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A NOTE ON THE STABILITY OF MORRIS-THORNE WORMHOLES

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Abstract

Two critical issues in the study of Morris-Thorne wormholes concern the stability of such structures, as well as their compatibility with quantum field theory. This note discusses an important subset characterized by zero tidal forces. It is shown that such wormholes (1) are in stable equilibrium using a criterion based on the Tolman-Oppenheimer-Volkoff (TOV) equation and (2) are not in direct conflict with quantum field theory.

1. Introduction

The Einstein field equations can be derived by means of the Hilbert-Einstein action

$$S_{\mathrm{HE}} = \frac{1}{2\kappa}\int\sqrt{-g}\,R\,d^4x, \tag{1}$$

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where R is the Ricci curvature scalar and $\kappa = 8\pi G$. (For notational convenience we will let $\kappa = 1$). A modification of Einstein's theory called $f(R)$ modified gravity replaces R by a nonlinear function $f(R)$ in Equation (1) to yield

$$S_{f(R)} = \frac{1}{2\kappa}\int \sqrt{-g}f(R)d^4x. \tag{2}$$

(For a review, see References [1, 2, 3].) Wormhole geometries in $f(R)$ modified gravitational theories are discussed in Ref. [4].

Here we need to recall that Morris and Thorne [5] proposed the following static and spherically symmetric line element for a wormhole space time:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

(We are using units in which $c = 1$, in addition to $\kappa = 1$). The function $b = b(r)$ is called the *shape function* and $\Phi = \Phi(r)$ is called the *redshift function*. For the shape function, we must have $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole. Other requirements are $b'(r_0) > 0$ and $b'(r_0) < 1$, called the *flare-out condition*, while $b(r) < r$ near the throat. In classical general relativity, the flare-out condition can only be satisfied by violating the null energy condition (NEC):

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{4}$$

for all null vectors k^μ, where $T_{\mu\nu}$ is the stress-energy tensor. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation becomes

$$T_{\mu\nu}k^\mu k^\nu = \rho + p_r < 0. \tag{5}$$

Here $T^t{}_t = -\rho$ is the energy density, $T^r{}_r = p_r$ is the radial pressure, and $T^{\theta}{}_{\theta} = T^{\phi}{}_{\phi} = p_t$ is the lateral pressure. In $f(R)$ modified gravity, by contrast, meeting the flare-out condition does not automatically result in a violation of the NEC; so condition (5) has to be checked separately.

Regarding the redshift function, we normally require that $\Phi(r)$ remain finite to prevent the occurrence of an event horizon. In this paper, we need to assume that $\Phi(r) \equiv \text{constant},$ so that $\Phi'(r) \equiv 0.$ Otherwise, according to Lobo and Oliveira [4], the analysis becomes intractable. Fortunately, the condition $\Phi'(r) \equiv 0$ is a highly desirable feature for a traversable wormhole since it implies that the tidal forces are zero.

We can see from Equations (1) and (2) that for the proper choice of $f(R),$ the modified gravity theory can be arbitrarily close to Einstein gravity. For this choice, any Morris-Thorne wormhole with zero tidal forces is found to be stable by satisfying a well-known equilibrium condition based on the Tolman-Oppenheimer-Volkov equation.

2. The Solution

As a first step, let us state the gravitational field equations in the form given in Ref. [4]:

$$\rho(r) = F(r)\frac{b'(r)}{r^2}, \tag{6}$$

$$p_r(r) = -F(r)\frac{b(r)}{r^3} + F'(r)\frac{rb'(r) - b(r)}{2r^2} - F''(r)\left(1 - \frac{b(r)}{r}\right), \tag{7}$$

and

$$p_t(r) = -\frac{F'(r)}{r}\left(1 - \frac{b(r)}{r}\right) + \frac{F(r)}{2r^3}[b(r) - rb'(r)], \tag{8}$$

where $F = \dfrac{df}{dR}$. The Ricci curvature scalar is given by

$$R(r) = \frac{2b'(r)}{r^2}. \tag{9}$$

In this paper, we are going to choose

$$f(R) = aR^{1\pm\varepsilon}, \quad \varepsilon \ll 1, \tag{10}$$

where a is a constant. Given that ε can be arbitrarily close to zero, the resulting $f(R)$ modified gravity can be arbitrarily close to Einstein gravity. Since $F = \frac{df}{dR}$, we get from Equation (9)

$$F = a(1 \pm \varepsilon)R^{\pm\varepsilon} = a(1 \pm \varepsilon)\left(\frac{2b'(r)}{r^2}\right)^{\pm\varepsilon}. \tag{11}$$

So from Equation (6),

$$b' = \frac{r^2\rho}{F} = \frac{r^2\rho}{a(1 \pm \varepsilon)(2/r^2)^{\pm\varepsilon}[(b'(r)]^{\pm\varepsilon}}.$$

Solving for b', we obtain

$$b'(r) = \left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{1}{1\pm\varepsilon}} r^2. \tag{12}$$

If $a = 1$ and $\varepsilon = 0$, we recover Equation (6). It also follows from Equation (11) that

$$F = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]. \tag{13}$$

Now observe that F' has the form

$$\frac{\pm\varepsilon}{1 \pm \varepsilon}\rho^{\frac{\pm\varepsilon}{1\pm\varepsilon}-1} \times \text{a constant.}$$

Since ε can be arbitrarily close to zero, we conclude that

$$F' \approx 0 \quad \text{and} \quad F'' \approx 0. \tag{14}$$

Our next task is to show that the NEC is violated at the throat, i.e., $\rho(r_0) + p_r(r_0) < 0$. (As already noted, meeting the flare-out condition is not enough in $f(R)$ modified gravity). To that end, we first obtain

$$\rho(r) = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\frac{b'(r)}{r^2}, \tag{15}$$

$$p_r(r) = -a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\frac{b(r)}{r^3}, \tag{16}$$

and

$$p_t(r) = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\frac{b(r) - rb'(r)}{2r^3}. \tag{17}$$

We now get from $b(r_0) = r_0$,

$$\rho(r_0) + p_r(r_0) = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\left(\frac{b'(r_0)}{r_0^2} - \frac{1}{r_0^2}\right) < 0 \tag{18}$$

since $b'(r_0) < 1$. The null energy condition is thereby violated.

