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A COUPLED VERSION OF THE CONVEX CONTRACTION MAPPING THEOREM IN BIPOLAR METRIC SPACE

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Abstract

A new proof of the convex contraction mapping theorem [1] was given by Ampadu [2]. In the present paper, motivated by certain results contained in Mutlu et al. [3], we obtain the coupled version of the convex contraction mapping theorem in bipolar metric spaces.

1. Introduction

A coupled version of the Banach contraction principle appeared in Mutlu et al. [3]. In the present paper, we address the following

Question 1.1. What is the coupled version of the convex contraction mapping theorem [2] in the setting of bipolar metric spaces [4]?

This paper is organized as follows. Section 2 contains some preliminary ideas that would be useful in the sequel. The main results are given in Section 3. An

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example is given to motivate the main result.

2. Preliminaries

Definition 2.1 (Mutlu and Gurdal [4]). A bipolar metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \mapsto \mathbb{R}^+$ is a function satisfying the following

(a) if $d(x, y) = 0$, then $x = y$,

(b) if $x = y$, then $d(x, y) = 0$,

(c) if $x, y \in X \bigcap Y$, then $d(x, y) = d(y, x)$,

(d) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$.

We say *d* is *a* bipolar metric on the pair (X, Y).

Definition 2.2 (Mutlu and Gurdal [4]). Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets and $f : X_1 \bigcup Y_1 \mapsto X_2 \bigcup Y_2$ be a given function. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, we say *f* is a covariant map from (X_1, Y_1) to (X_2, Y_2) and write $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$. If $f(X_1) \subseteq Y_2$, and $f(Y_1) \subseteq X_2$, we say *f* is a contra-variant map from (X_1, Y_1) to (X_2, Y_2) and in this paper, we shall write $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$.

Remark 2.3. If d_1, d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we shall sometimes write $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

Definition 2.4 (Mutlu and Gurdal [4]). Let (X, Y, d) be a bipolar metric space

(a) A point $u \in X \bigcup Y$ is called a left point if $u \in X$, a right point if $u \in Y$, and a central point if it is both a left and right point.

(b) A sequence $\{x_n\} \in X$ is called a left sequence, and a sequence $\{y_n\} \in Y$ is called a right sequence. In a bipolar metric space, a left or right sequence, is simply called a sequence.

(c) A sequence $\{u_n\}$ is said to be convergent to a point u, iff $\{u_n\}$ is a left sequence, u is a right point and $\lim_{n\to\infty} d(u_n, u) = 0$; or $\{u_n\}$ is a right sequence, u is a left point and $\lim_{n\to\infty} d(u, u_n) = 0$.

(d) A bi-sequence $\{(x_n, y_n)\}$ on (X, Y, d) is a sequence on the set $X \times Y$. If the sequence $\{x_n\}$ and $\{y_n\}$ are convergent, then the bi-sequence $\{(x_n, y_n)\}$ is said to be convergent, and if $\{x_n\}$ and $\{y_n\}$ converge to a common fixed point, then $\{(x_n, y_n)\}$ is said to be bi-convergent.

(e) $\{(x_n, y_n)\}$ is called a Cauchy bi-sequence if $\lim_{n, m\to\infty} d(x_n, x_m) = 0$.

(f) A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent, hence bi-convergent.

Definition 2.5 (Mutlu and Gurdal [4])**.** Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces

(a) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$ implies $d_2(fx_0, fy) < \varepsilon$ for all $y \in Y_1$.

(b) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$ implies $d_2(fx, fy_0) < \varepsilon$ for all $x \in X_1$.

(c) A map f is called continuous, if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.

(d) A contra-variant map $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is continuous iff it is continuous as a covariant map $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

Remark 2.6 (Mutlu and Gurdal [4])**.** A covariant or contra-variant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous iff $\{u_n\} \to v$ on (X_1, Y_1, d_1) implies $\{f(u_n)\} \to f(v)$ on (X_2, Y_2, d_2).

Definition 2.7 (Mutlu and Gurdal [4])**.** Let (X, Y, d) be a bipolar metric space, $F : (X^2, Y^2) \rightrightarrows (X, Y)$ be a covariant mapping. $(a, b) \in X^2 \bigcup Y^2$ is said to be a coupled fixed point of F if $F(a, b) = a$ and $F(b, a) = b$.

