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A NONCONVEX QUADRILATERAL AND SEMI GERGONNE POINTS ON IT: SOME RESULTS AND ANALYSIS

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Abstract

Suppose *ABCD* is a nonconvex quadrilateral. We can construct 3 semi-circles outer tangent and 2 in the semi circle tangent of the quadrilateral. In this paper, we will discuss how to calculate length of the radius of the circle tangent of a nonconvex quadrilateral. Furthermore, we will discuss how to calculate lengths of the new sides formed by construction of the semi Gergonne points on it.

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1. Introduction

Two-dimensional figure of triangle is one of the scopes of geometry that can always be used to construct tangent circles inside and outside tangent circles [2, 8, 9, 13, 14]. If a line is drawn from the third vertex of the triangle to the point of tangency of the circle in (*incircle*) then the third line will intersect at one point (*concurrent*), this is called Gergonne point inside the triangle and if there is a circle outside the triangle tangent (*excircle*) with the center point of tangency circle outside the triangle (*excenter*), Gergonne point can also be formed outside the triangle derived from the circle tangent [1, 9, 13].

But this does not apply to the quadrilaterals because not all quadrilaterals can be used to construct a tangent circle inside and outer circle tangent. Circle tangent in the quadrilateral [4, 6, 12] is a circle inside a convex quadrilateral and the quadrilateral four sides of the offensive, while the outer circle tangent quadrilateral [5, 10, 11] is a circle that is offensive or lengthening the other sides of a quadrilateral.

Mashadi et al. [10], have discussed about how to calculate the length of the radius of circle tangent to the convex quadrilateral, lengths of new sides formed from the construction and how to construct semi Gergonne point on a convex quadrilateral. Josefsson [7] also talked about the relationship between the lengths of the fingers outside the circle tangent to the convex quadrilateral. In the other article, Mashadi et al. [11] also discussed the necessary and sufficient conditions in order that a quadrilateral has a tangent circle outside, calculated the lengths of the new sides and the lengths of the fingers and the relationship between the lengths of the fingers formed.

In addition to a convex quadrilateral, there is a quadrilateral having a single angle of more than $180°$. This is called nonconvex quadrilateral [3]. In a nonconvex quadrilateral, we can construct 3 tangent circles outer edges and 2 in the semi circle tangent. In this paper, authors discuss the lengths of the fingers in a semi circle tangent and semi circle tangent outside as well as the relationship between the lengths of the radii of the semi circle tangent. Lengths of the new sides formed from constructing the semi-circle tangent will also be determined.

To be more clear about nonconvex quadrilateral which has a semi incircle and semi excircle tangent to tangent outside, consider Figure 1.1.

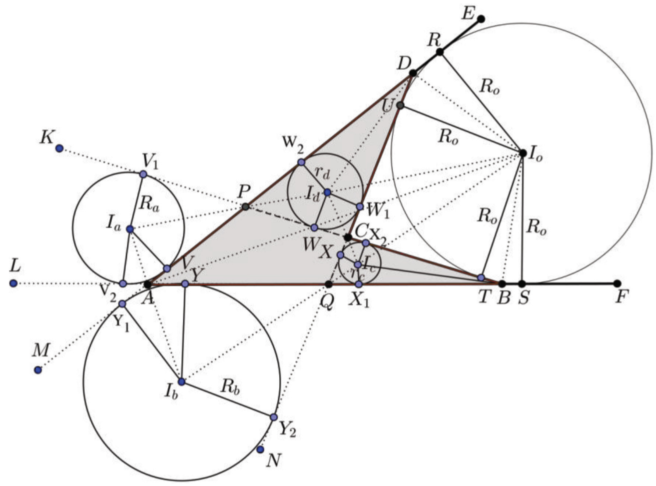

Figure 1.1.

