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A COMPLETE CHARACTERIZATION OF TOPOLOGICAL PRODUCT PROPERTIES

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Abstract

Within this paper, recent discoveries in the investigation of weakly *P*o spaces and properties, and product properties are used to completely characterize topological product properties.

1. Introduction and Preliminaries

Topological product properties were introduced in 1930 [11].

Definition 1.1. Let *P* be a topological property. Then *P* is a product property iff a product space, with the Tychonoff topology, has property *P* iff each factor space has property *P*.

The 1930 definition began the search within topology for product properties and not product properties and great progress has been made.

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In 1936, T_0-identification spaces were introduced and used to jointly characterize each of pseudometrizable and metrizable [10].

Definition 1.2. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space is pseudometrizable iff its T_0-identification space is metrizable.*

T_0-identification spaces were cleverly created to add T_0 to an externally generated, strongly (X, T) related T_0-identification space of (X, T), making T_0-identification spaces a strong, useful topological tool [10], as established below.

In a similar manner to that of pseudometrizable and metrizable, T_0-identification spaces were used in 1975 [9] to jointly characterize R_1 and Hausdorff.

Definition 1.3. A space (X, T) is R_1 iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

In a 2015 paper [2], pseudometrizable and R_1 were generalized to weakly Po.

Definition 1.4. Let P be a topological property for which $P\text{o} = (P \text{ and } T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

In the initial investigation of weakly Po spaces and properties, it was shown that for a topological property P for which weakly Po exists, weakly Po is a unique topological property and weakly Po is simultaneously shared by both a space and its

T_0-identification space, motivating the introduction of T_0-identification P properties [3].

Definition 1.5. Let S be a topological property. Then S is a T_0-identification P property iff a space and its T_0-identification space simultaneously satisfy property S.

Within that paper [3], it was shown that for a topological property S, S is a T_0-identification P property iff S = weakly So, which for awhile clouded the obvious: A topological property S is weakly Po iff it is a T_0-identification P property.

Within the 2015 paper [2], the search for topological properties that are not weakly Po led to the use of T_0 and "not-T_0". Thus it was discovered that T_0 played another foundational role in the study of topology and "not-T_0" proved to be a useful topological property, motivating the addition of "not-P", where P is a topological property for which "not-P", exists, into the study of topology [2]. The addition and use of the many new properties provided tools not before studied and used in the study of topology and, in a short time period, has exposed a mathematically fertile, never before imagined territory long overlooked within topology that has already changed and expanded the study of topology.

For example, in the paper [4], the use of "not-T_0" and "not-P", where "not-P", exists, not only provided needed tools to prove the existence of the never before imagined least of all topological properties L, but, also, provided the needed tools for a quick, easily understood proof of its existence.

Theorem 1.2. *L, the least of all topological properties, is given by $L = (T_0$ or "not-T_0"$) = (P$ or "not-P"), where P is a topological property for which "not-P", exists.*

Also, "not-P", where P is a topological property for which "not-P" exists, was used to easily show there is no strongest topological property [5].

Within the paper [4], it was shown that every space has property L. Thus each product space and each of its factor spaces simultaneously share property L, regardless of how diverse or even if factor spaces have properties that are known not product properties, and, by the 1930 definition, L is a product property, a reality far different than the intent of product properties in 1930.

Thus the discovery of L in the study of weakly *P*o created a disconnect in the study of product properties and, if possible, needed fixing. A quick, easy fix to restoring continuity in the study of product properties was the removal of L as a product property.

Definition 1.6. Let P be a topological property. Then P is a product property iff $P \neq L$ and a product space with the Tychonoff topology has property P iff each factor space has property P [4].

Within this paper, Definition 1.6 will be used as the definition of product properties. Thus, among the 1930 defined product properties, L is unique. It is the only 1930 defined product property that needed to be removed to end the discontinuity in the study of product properties.

Hence, a first connection between product properties and weakly *P*o spaces and properties was made, revealing T_0-identification spaces to be more powerful and useful than earlier imagined.

