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A CHARACTERIZATION OF COMPLEX HYPERQUADRICS BY SECTIONS OF LINE BUNDLES

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Abstract

Let M be an n-dimensional compact irreducible complex space with an ample line bundle L. Suppose that $\dim H^0(M, L) = n + 2$ and that the common zeros of any n linearly independent irreducible global sections of L consist of two distinct points or a single point with multiplicity two. Then M is biholomorphic to a hyperquadric in a complex projective space P^{n+1} of dimension $n + 1$.

1. Introduction

Kobayashi and Ochiai [1], Fujita [2] and Miyaoka [3] have given characterizations of the hyperquadrics, respectively. The purpose of this paper is to give a slightly different characterization of the hyperquadrics.

Results which can be found in [4], [5] and [6] are used freely often without

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explicit references. Let M be a complex space with a line bundle L. ϑ is the sheaf of germs of holomorphic functions, $\vartheta(L)$ is the sheaf of germs of holomorphic sections of L. $H^0(M, L)$ means $H^0(M, \vartheta(L))$.

2. Characterization of the Hyperquadrics

We have given a characterization of complex projective space in [7]. In this paper, a characterization of the complex hyperquadrics will be given.

Suppose that the space $H^0(M, L)$ of global holomorphic sections of L is base point free. Let $\varphi_1, \ldots, \varphi_{N+1}$ be a basis for $H^0(M, L)$. Then a holomorphic map f from M into the N-dimensional complex projective space P^N can be defined by $f(x) = (\varphi_1(x), \ldots, \varphi_{N+1}(x))$. It is interesting to know the structures of the image $f(M)$ and the relations between M and $f(M)$. Now we give the main result of this paper.

Theorem. *Let M be an n-dimensional compact irreducible complex space with an ample line bundle L. Suppose that* $\dim H^0(M, L) = n + 2$ *and that for any n linearly independent irreducible sections* $\varphi_1, \ldots, \varphi_n$ *in* $H^0(M, L)$, *their common zeros consist of two distinct points or a single point with multiplicity two. Then M is biholomorphic to a hyperquadric in a complex projection space P^{n+1} of dimension* $n + 1$.

Proof. Given any $n + 2$ linearly independent irreducible sections $\varphi_1, \ldots, \varphi_{n+2} \in H^0(M, L)$, set $V_{n-i} = Z(\varphi_1, \ldots, \varphi_i)$ denotes the common zeros of $\varphi_1, \ldots, \varphi_i$. We have complex subspaces

$$M = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_0.$$

Let d be the largest integer such that $V_n = M$, $V_1, \ldots, V_{n-d}$ are all irreducible. By the assumption, $V_0 = Z(\varphi_1, \ldots, \varphi_n)$ consists of two distinct points or a single point with multiplicity two, and so V_0 is reducible. Thus $d \leq n - 1$. According to the claim 2 in the proof of Theorem [7], we have an exact sequence

$$0 \to (\varphi_1, ..., \varphi_d) \to H^0(M, L) \overset{\beta}{\to} H^0(V_{n-d}, L), \tag{1}$$

where β is the restriction map.

If $\varphi_{d+1} \in Ker\beta$, that is, $\varphi_{d+1} = 0$ on V_{n-d}, then $\varphi_{d+1} \in (\varphi_1, ..., \varphi_d)$ by the sequence (1). This shows that φ_{d+1} is a linear combination of $\varphi_1, ..., \varphi_d$. But by the assumption, $\varphi_1, ..., \varphi_{d+1}$ are linearly independent, a contradiction. Thus φ_{d+1} is not trivial on V_{n-d}.

Let $V = V_{n-d} \cap Z(\varphi_{d+1})$. By [8, Theorem 11, Theorem 14 in Ch. III], we know that $\dim V = \dim V_{n-1} - 1 = n - d - 1$, and that V is of pure dimension $n - d - 1$. It follows that each irreducible component of V is of dimension $n - d - 1$.

Let $V = \sum W_i$, where each W_i is an irreducible component of V. Then each W_i is a positive divisor of V_{n-d}.

$Z(\varphi_1), ..., Z(\varphi_n)$ are divisors of M, their intersection number $r = Z(\varphi_1) \cdot Z(\varphi_2) \cdots Z(\varphi_n)$ is equal to the number of points (multiplicities counted) in $V_0 = Z(\varphi_1, ..., \varphi_n)$ by [5].