In Figure 1.1, there are some tangent circles centered at the points I_o, I_a, I_b, I_c and I_d. Circles centered at the points I_o, I_a and I_b are called semi-circle outside tangent of a nonconvex quadrilateral $ABCD$, while circles centered at points I_c and I_d are called semi incircle tangent of a nonconvex quadrilateral $ABCD$. Besides, there are some tangency circle radii outside are R_o, R_a, and R_b and the radii of the incircle tangents are r_c and r_d. There is also a relationship between the lengths of the fingers formed, that is, $R_a \cdot r_d = R_b \cdot r_c$.

2. Semi Gergonne Point and Long's Fingers

Gergonne points can be constructed in any of the triangles, that is, 1 Gergonne point in the circle inside and 3 Gergonne points in the circle outside [1, 9, 13], but in the nonconvex quadrilateral, it can not be constructed. In nonconvex quadrilateral only can be constructed circle of offending one side and the extension of the other two sides that will form 3 pieces circles tangent outside on ΔABP on the side of BP and AB and the circle tangent outside on ΔAQD on the side of AD and two circles tangent in the ΔABP and ΔAQD. Circle tangent formed on the quadrilateral is called semi circle tangent at nonconvex quadrilateral $ABCD$.

If a line is drawn from each vertex to the point of tangency in front of him, then the intersection of the three lines is called the point of the semi Gergonne point on nonconvex quadrilateral $ABCD$. In Figure 2.1, there are six points of semi Gergonne constructed at nonconvex quadrilateral $ABCD$, that is, Ge_1, Ge_2, Ge_3, Ge_4, Ge_5, and Ge_6.

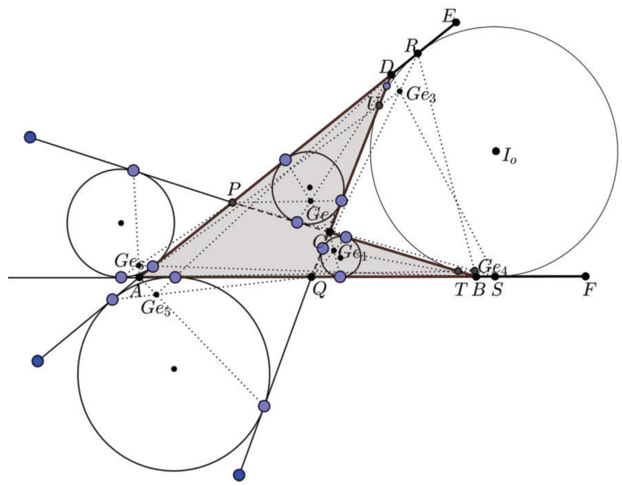

Figure 2.1.

A nonconvex quadrilateral $ABCD$ has three outer edges tangent circles and 2 in the semi circle tangent. Lengths of fingers for each circle tangent can be calculated.

Theorem 2.1. *Suppose $ABCD$ is a nonconvex quadrilateral with the lengths of sides are a, b, c and d. Then the length of the radius of semi-circle tangent at the front corner of C is*

$$R_o = \frac{L\square ABCD}{a - c} = \frac{L\square ABCD}{d - b}.$$

Proof. Let R_o be the length of the radius of semi-incircle tangent in front of the corner C as in Figure 1.1. Area of the nonconvex quadrilateral $ABCD$ can be calculated as

$$L\square ABCD = L\Delta ABO + L\Delta ADO - L\Delta CBO - L\Delta DCO$$

$$= \frac{1}{2} \cdot a \cdot R_o + \frac{1}{2} \cdot d \cdot R_o - \frac{1}{2} \cdot b \cdot R_o - \frac{1}{2} \cdot c \cdot R_o,$$

$$L\square ABCD = \frac{1}{2}(a + d - b - c) \cdot R_o,$$

$$R_o = \frac{2L\square ABCD}{a + d - b - c}.$$

Because $a + b = c + d$, we will obtain

$$R_o = \frac{L\square ABCD}{a - c} \quad \text{or} \quad R_o = \frac{L\square ABCD}{d - b}.$$

From Theorem 2.1, then will be calculated $L\square ABCD$. We can determine $L\square ABCD$ as follows.