3. Stability Analysis

In this section, we examine the stability of a zero-tidal force wormhole by employing an equilibrium condition obtained from the Tolman-Oppenheimer-Volkov (TOV) Equation [6, 7]

$$\frac{dp_r}{dr} + \Phi'(\rho + p_r) + \frac{2}{r}(p_r - p_t) = 0. \tag{19}$$

The equilibrium state of a structure is determined from the three terms in this equation, defined as follows: the gravitational force

$$F_g = -\Phi'(\rho + p_r), \tag{20}$$

the hydrostatic force

$$F_h = -\frac{dp_r}{dr}, \tag{21}$$

and the anisotropic force

$$F_a = \frac{2(p_t - p_r)}{r} \tag{22}$$

due to the anisotropic pressure in a Morris-Thorne wormhole. Equation (19) then yields the following equilibrium condition:

$$F_g + F_h + F_a = 0. \tag{23}$$

Since $\Phi' \equiv 0,$ the equilibrium condition becomes

$$F_h + F_a = 0. \tag{24}$$

So from Equations (21) and (16), we obtain

$$F_h = -\frac{dp_r}{dr} = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\frac{r^3 b'(r) - 3r^2 b(r)}{r^6}, \tag{25}$$

while Equations (16) and (17) yield

$$F_a = \frac{2}{r}(p_t - p_r)$$

$$= \frac{2}{r}a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]\left(\frac{b(r) - rb'(r)}{2r^3} + \frac{b(r)}{r^3}\right). \tag{26}$$

The result is

$$F_h + F_a = a(1 \pm \varepsilon)\left[2^{\pm\varepsilon}\left(\frac{\rho}{2^{\pm\varepsilon}a(1 \pm \varepsilon)}\right)^{\frac{\pm\varepsilon}{1\pm\varepsilon}}\right]$$

$$\times\left(\frac{rb'(r) - 3b(r)}{r^4} + \frac{b(r) - rb'(r)}{r^4} + \frac{2b(r)}{r^4}\right) = 0. \qquad (27)$$

The equilibrium condition is satisfied, thereby yielding a stable wormhole. Since our modified theory, based on Equation (10), can be arbitrarily close to Einstein's theory, the conclusion carries over to Morris-Thorne wormholes.

As a final comment, recall that the condition $\Phi(r) \equiv \text{constant}$ was introduced for purely technical reasons: without this condition, the analysis becomes intractable. So while the zero-tidal force assumption proved to be a sufficient condition for stability, it is not a necessary condition.

4. Compatibility with Quantum field Theory

That quantum field theory may place severe constraints on Morris-Thorne wormholes was first shown by Ford and Roman [8]. The sought-after compatibility depends on a quantum inequality in an inertial Minkowski space time without boundary. If u^μ is the observer's four-velocity and $\langle T_{\mu\nu}u^\mu u^\nu \rangle$ is the expected value of the local energy density in the observer's frame of reference, then

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^4}, \qquad (28)$$

where τ is the observer's proper time and τ_0 the duration of the sampling time. (See Ref. [8] for details). The inequality can be applied in a curved space time as long as τ_0 is small compared to the local proper radius of curvature. The desired estimates of the local curvature are obtained from the components of the Riemann curvature tensor in classical general relativity, discussed further in Ref. [9].

If we now let $F \equiv 1$ in Equations (6)-(8), we obtain the Einstein field equations with $\Phi' \equiv 0.$ As a result, the equilibrium condition in Section 3 is automatically satisfied. Unfortunately, according to Ref. [9], whenever $\Phi' \equiv 0,$ the quantum inequality is no longer satisfied and the wormhole cannot exist on a macroscopic scale.

Now the appeal to $f(R)$ modified gravity becomes clear: as already noted, the estimates of the local curvature needed to apply Equation (28) come from Einstein's theory, not from the modified theory. So the previous objections do not apply. More precisely, in the equation $f(R) = aR^{1\pm\varepsilon},$ ε is always positive. So even if the modified theory is arbitrarily close to Einstein's theory, it remains an $f(R)$ theory, thereby avoiding a direct conflict with quantum field theory.

5. Conclusion

While wormholes are a valid prediction of Einstein's theory, their possible existence faces additional challenges such as the question of the stability of such structures, as well as their compatibility with quantum field theory. This note addresses these issues indirectly by invoking $f(R)$ modified gravity. More precisely, the function $f(R)$ in Equation (10) leads to a modification that can be arbitrarily close to Einstein gravity. It is subsequently shown that a Morris-Thorne wormhole with zero tidal forces is stable by satisfying an equilibrium condition based on the Tolman-Oppenheimer-Volkov equation. Since the modified theory can be arbitrarily close to Einstein's theory, the conclusion carries over to Morris-Thorne wormholes. In classical general relativity, however, the zero-tidal force assumption is incompatible with quantum field theory. The $f(R)$ modified theory based on Equation (10) avoids this problem since the constraints stemming from Equation (28), being purely classical, do not apply.

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