Definition 2.8. Let (X, Y, d) be a bipolar metric space, $F : (X^2, Y^2) \rightrightarrows (X, Y)$ be a covariant mapping, and k_1, k_2, k_3 be non-negative constants. If F satisfies the condition

$$d(F^2(a, b), F^2(p, q)) \leq k_1 d(a, p) + k_2 d(b, q)$$

$$+k_3 d(F(a, b), F(p, q)), k_1 + k_2 + k_3 < 1$$

for all $a, b \in X$ and $p, q \in Y$, then we say $F : X^2 \bigcup Y^2 \mapsto X \bigcup Y$ is a coupled convex contraction mapping of order 2.

3. Main Results

Theorem 3.1. *Let* (X, Y, d) *be a complete bipolar metric space, and F be a map satisfying Definition* 2.8, *then F has a unique coupled fixed point.*

Proof. Let $a_0, b_0 \in X$ and $p_0, q_0 \in Y$. Take $a_1, b_1 \in X$ and $p_1, q_1 \in Y$ with $a_1 = F(a_0, b_0)$, $b_1 = F(b_0, a_0)$, $p_1 = F(p_0, q_0)$, $q_1 = F(q_0, p_0)$. Continuing, we obtain bi-sequences $\{(a_n, b_n)\}$ and $\{(p_n, q_n)\}$ such that

$$a_n = F(a_{n-1}, b_{n-1}), a_{n+1} = F(a_n, b_n) = F^2(a_{n-1}, b_{n-1}),$$

$$b_n = F(b_{n-1}, a_{n-1}), b_{n+1} = F(b_n, a_n) = F^2(b_{n-1}, a_{n-1}),$$

$$p_n = F(p_{n-1}, q_{n-1}), p_{n+1} = F(p_n, q_n) = F^2(p_{n-1}, q_{n-1}),$$

$$q_n = F(q_{n-1}, p_{n-1}), q_{n+1} = F(q_n, p_n) = F^2(q_{n-1}, p_{n-1})$$

for all $n \in \mathbb{N}$. Let $\gamma = k_1 + k_2 + k_3$. By definition of F, we have

$$d(a_{n+1},\ p_{n+2}) = d(F^2(a_{n+1},\ b_{n-1}),\ F^2(p_n,\ q_n))$$

$$\leq k_1 d(a_{n-1},\ p_n) + k_2 d(b_{n-1},\ q_n)$$

$$+ k_3 d(F(a_{n-1},\ b_{n-1}),\ F(p_n,\ q_n))$$

and,

$$d(b_{n+1},\ q_{n+2}) = d(F^2(b_{n-1},\ a_{n-1}),\ F^2(q_n,\ p_n))$$

$$\leq k_1 d(b_{n-1},\ q_n) + k_2 d(a_{n-1},\ p_n)$$

$$+ k_3 d(F(b_{n-1},\ a_{n-1}),\ F(q_n,\ p_n))$$

for all $n \in \mathbb{N}$. and $\gamma < 1$. Now set $e_{n+1} = d(a_{n+1},\ p_{n+2}) + d(b_{n+1},\ q_{n+2})$, and observe that

$$e_{n+1} \leq (k_1 + k_2)[d(b_{n-1},\ q_n) + d(a_{n-1},\ p_n)] + k_3[d(a_n,\ p_{n+1}) + d(b_n,\ q_{n+1})]$$

$$\leq (k_1 + k_2)[d(b_{n-1},\ q_n) + d(a_{n-1},\ p_n)] + k_3 e_n$$

$$\leq (k_1 + k_2)e_{n-1} + k_3 e_n$$

$$\leq (k_1 + k_2 + k_3)e_n$$

$$= \gamma e_n.$$

Now it follows that $0 \leq e_{n+1} \leq \gamma e_n \leq \gamma^2 e_{n-1} \leq \cdots \leq \gamma^n e_1$. Now observe that