Theorem 2.2. *Let $ABCD$ is a nonconvex quadrilateral with the lengths of sides are $AB = a$, $BC = b$, $CD = c$ and $AD = d$, then*

$$L\square ABCD = \sqrt{abcd}\,\sin\gamma, \qquad \textit{where } \gamma = \frac{\angle B + \angle D}{2}.$$

Proof. See Figure 2.1, pull the AC line so that there are two triangles ΔABC and ΔACD. Applying cosine rule in ΔABC, we have

$$AC^2 = a^2 + b^2 - 2ab\cos\angle B. \tag{2.1}$$

Also applying cosine rule in ΔACD, we have

$$AC^2 = c^2 + d^2 - 2cd\cos\angle D. \tag{2.2}$$

Based on the equations (2.1) and (2.2), we will obtain

$$a^2 + b^2 - c^2 - d^2 = 2ab\cos\angle B - 2cd\cos\angle D,$$

$$(a^2 + b^2 - c^2 - d^2)^2 = 4(ab\cos\angle B - cd\cos\angle D)^2. \tag{2.3}$$

Now $L\square ABCD$ can be written as

$$L\square ABCD = L\Delta ABC + L\Delta ACD,$$

$$L\square ABCD = \frac{1}{2}ab\sin\angle B + \frac{1}{2}cd\sin\angle D,$$

$$16(L\square ABCD)^2 = (2ab\sin\angle B + 2cd\sin\angle D)^2. \tag{2.4}$$

Adding equation (2.3) and equation (2.4), will be obtained

$$16(L\square ABCD)^2 + (a^2 + b^2 - c^2 - d^2)^2$$

$$= 4(ab\sin\angle B + cd\sin\angle D)^2 + 4(ab\cos\angle B - cd\cos\angle D)^2$$

$$= 4a^2b^2\sin^2\angle B + 8abcd\sin\angle B\sin\angle D + 4c^2d^2\sin^2\angle D$$

$$+ 4a^2b^2\cos^2\angle B - 8abcd\cos\angle B\cos\angle D + 4c^2d^2\cos^2\angle D$$

$$= 4a^2b^2 + 4c^2d^2 + 8abcd(\sin\angle B\sin\angle D - \cos\angle B\cos\angle D)$$

$$= 4a^2b^2 + 4c^2d^2 + 8abcd\cos(\angle B + \angle D)$$

$$= (2ab + 2cd)^2 - 16abcd\cos^2\gamma,$$

$$16(L\square ABCD)^2 + (a^2 + b^2 - c^2 - d^2)^2 = (2ab + 2cd)^2 - 16abcd\cos^2\gamma,$$

$$16(L\square ABCD)^2 = (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd\cos^2\gamma$$

$$= [(2ab + 2cd) + (a^2 + b^2 - c^2 - d^2)][(2ab + 2cd) - (a^2 + b^2 - c^2 - d^2)]$$

$$- 16abcd\cos^2\gamma$$

$$= (a + b + c - d)(a + b - c + d)(c + d + a - b)(c + d - a + b) - 16abcd\cos^2\gamma$$

$$= (2c)(2d)(2a)(2b) - 16abcd\cos^2\gamma$$

$$= 16abcd(1 - \cos^2 \gamma),$$

$$16(L\square ABCD)^2 = 16abcd \sin^2 \gamma,$$

$$L\square ABCD = \sqrt{abcd} \sin \gamma.$$

Based on Theorems 2.1 and 2.2, it can be determined that the length of the radius of tangent circle outside in front of the corner C is

$$R_o = \frac{\sqrt{abcd}}{a - c} \sin \gamma \quad \text{or} \quad R_o = \frac{\sqrt{abcd}}{d - b} \sin \gamma.$$

In a convex quadrilateral there is a relationship between the lengths of the radii construction results. Here is given a long relationship between the radii of a circle tangent at nonconvex quadrilateral $ABCD$.