By the assumption, V_0 consists of two distinct points or a single point with multiplicity two, which implies that $r = 2$. On the other hand, according to the properties of intersection numbers of divisors [5] and [9], it follows that the self-intersection of V as a divisor of V_{n-d} is equal to the intersection number of $Z(\varphi_1), ..., Z(\varphi_n)$, we have

$$2 = V \cdot V \cdots V = \left(\sum W_i\right) \cdots \left(\sum W_i\right) \geq \sum_i W_i \cdot W_i \cdots W_i.$$

This shows that $V = W_1 + W_2$. Suppose that $d \leq n - 2$. If $W_1 = W_2$, then from the self-intersection of V, we have

$$2 = V \cdot V \cdots V = (2W_1) \cdots (2W_1) = 2^{n-d} W_1 \cdots W_1 \geq 2^{n-d} \geq 4$$

an absurd. Thus $W_1 \neq W_2$. $V = V_{n-d} \cap Z(\varphi_{d+1})$ is the set of zeros of φ_{d+1} on the irreducible complex space V_{n-d}. By [1, Lemma 1, Sect. 1] and [3, p. 130], we have an exact sequence

$$0 \to \vartheta_{V_{n-d}} \to \vartheta_{V_{n-d}}(L) \overset{\beta}{\to} \vartheta_V(L) \to 0,$$

where β is the restriction map.

Then the following sequence is exact:

$$0 \to H^0(V_{n-d}, \vartheta_{V_{n-d}}) \to H^0(V_{n-d}, L) \overset{\beta}{\to} H^0(V, L).$$

Since V_{n-d} is irreducible, $\dim H^0(V_{n-d}, \vartheta_{V_{n-d}}) = 1$. It shows that $H^0(V_{n-d}, \vartheta_{V_{n-d}})$ is generated by φ_{d+1}. Combining with sequence (1), we obtain an exact sequence

$$0 \to (\varphi_1, ..., \varphi_{d+1}) \to H^0(M, L) \overset{\beta}{\to} H^0(V, L),$$

where β is the restriction map.

On the other hand, similarly to the proof of Theorem in [7], it can be shown that $H^0(W_1, L)$ and $H^0(W_2, L)$ are base point free.

Consequently, when $d \leq n-2$, it follows that $H^0(M, L)$ is base point free.

When $d = n-1$, $V_1 = Z(\varphi_1, ..., \varphi_{n-1})$ is an irreducible curve. We have an exact sequence

$$0 \to (\varphi_1, ..., \varphi_{n-1}) \to H^0(M, L) \overset{\beta}{\to} H^0(V_1, L), \qquad (2)$$

where β is the restriction map.

We have

$$\dim H^0(V_1, L) \geq \dim H^0(M, L) - \dim(\varphi_1, ..., \varphi_{n-1}) = (n+2) - (n-1) = 3.$$

Taken complex numbers a, b, c such that $a\varphi_n + b\varphi_{n+1} + c\varphi_{n+2}$ vanishes on V_1, it follows from the sequence (2) that $a\varphi_n + b\varphi_{n+1} + c\varphi_{n+2}$ is contained in $(\varphi_1, ..., \varphi_{n-1})$, that is, there are complex numbers $a_1, ..., a_{n-1}$ such that $a\varphi_n + b\varphi_{n+1} + c\varphi_{n+2} = a_1\varphi_1 + \cdots + a_{n-1}\varphi_{n-1}$. Observed that $\varphi_1, ..., \varphi_{n+2}$ are linearly independent on V_1, which implies that $a = b = c = 0$. Thus $\varphi_n, \varphi_{n+1}, \varphi_{n+2}$ are linearly independent on V_1.

By the assumption, V_0 consists of two distinct points or a single point with multiplicity two, and so $H^0(M, L)$ has at most two base points. $V_1 = Z(\varphi_1, ..., \varphi_{n-1})$ is an irreducible curve, we may select two distinct points *u* and *v* in V_1 such that *u* and *v* are not base points of $H^0(M, L)$. Since $\varphi_i(u) = \varphi_i(v) = 0$ for $i = 1, ..., n-1$, it follows that $\varphi_n(u), \varphi_{n+1}(u), \varphi_{n+2}(u)$ and $\varphi_n(v), \varphi_{n+1}(v), \varphi_{n+2}(v)$ are not all zero, respectively.