$$d(a_{n+2},\ p_{n+1}) = d(F^2(a_n,\ b_n),\ F^2(p_{n-1},\ q_{n-1})$$

$$\leq k_1 d(a_n,\ p_{n-1}) + k_2 d(b_n,\ q_{n-1})$$

$$+ k_3 d(F(a_n,\ b_n),\ F(p_{n-1},\ q_{n-1}))$$

and,

$$d(b_{n+2},\ q_{n+1}) = d(F^2(b_n,\ a_n),\ F^2(q_{n-1},\ p_{n-1})$$

$$\leq k_1 d(b_n, q_{n-1}) + k_2 d(a_n, p_{n-1})$$

$$+k_3 d(F(b_n, a_n), F(q_{n-1}, p_{n-1}))$$

for all $n \in \mathbb{N}$ and $\gamma < 1$. Put $s_{n+1} = d(a_{n+2}, p_{n+1}) + d(b_{n+2}, q_{n+1})$, and observe that

$$s_{n+1} \leq (k_1 + k_2)[d(a_n, p_{n-1}) + d(b_n, q_{n-1})] + k_3[d(a_{n+1}, p_n) + d(b_{n+1}, q_n)]$$

$$\leq (k_1 + k_2)s_{n-1} + k_3 s_n$$

$$\leq (k_1 + k_2 + k_3)s_n$$

$$= \gamma s_n.$$

Now it follows that $0 \leq s_{n+1} \leq \gamma s_n \leq \gamma^2 s_{n-1} \leq \cdots \leq \gamma^2 s_1$. Now observe that

$$d(a_{n+1}, p_{n+1}) = d(F^2(a_{n-1}, b_{n-1}), F^2(p_{n-1}, q_{n-1}))$$

$$\leq k_1 d(a_{n-1}, p_{n-1}) + k_2 d(b_{n-1}, q_{n-1})$$

$$+k_3 d(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1}))$$

and,

$$d(b_{n+1}, q_{n+1}) = d(F^2(b_{n-1}, a_{n-1}), F^2(q_{n-1}, p_{n-1}))$$

$$\leq k_1 d(b_{n-1}, q_{n-1}) + k_2 d(a_{n-1}, p_{n-1})$$

$$+k_3 d(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1}))$$

for all $n \in \mathbb{N}$ and $\lambda < 1$. Now set $t_{n+1} = d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})$ and observe that

$$t_{n+1} \leq (k_n + k_2)[d(a_{n-1}, p_{n-1}) + d(b_{n-1}, q_{n-1})] + k_3[d(a_n, p_n) + d(b_n, q_n)]$$

$$\leq (k_1 + k_2)t_{n-1} + k_3 t_n$$

$$\leq (k_1 + k_2 + k_3)t_n$$

$$= \gamma t_n.$$

Now it follows that $0 \le t_{n+1} \le \gamma t_n \le \gamma^2 t_{n-1} \le \cdots \le \gamma^n t_1$. By Definition 2.1(d), we have

$$d(a_{n+1}, p_{m+1}) \le d(a_{n+1}, p_{n+2}) + d(a_{n+2}, p_{n+2}) + \cdots + d(a_m, p_{m+1}),$$

$$d(b_{n+1}, q_{m+1}) \le d(b_{n+1}, q_{n+2}) + d(b_{n+2}, q_{n+2}) + \cdots + d(b_m, q_{m+1}),$$

$$d(a_{m+1}, p_{n+1}) \le d(a_{m+1}, p_m) + d(a_m, p_m) + \cdots + d(a_{n+2}, p_{n+1}),$$

$$d(b_{m+1}, q_{n+1}) \le d(b_{m+1}, q_m) + d(b_m, q_m) + \cdots + d(b_{n+2}, q_{n+1})$$

for each $n, m \in \mathbb{N}$, $n < m$. Consequently, we have

$$\begin{aligned} & d(a_{n+1}, p_{m+1}) + d(b_{n+1}, q_{m+1}) \\ & \le [d(a_{n+1}, p_{n+2}) + d(b_{n+1}, q_{n+2})] + [d(a_{n+2}, p_{n+2}) + d(b_{n+2}, q_{n+2})] \\ & \quad + \cdots + [d(a_m, p_{m+1}) + d(b_m, q_{m+1})] \\ & = e_{n+1} + t_{n+2} + e_{n+2} + \cdots + t_m + e_m \\ & \le \gamma^{n+1} e_1 + \gamma^{n+2} t_1 + \gamma^{n+2} e_1 + \cdots + \gamma^m e_1 + \gamma^m t_1 \\ & \le (\gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^m) e_1 + (\gamma^{n+2} + \gamma^{n+3} + \cdots + \gamma^m) t_1 \\ & \le \frac{\gamma^{n+1}}{1-\gamma} e_1 + \frac{\gamma^{n+2}}{1-\gamma} t_1 \end{aligned}$$

