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A NON-UNITAL ALGEBRA HAS UUNP IFF ITS UNITIZATION HAS UUNP

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Abstract

Let A be a non-unital Banach algebra, S. J. Bhatt and H. V. Dedania showed that A has the unique uniform norm property (UUNP) if and only if its unitization has UUNP. Here we prove this result for any non-unital algebra.

1. Preliminaries

Let A be a non-unital algebra and let $A_e = \{a + \lambda e : a \in A, \lambda \in C\}$ be the unitization of A with the identity denoted by e. For an algebra norm $\|.\|$ on A, define $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\}$ and $\|a + \lambda e\|_1 = \|a\| + |\lambda|$ for all $a + \lambda e \in A_e$. $\|.\|_{op}$ is an algebra seminorm on A_e, and $\|.\|_1$ is an algebra norm on A_e. An algebra norm $\|.\|$ on A is called regular if $\|.\|_{op} = \|.\|$ on A. A uniform norm $\|.\|$ on A is an algebra norm satisfying the square property $\|a^2\| = \|a\|^2$ for all

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$a \in A$; and in this case, $\|.\|$ is regular and $\|.\|_{op}$ is a uniform norm on A_e. An algebra has the unique uniform norm property (UUNP) if it admits exactly one uniform norm.

2. The Result

Theorem. *A non-unital algebra A has UUNP if and only if its unitization A_e has UUNP.*

Proof. Let $\|.\|$ and $|||.|||$ be two uniform norms on A_e, then $\|.\| = |||.|||$ on A since A has UUNP, and so $\|.\|_{op} = |||.|||_{op}$ on A_e. By [3, Corollary 2.2(1)] and since two equivalent uniform norms are identical, it follows that $(\|.\| = \|.\|_{op}$ or $\|.\| \cong \|.\|_1)$ and $(|||.||| = |||.|||_{op} = \|.\|_{op}$ or $|||.||| \cong |||.|||_1 = \|.\|_1)$; equivalently, at least one of the following holds:

(i) $\|.\| = \|.\|_{op}$ and $|||.||| = |||.|||_{op} = \|.\|_{op}$;

(ii) $\|.\| = \|.\|_{op}$ and $|||.||| \cong |||.|||_1 = \|.\|_1$;

(iii) $\|.\| \cong \|.\|_1$ and $|||.||| = |||.|||_{op} = \|.\|_{op}$;

(iv) $\|.\| \cong \|.\|_1$ and $|||.||| \cong |||.|||_1 = \|.\|_1$.

If either (i) or (iv) is satisfied, then $\|.\| = |||.|||$. By noting that (ii) and (iii) are similar by interchanging the roles of $\|.\|$ and $|||.|||$, it is enough to assume (ii). Let $(c(A). \|.\|^{\sim})$ be the completion of $(A, \|.\|)$, we distinguish two cases:

(1) $c(A)$ has not an identity:

$\|.\|^{\sim}$ is regular since it is uniform. By [1, Corollary 2], $\|.\|_{op}^{\sim} \leq \|.\|_1^{\sim} \leq 3\, \|.\|_{op}^{\sim}$ on $c(A)_e$ (unitization of $c(A)$). Let $a + \lambda e \in A_e \subset c(A)_e$, $\|a + \lambda e\|_1^{\sim} = \|a\|^{\sim} + |\lambda| = \|a\| + |\lambda| = \|a + \lambda e\|_1$ and $\|a + \lambda e\|_{op}^{\sim} = \sup\{\|(a + \lambda e)b\|^{\sim} : b \in c(A), \|b\|^{\sim} \leq 1\}$ $= \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\} = \|a + \lambda e\|_{op}$. Therefore $\|.\|_{op} \leq \|.\|_1 \leq 3\|.\|_{op}$. By (ii), $\|.\|$ and $|||.|||$ are equivalent uniform norms, and so $\|.\| = |||.|||$.

(2) $c(A)$ has an identity e :

Let $(c(A_e), |||.|||^{\sim})$ be the completion of $(A_e, |||.|||)$. Since $\|.\| = |||.|||$ on A, $c(A)$ can be identified to the closure of A in $(c(A_e), |||.|||^{\sim})$ so that $\|.\|^{\sim} = |||.|||^{\sim}$ on $c(A)$. Let $a + \lambda e \in A_e \subset c(A)$,

$\|a + \lambda e\| = \|a + \lambda e\|_{op}$ by (ii)

$= \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\}$

$= \sup\{\|(a + \lambda e)b\|^{\sim} : b \in c(A), \|b\|^{\sim} \leq 1\}$

$= \|a + \lambda e\|^{\sim}$ since $c(A)$ is unital

$= |||a + \lambda e|||^{\sim} = |||a + \lambda e|||$. Thus $\|.\| = |||.|||$.

Conversely, let $\|.\|$ and $|||.|||$ be two uniform norms on A, then $\|.\|_{op}$ and $|||.|||_{op}$ are uniform norms on A_e, hence $\|.\|_{op} = |||.|||_{op}$ since A_e has UUNP. Therefore $\|.\| = \|.\|_{op} = |||.|||_{op} = |||.|||$ on A since $\|.\|$ and $|||.|||$ are regular.

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