There exist complex numbers a, b, c with not all zero satisfying

$$a\varphi_n(u) + b\varphi_{n+1}(u) + c\varphi_{n+2}(u) = 0,$$

$$a\varphi_n(v) + b\varphi_{n+1}(v) + c\varphi_{n+2}(v) = 0.$$

Let $\varphi = a\varphi_n + b\varphi_{n+1} + c\varphi_{n+2}$. Since $\varphi_1, ..., \varphi_{n+2}$ are linearly independent, we have that $\varphi_1, ..., \varphi_{n+1}, \varphi$ are also linearly independent.

Let $V_{0.\varphi} = V_1 \cap Z(\varphi) = Z(\varphi_1, ..., \varphi_{n+1}, \varphi)$. By the assumption, $V_{0.\varphi}$ contains at most two points, it is clear that u, v are contained in $V_{0.\varphi}$. Thus $V_{0.\varphi} = \{u, v\}$.

Suppose that $H^0(M, L)$ has a base point *w*. Then $\varphi_1(w) = \cdots = \varphi_{n+2}(w) = 0$, which implies that $\varphi(w) = a\varphi_n(w) + b\varphi_{n+1}(w) + c\varphi_{n+2}(w) = 0$ and so $w \in V_{0.\varphi} = \{u, v\}$, that is, *u* or *v* is a base point, a contradiction.

Therefore, we have proved that $H^0(M, L)$ is always base point free.

We may define a holomorphic map *f* from *M* into a complex projective space P^{n+1} of dimension $n + 1$ by $f(x) = (\varphi_1(x), ..., \varphi_{n+2}(x))$ for $x \in M$.

For any point $p \in P^{n+1}$, let *V* be a connected component of $f^{-1}(p)$. Then *L* is trivial restricted on V. But L is an ample line bundle on M, and so L is also ample restricted on V. This implies that V must be a point. Thus $f^{-1}(p)$ is a finite set. It follows that f is a finite map. Set $Q = f(M)$ is the image of M in P^{n+1} under f. Since M is compact and irreducible, Q is an irreducible closed complex subspace in P^{n+1} by [8]. Noted that f is a finite map, we obtain that $\dim Q = \dim M = n$, which implies that Q is an n-dimensional hypersurface in P^{n+1}. Thus there exists a homogeneous polynomial F such that $Q = \{y \in P^{n+1}; F(y) = 0\}$. If Q is a hyperplane, that is, $\deg F = 1$, then there exist complex numbers $a_1, ..., a_{n+2}$ with some nonzero such that $F = a_1 Y_1 + \cdots + a_{n+2} Y_{n+1}$, and thus $Q = \{(y_1, ..., y_{n+2}) \in P^{n+1}; a_1 y_1 + \cdots + an + 2yn + 2 = 0$.

On the other hand, $Q = \{(\varphi_1(x), ..., \varphi_{n+2}(x)) \in P^{n+1}; x \in M\}$, we have $a_1\varphi_1(x) + \cdots + a_{n+2}\varphi_{n+2}(x) = 0$ for all $x \in M$. This shows that $\varphi_1, ..., \varphi_{n+2}$ are linearly dependent, a contradictory to the assumption.

Therefore the hypersurface Q has degree $m = \deg F \geq 2$. Since f is a finite map, the rank of f at every point of M is equal to $\dim M = \dim Q = n$, and so $f : M \to Q$ is a holomorphic finite map of rank n. By [10, Satz 28], $f : M \to Q$ is an open map.

Given any point $p \in Q$, let t_p denote the number of points in $f^{-1}(p)$, then t_p is a lower semi-continuous function of p.

Since $\deg F = m$, a generic complex line l in P^{n+1} meets Q at m points $p_1, ..., p_m$. Then l meets M at $t_{p_1} + \cdots + t_{p_m}$ points under f. Without loss of generality, we may take $l = \{(0, ..., 0, y_{n+1}, y_{n+2}) \in P^{n+1}\}$, then $\varphi_1, ..., \varphi_n$ have at least $t_1 + \cdots + t_m$ common zeros. But $V_0 = Z(\varphi_1, ..., \varphi_n)$ contains at most two points. It follows that $t_1 + \cdots + t_m \leq 2$. Since $m \geq 2$, this implies that $t_1 + t_2 = 2$ and $t_1 = t_2 = 1$. Thus $m = 2$. Therefore Q is a hyperquadric in P^{n+1}, and $t_p = 1$ for any generic point $p \in P^{n+1}$. Since t_p is a lower semi-continuous function of p, we have $t_q = 1$ for any $q \in Q$. Thus $f : M \to Q$ is bijective. By [10, Satz 32], we know that $f^{-1} : Q \to M$ is also holomorphic. Consequently, M is biholomorphic to a hyperquadric in P^{n+1} as desired.

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