and

$$\begin{aligned} & d(a_{m+1}, p_{n+1}) + d(b_{m+1}, q_{n+1}) \\ & \le [d(a_{m+1}, p_m) + d(b_{m+1}, q_m)] + [d(a_{m+2}, p_m) + d(b_{m+2}, q_m)] \\ & \quad + \cdots + [d(a_{n+2}, p_{n+1}) + d(b_{n+2}, q_{n+1})] \\ & = s_m + t_m + \cdots + s_{n+2} + t_{n+2} + s_{n+1} \\ & \le \gamma^m s_1 + \gamma^m t_1 + \cdots + \gamma^{n+2} s_1 + \gamma^{n+2} t_1 + \gamma^{n+1} s_1 \\ & = (\gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^m) s_1 + (\gamma^{n+2} + \gamma^{n+3} + \cdots + \gamma^m) t_1 \end{aligned}$$

$$\leq \frac{\gamma^{n+1}}{1-\gamma} s_1 + \frac{\gamma^{n+2}}{1-\gamma} t_1$$

for $n < m$. Since for an arbitrary $\varepsilon > 0$, there exists n_1 such that

$$\frac{\gamma^{n_1+1}}{1-\gamma} e_1 + \frac{\gamma^{n_1+2}}{1-\gamma} t_1 < \frac{\varepsilon}{3}$$

and

$$\frac{\gamma^{n_1+1}}{1-\gamma} s_1 + \frac{\gamma^{n_1+2}}{1-\gamma} t_1 < \frac{\varepsilon}{3},$$

then for each $n, m \geq n_1$, we deduce that

$$d(a_{n+1}, p_{m+1}) + d(b_{n+1}, q_{m+1}) < \frac{\varepsilon}{3},$$

$$d(a_{m+1}, p_{n+1}) + d(b_{m+1}, q_{n+1}) < \frac{\varepsilon}{3}.$$

It follows that the sequences $\{(a_n, p_n)\}$ and $\{(b_n, q_n)\}$ are Cauchy bi-sequences. By completeness of (X, Y, d), there exists $a, b \in X$ and $p, q \in Y$ with $\lim_{n\to\infty} a_n = p$, $\lim_{n\to\infty} b_n = q$, $\lim_{n\to\infty} p_n = a$, and $\lim_{n\to\infty} q_n = b$. Now observe there exists $n_2 \in \mathbb{N}$ with $d(a_n, p) < \frac{\varepsilon}{3}$, $d(b_n, q) < \frac{\varepsilon}{3}$, $d(a, p_n) < \frac{\varepsilon}{3}$, $d(b, q_n) < \frac{\varepsilon}{3}$ for all $n \geq n_2$ and every $\varepsilon > 0$. Since $\{(a_n, p_n)\}$ and $\{(b_n, q_n)\}$ are Cauchy bi-sequences, we get $d(a_n, p_n) < \frac{\varepsilon}{3}$, and $d(b_n, q_n) < \frac{\varepsilon}{3}$. By the contractive condition of the theorem, we have

$$d(F^2(a, b), p)$$

$$\leq d(F^2(a, b), p_{n+2}) + d(a_{n+2}, p_{n+2}) + d(a_{n+2}, p)$$

$$\leq d(F^2(a, b), F^2(p_n, q_n)) + d(a_{n+2}, p_{n+2}) + d(a_{n+2}, p)$$

$$\leq k_1 d(a, p_n) + k_2 d(b, q_n) + k_3 d(F(a, b), F(p_n, q_n))$$

$$+d(a_{n+2},\ p_{n+2}) + d(a_{n+2},\ p)$$

$$\leq (k_1 + k_3)\frac{\varepsilon}{3} + k_2\frac{\varepsilon}{3} + 2\frac{\varepsilon}{3}$$

$$= (\gamma)\frac{\varepsilon}{3} + 2\frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3}$$

$$= \varepsilon$$

for each $n \in \mathbb{N}$ and $\gamma := k_1 + k_2 + k_3 < 1$. So $d(F^2(a, b),\ p) = 0$, that is, $F^2(a, b) = p$. Similarly, $F^2(a, b) = q$, $F^2(p, q) = a$, $F^2(q,\ p) = b$. Since

$$d(a,\ p) = d(\lim_{n\to\infty} p_n,\ \lim_{n\to\infty} a_n) = \lim_{n\to\infty} d(a_n,\ p_n) = 0,$$

$$d(b,\ q) = d(\lim_{n\to\infty} q_n,\ \lim_{n\to\infty} b_n) = \lim_{n\to\infty} d(b_n,\ q_n) = 0.$$