With the addition of the topological property weakly *P*o and the continual search for product properties, a natural question to ask is whether for a product property P, is weakly *P*o a product property? The investigation of that question led to the following result.

Theorem 1.3. *Let P be a product property such that weakly P*o *exists. Then weakly P*o *and P*o *are product properties* [3].

Thus a second, strong connection between product properties and weakly *P*o spaces and properties was discovered.

In the initial investigations of weakly *P*o, it was unknown which topological properties are weakly *P*o and, as seen above, the existence of weakly *P*o was required to move forward with the investigation, greatly hindering the exploration of the recently discovered mathematically fertile territory within topology. Major progress was made in overcoming that difficulty in a 2017 paper [6].

Answer 1.1. {*Q*o| *Q* is a T_0-identification *P* property} = {*Q*o| *Q*o is a weakly *P*o property} = {*Q*o| *Q* is a topological property and *Q*o exists}.

Thus the uncertainty of where to start the search for weakly *P*o properties was replaced by certainty. In addition, the uncertainty of the existence of a topological property *W* such that *W* = weakly *Q*o for each topological property *Q* for which *Q*o exists was replaced by certainty.

As cited above, the existence of *L* created a discontinuity in the study of product properties, but, on the positive side, as given above, the discontinuity was easily fixed with the removal of *L* as a product property. Also, the discovery and use of *L* and its properties, and the use of "not-*P*" has revealed much new knowledge within the study of both topology and product properties. Through the many years of study of product properties, it was unknown whether for product properties *P* and *Q* if (*P* and *Q*) exists. In the paper [7], properties of *L* were used to show that for product properties *P* and *Q*, (*P* and *Q*) exists and is thus a product property, giving many additional product properties.

With the addition and use of "not-*P*", where *P* is a topological property different from *L* and "not-*P*" is the negation of *P*, there would be a need to define "not-*P*", where *P* is a product property. Since *L* is the only topological property *P* for which "not-*P*" does not exist [7] and *L* is not a product property, as defined above, then for each product property *P*, "not-*P*" exists.

Definition 1.7. Let *P* be a product property. Then a product space with the Tychonoff topology has property "not-*P*" iff there exists a factor space with property "not-*P*" [4].

In the paper [4], it was shown that for each product property *P*, "not-*P*" is not a product property and for product properties *P* and *Q* for which (*P* and "not-*Q*") exists, (*P* and "not-*Q*") is not a product property. Thus, in future studies of product properties, product properties and not-product properties can be simultaneously studied and known product properties and not-product properties can be used to give additional product and not-product properties.

Below the properties above are used to further investigate weakly *P*o and completely characterize product properties.

2. Additional Connecting Properties

Within the 2017 paper [6], for a topological property *W* for which *W*o exists, a property *WNO* was defined. Let *W* be a topological property such that *W*o exists. A space (X, T) has property *WNO* iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property *W*o. In that paper [6], it was shown that for a topological property *W* for which *W*o exists, *WNO* exists and is a topological property, and a space has property (*W*o or *WNO*) iff its T_0-identification space has property (*W*o or *WNO*). Thus for a topological property *W* for which *W*o exists, (*W*o or *WNO*) is a T_0-identification P property and (*W*o or *WNO*) = weakly (*W*o or *WNO*)o. Since *WNO* is "not-T_0", then (*W*o or *WNO*)o = *W*o and (*W*o or *WNO*) = weakly (*W*o or *WNO*)o = weakly *W*o. Hence, very substantial progress was made in the 2017 paper [6] concerning T_0-identification P properties or equivalently, topological properties that are weakly *P*o.

By the results above, weakly *Q*o is a T_0-identification P property that is weakly *P*o.

Theorem 2.1. *Let Q be a topological property such that* $Q \neq Q$o. *Then Q is a product property iff Q = weakly Q*o *is a* T_0-identification P *product property that is weakly P*o.

