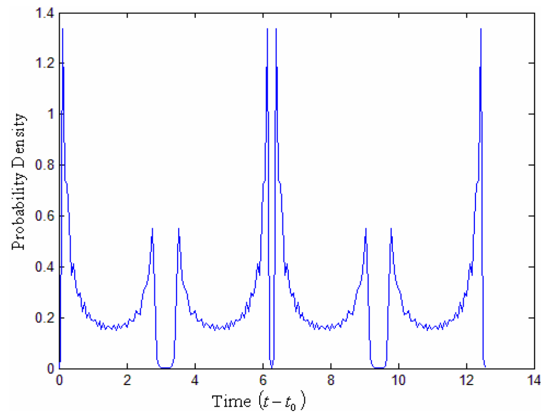

Figure 1. The probability density $|\psi(x, t)|^2$ of quantum mechanical oscillator as a function of time at half the classical amplitude $x = x_0 /2$. The amplitude x_0 is taken such that it is nearly an eigenvalue of the position operator acting on the wave-function.

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