Theorem 2.3. *Let $ABCD$ be a nonconvex quadrilateral having semi-circle tangent with the lengths of the radii R_a, R_b, r_c and r_d, then it can be shown that*

$$R_a r_d = R_b r_c.$$

Proof. See Figure 1.1. Let a noncenvex quadrilateral $ABCD$ has semi-circle tangent with the lengths of the radii R_a, R_b, r_c and r_d. By using trigonometry rules on ΔAI_oS, one can obtain

$$\operatorname{tg} \frac{A}{2} = \frac{R_o}{AS},$$

$$AS = R_o \times \operatorname{ctg} \frac{A}{2}. \tag{2.5}$$

And note ΔBI_oS in Figure 1.1,

$$\operatorname{tg}\left(\frac{180° - B}{2}\right) = \frac{R_o}{BS},$$

$$BS = R_o \times \operatorname{tg} \frac{B}{2}. \tag{2.6}$$

From equations (2.5) and (2.6), we will obtain

$$AB = AS - BS,$$

$$a = R_o\left(\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{B}{2}\right). \tag{2.7}$$

Note ΔBI_aV_2 in Figure 1.1,

$$\operatorname{tg}\frac{B}{2} = \frac{R_a}{BV_2},$$

$$BV_2 = R_a \times \operatorname{ctg}\frac{B}{2}. \tag{2.8}$$

Note ΔAI_aV_2 in Figure 1.1,

$$\operatorname{tg}\left(\frac{180° - A}{2}\right) = \frac{R_a}{AV_2},$$

$$AV_2 = R_a \times \operatorname{tg}\frac{A}{2}. \tag{2.9}$$

From equations (2.7) and (2.8), we will obtain

$$AB = BV_2 - AV_2,$$

$$a = R_a\left(\operatorname{ctg}\frac{B}{2} - \operatorname{tg}\frac{A}{2}\right). \tag{2.10}$$

From equations (2.7) and (2.10), we will be obtain

$$R_a = R_o \times \frac{\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{B}{2}}{\operatorname{ctg}\frac{B}{2} - \operatorname{tg}\frac{A}{2}}. \tag{2.11}$$

In the same way, we will obtain long-radius semi-circle outside the tangent offensive side of the extension sides AB and AD and BC in nonconvex quadrilateral $ABCD$ as

$$R_b = R_o \times \frac{\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{D}{2}}{\operatorname{ctg}\frac{D}{2} - \operatorname{tg}\frac{A}{2}}. \tag{2.12}$$

The length of the radius of the semi incircle the offensive side tangent AB, BC and CD at the extension side of a nonconvex quadrilateral $ABCD$ is

$$r_c = R_o \times \frac{\operatorname{tg}\frac{B}{2} - \operatorname{ctg}\frac{C}{2}}{\operatorname{ctg}\frac{B}{2} - \operatorname{tg}\frac{C}{2}}. \tag{2.13}$$

The length of the radius of the semi incircle the offensive side tangent AD, CD and BC at the extension side of a nonconvex quadrilateral $ABCD$ is

$$r_d = R_o \times \frac{\operatorname{tg}\frac{D}{2} - \operatorname{ctg}\frac{C}{2}}{\operatorname{ctg}\frac{D}{2} - \operatorname{tg}\frac{C}{2}}. \tag{2.13}$$

By multiplying the equations (2.11) to (2.14), we obtain

$$R_a r_d = R_o \times \frac{\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{B}{2}}{\operatorname{ctg}\frac{B}{2} - \operatorname{tg}\frac{A}{2}} \times R_o \times \frac{\operatorname{tg}\frac{D}{2} - \operatorname{ctg}\frac{C}{2}}{\operatorname{ctg}\frac{D}{2} - \operatorname{tg}\frac{C}{2}},$$