Proof. Suppose *Q* is a product property. Then each of *Q* and T_0 are product properties, which implies $(Q \text{ and } T_0) = Q$o exists. Thus, by the results above, weakly *Q*o exists and weakly *Q*o is a product property. If *Q* = weakly *Q*o, then *Q* is a T_0-identification P product property that is weakly *P*o. Thus, consider the case that $Q \neq$ weakly *Q*o. Since weakly *Q*o is the least element of $\mathcal{S} = \{P \mid P$ is a topological property, Po exists, and Po implies Qo$\}$ [2], and Q is a topological property such that Qo exists and Qo implies Qo, then Q is stronger than or equal to weakly Qo. Since $Q \neq$ weakly Qo, then Q is stronger than weakly Po and weakly Qo = (Q or ((weakly Qo) and "not-Q")), where ((weakly Qo) and "not-Q") exists and is not a product property, which contradicts weakly Qo is a product property. Thus Q = weakly Qo and Q is a T_0-identification P product property that is weakly Po.

Clearly, the converse is true.

Theorem 2.2. *Let Q be a product property such that $Q \neq Q$o. Then Qo $\neq T_0$ and weakly Qo $\neq L$.*

Proof. Since Q is a product property, then $Q \neq L$ and since Q = weakly Qo, then weakly $Qo \neq L$. Since weakly Po is a unique topological property and L = weakly Lo = weakly T_0 [8], if Qo $= T_0$, then weakly Qo $= L$, which is a contradiction.

Thus, within the study of product properties, L and T_0 are unique. As given above, L is the only initially defined product property that is no longer a product property and T_0 is a product property, using either definition, that is not a weakly Po property of a current product property.

Theorem 2.3. *Let Q be a product property such that $Q \neq Q$o. Then Q = (Qo or (Q and "not-T_0)) = weakly Qo = (Qo or QNO) and (Q and "not-T_0) = QNO.*

Proof. Since Q is a product property, then, as above, Qo exists. Since Q = (Q and L) = (Q and (T_0 or "not-T")) = (Qo or (Q and "not-T_0 ")) and $Q \neq Q$o, then (Q and "not-T_0 ") exists. Thus Q = (Qo or (Q and "not-T_0 ")) = weakly Qo = (Qo or QNO). Since Qo is independent of each of (Q and "not-T_0 ") and QNO, then (Q and "not-T_0 ") = ((Qo or (Q and "not-T_0 ")) \ Qo) = ((Qo or QNO) \ Qo) = QNO.

Theorem 2.4. *Let* Q *be a product property for which* $Q = Q\text{o}$. *Then* $Q = Q\text{o}$ *is a weakly* $P\text{o}$ *product property different from* T_0.

Proof. Since $Q = Q\text{o}$ exists, then weakly $Q\text{o}$ exists and $Q = Q\text{o}$ is a weakly $P\text{o}$ property. Thus $Q = Q\text{o}$ is a weakly $P\text{o}$ product property and, by Theorem 2.2, is not equal to T_0.

3. The Characterization

Corollary 3.1. $\{Q \mid Q$ *is a product property* $\} = \{Q \mid Q$ *is a product property and* $Q = Q\text{o}$ *exists*$\} \cup \{Q \mid Q$ *is a product property*, $Q\text{o}$ *exists*, *and* $Q \neq Q\text{o}\} = \{T_0\} \cup \{Q \mid Q = Q\text{o}$ *and* $Q\text{o}$ *is a weakly* $P\text{o}$ *product property*$\}$ $\cup \{Q \mid Q\text{o}$ *exists*, $Q \neq Q\text{o}$, *and* Q *is a* T_0*-identification* P *product property that is weakly* $P\text{o}\}$.

Thus the addition of the new topological categories weakly $P\text{o}$ and weakly $P\text{o}$ properties and new properties and tools revealed in the investigation of the two new categories has provided both the categories and the needed tools to give the complete, valuable characterization of product properties above. If for a topological property W, $W\text{o}$ does not exist, then, immediately it is known that W is not a product property. If $W\text{o}$ exists, but W is not a T_0-identification P property and $W\text{o}$ is not a weakly $P\text{o}$ property, then, again, it is immediately known that W is not a product property.

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