It follows that $a = p$ and $b = q$. Therefore $(a, b) \in X^2 \cap Y^2$ is a coupled fixed point. For uniqueness, take another coupled fixed point $(a_1, b_1) \in X^2 \cup Y^2$. If $(a_1, b_1) \in X^2$, then we get

$$d(a_1, a) = d(F^2(a_1, b_1), F(a, b))$$

$$\leq k_1 d(a_1, a) + k_2 d(b_1, b) + k_3 d(F^2(a_1, b_1),\ F(a, b))$$

$$\leq (k_1 + k_3) d(a_1, a) + k_2 d(b_1, b)$$

and

$$d(b_1, b) = d(F^2(b_1, a_1), F(b, a))$$

$$\leq k_1 d(b_1, b) + k_2 d(a_1, a) + k_3 d(F^2(b_1, a_1),\ F(b, a))$$

$$\leq (k_1 + k_3) d(b_1, b) + k_2 d(a_1, a).$$

From the chain of two inequalities immediately above, we deduce that

$$d(a_1, a) + d(b_1, b) \leq (k_1 + k_2 + k_3)[d(a_1, a) + d(b_1, b)]$$

$$= \gamma[d(a_1, a) + d(b_1, b)] < d(a_1, a) + d(b_1, b)$$

which is a contradiction, thus $d(a_1, a) + d(b_1, b) = 0,$ and so $a_1 = a$ and $b_1 = b,$ that is, $(a_1, b_1) = (a, b).$ It follows that the coupled fixed point is unique.

If all the constants in the previous theorem are equal, then we obtain the following

Corollary 3.2. *Let* (X, Y, d) *be a complete bipolar metric space, and* $F : (X^2, Y^2) \rightrightarrows (X, Y)$ *be a covariant mapping, and* $k < 1$ *be a non-negative constant. If F satisfies the condition*

$$d(F^2(a, b), F^2(p, q)) \leq \frac{k}{3}[d(a, p) + d(b, q) + d(F(a, b), F(p, q))]$$

for all $a, b \in X$ *and* $p, q \in Y,$ *then F has a unique coupled fixed point.*

Now we have the following, illustrating the main ideas of this paper

Example 3.3. Let $U_n(\mathbb{R}), L_n(\mathbb{R}), d,$ and F be defined as in Example 1 [3]. Now fix $k := \frac{1}{2},$ and observe that

$$\frac{2}{9}\sum_{i,j=1}^{n}|(a_{ij} - c_{ij}) + (b_{ij} - d_{ij})|$$

$$= d(F^2(A, B), F^2(C, D))$$

$$\leq \frac{k}{3}\sum_{i,j=1}^{n}\left|\frac{4}{3}(a_{ij} - c_{ij}) + \frac{4}{3}(b_{ij} - d_{ij})\right|$$

$$\leq \frac{k}{3}\sum_{i,j=1}^{n}\left|\frac{1}{3}(a_{ij} - c_{ij}) + \frac{1}{3}(b_{ij} - d_{ij})\right| + \frac{k}{3}\sum_{i,j=1}^{n}|(a_{ij} - c_{ij}) + (b_{ij} - d_{ij})|$$

$$\leq \frac{k}{3}\sum_{i,j=1}^{n}\left|\frac{1}{3}(a_{ij} - c_{ij}) + \frac{1}{3}(b_{ij} - d_{ij})\right| + \frac{k}{3}\sum_{i,j=1}^{n}|(a_{ij} - c_{ij})| + \frac{k}{3}\sum_{i,j=1}^{n}|(b_{ij} - d_{ij})|$$

$$= \frac{k}{3}[d(F(A, B), F(C, D)) + d(A, C) + d(B, D)].$$

Clearly, all the conditions of the previous Corollary hold, and the unique coupled fixed point is $(0_{n\times n}, 0_{n\times n}) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$, where $0_{n\times n}$ is the null matrix.

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