$$R_a r_d = R_o^2 \times \frac{\left(\dfrac{\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}}\right)\left(\dfrac{\sin\frac{D}{2}\cos\frac{C}{2}}{\cos\frac{D}{2}\sin\frac{C}{2}}\right)}{\left(\dfrac{\cos\frac{B}{2}\sin\frac{A}{2}}{\sin\frac{B}{2}\cos\frac{A}{2}}\right)\left(\dfrac{\cos\frac{D}{2}\sin\frac{C}{2}}{\sin\frac{D}{2}\cos\frac{C}{2}}\right)},$$

$$R_a r_d = -R_o^2 \times \frac{\sin\frac{B}{2}\cos\frac{A}{2}\sin\frac{D}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{D}{2}}. \tag{2.15}$$

And then, by multiplying the equation (2.12) to (2.13), we obtain

$$R_b r_c = R_o \times \frac{\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{D}{2}}{\operatorname{ctg}\frac{D}{2} - \operatorname{tg}\frac{A}{2}} \times R_o \times \frac{\operatorname{tg}\frac{B}{2} - \operatorname{ctg}\frac{C}{2}}{\operatorname{ctg}\frac{B}{2} - \operatorname{tg}\frac{C}{2}},$$

$$R_b r_c = R_o^2 \times \frac{\left(\dfrac{\cos\frac{A}{2}\sin\frac{D}{2}}{\sin\frac{A}{2}\cos\frac{D}{2}}\right)\left(\dfrac{\sin\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}\sin\frac{C}{2}}\right)}{\left(\dfrac{\cos\frac{D}{2}\sin\frac{A}{2}}{\sin\frac{D}{2}\cos\frac{A}{2}}\right)\left(\dfrac{\cos\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}\cos\frac{C}{2}}\right)},$$

$$R_b r_c = -R_o^2 \times \frac{\sin\frac{D}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{D}{2}\cos\frac{B}{2}\sin\frac{C}{2}}. \tag{2.16}$$

By comparing equations (2.15) and (2.16), we obtain

$$\frac{R_a r_d}{R_b r_c} = \frac{-R_o^2 \times \dfrac{\sin\frac{B}{2}\cos\frac{A}{2}\sin\frac{D}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{D}{2}}}{-R_o^2 \times \dfrac{\sin\frac{D}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{D}{2}\cos\frac{B}{2}\sin\frac{C}{2}}},$$

$$\frac{R_a r_d}{R_b r_c} = 1, \text{ then } R_a r_d = R_b r_c.$$

3. Long Side New Construction Results

There are some points of tangency circle centered a nonconvex quadrilateral $ABCD$ at point I_o, including the points S, T, U and R (See Figure 3.1). Let $BS = x$, $CT = z$, and $DR = y$. The lengths of the tangent lines can be determined based on the following theorem:

Theorem 3.1. *Let* $ABCD$ *be a nonconvex quadrilateral, then the length of the* *tangent of a nonconvex quadrilateral in front corner of* C *on the extension side* AB is

$$x = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma \times \operatorname{ctg}\frac{1}{2}A - a.$$

Proof. See Figure 3.1.

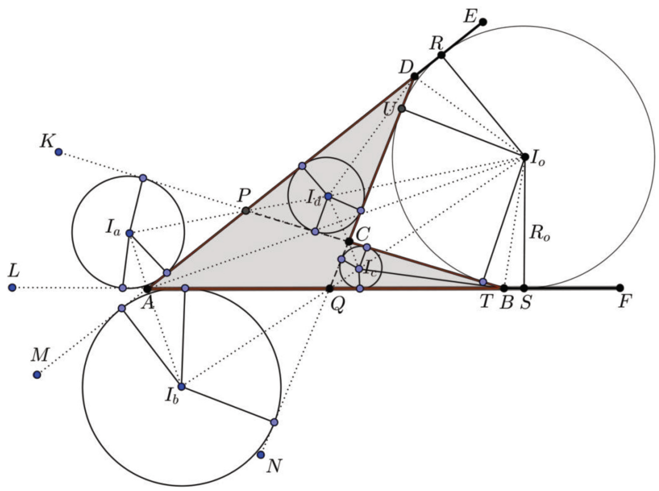

Figure 3.1.

Let the lengths of sides AB, BC, CD and AD be, respectively, a, b, c and d. By using the concept of trigonometry on Figure 3.1, we obtain

$$\operatorname{tg}\frac{1}{2}A = \frac{I_oS}{AS} \text{ or } AS = \frac{I_oS}{\operatorname{tg}\frac{1}{2}A}, \text{ then } AS = R_o \times \operatorname{ctg}\frac{1}{2}A, \text{ and so}$$

$$AB + BS = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma \times \operatorname{ctg}\frac{1}{2}A,$$

$$x = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma \times \operatorname{ctg}\frac{1}{2}A - a.$$

Remark 3.1. In the same way, we will obtain

(a) The length of a tangent of a nonconvex quadrilateral in the front corner of C on the side BC is

$$z = a + b - \frac{\sqrt{abcd}}{a-c} \times \sin\gamma \times \text{ctg}\,\frac{1}{2}A.$$

(b) The length of a tangent of nonconvex quadrilateral in the front corner of C on the extension of the side AD is

$$y = c - (a+b) + \frac{\sqrt{abcd}}{a-c} \times \sin\gamma \times \text{ctg}\,\frac{1}{2}A.$$

Theorem 3.2. *Let $ABCD$ be a nonconvex quadrilateral and the point Q an intersection between the lines AB with CD line extension, the length of the line AQ is*

$$AQ = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma\left(\text{ctg}\,\frac{A}{2} - \text{tg}\,\frac{Q}{2}\right).$$

Proof. See Figure 3.1. Let the lengths of sides AB, BC, CD and AD be, respectively, a, b, c and d. Then $\angle QAP = \angle A$, $\angle AQC = \angle Q$, and $\angle APC = \angle P$. By using trigonometry comparison to tangent the length of side AQ obtained at ΔASI_o is

$$\text{tg}\angle SAI_o = \text{tg}\,\frac{\angle QAP}{2},$$

$$\text{tg}\,\frac{\angle QAP}{2} = \frac{\text{the length of side } SI_o}{\text{the length of side } AS},$$

$$\text{tg}\,\frac{A}{2} = \frac{R_o}{AS}, \text{ then } AS = R_o \times \text{ctg}\,\frac{A}{2},$$

and

$$tg\angle SQI_o = \text{tg}\left(\frac{180° - \angle AQC}{2}\right),$$

$$\text{tg}\left(\frac{180° - \angle AQC}{2}\right) = \frac{\text{the length of side } SI_o}{\text{the length of side } QS},$$

$$\text{ctg}\,\frac{Q}{2} = \frac{R_o}{QS}, \text{ such that } QS = R_o \times \text{tg}\,\frac{Q}{2},$$

so that

$$AQ + QS = AS,$$

$$AQ + QS = R_o \times \operatorname{ctg}\frac{A}{2},$$

$$AQ + R_o \times \operatorname{tg}\frac{Q}{2} = R_o \times \operatorname{ctg}\frac{A}{2},$$

$$AQ = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma\left(\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{Q}{2}\right).$$

Remark 3.2. In the same way, we will obtain

(a) The length of the side AP, P being intersection of the lines AD in the extension of BC is

$$AP = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma\left(\operatorname{ctg}\frac{A}{2} - \operatorname{tg}\frac{P}{2}\right).$$

(b) The length of the side CQ which is an extension of the line BC to the point Q is

$$CQ = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma\left(\operatorname{tg}\frac{Q}{2} + \operatorname{ctg}\frac{1}{2}A - (a+b)\right).$$

(c) The length of the side CP which is an extension of the line BC to the point P is

$$CP = \frac{\sqrt{abcd}}{a-c} \times \sin\gamma\left(\operatorname{tg}\frac{P}{2} + \operatorname{ctg}\frac{1}{2}A - (a+b)\right).$$

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