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NEW TRANSFORMATION FOR A NON-IDEAL BOSE SYSTEM

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Abstract

We present a new simple transformation for a dilute Bose gas with weak interactions between the atoms, used to describe superfluid liquid helium. This transformation, together with the Fröhlich unitary transformation, leads to the formation of Bogoliubov quasiparticles. In this respect, we find the equation for density atoms in the condensate which coincides with Bogoliubov's famous equation.

1. Introduction

In 1938, London [1] made the connection between the ideal Bose gas and superfluidity in liquid $^4\mathrm{He}$. The ideal Bose gas undergoes a phase transition at very low temperatures to a condition in which the zero-momentum quantum state is occupied by a finite fraction of the atoms. In 1941, Landau described the properties of superfluid $^4\mathrm{He}$ in terms of collective excitations, identified as phonons and rotons. The purely microscopic theory most widely used was first described by Bogoliubov [3] within a model of weak non-ideal Bose-gas, with the inter-particle S-wave scattering. The Bogoliubov model for superfluid helium gives a description of the gas of helium atoms via Bogoliubov's quasiparticles, which reproduce phonons at lower momenta (Landau prediction) as well as free atoms at higher momenta (London's model of ideal gas). In his theory, Bogoliubov introduced the original famous transformation for the Bose system which represents as transformation of the

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non-ideal Bose gas of helium atoms to the ideal Bose gas of Bogoliubov's quasiparticles. To diagonalize the Bogoliubov's Hamiltonian, we may also meet with the Bogoliubov transformation in terms of the pseudorotations (Lorentz transformations) [4]. Hence also, we may remark a letter [5], where Girardeeau and Arnowitt first used a canonical transformation for non-ideal Bose gas of helium atoms within unitary transformation.

In this paper, we present a new simple transformation of non-ideal Bose system of atoms which reproduces Bogoliubov's quasiparticles within introduction of the Fröhlich's transformation [6, 7]. In this respect, we find a famous formula for density atoms in the condensate for weak non-ideal Bose-gas, which was first discovered by Bogoliubov.

2. Bogoliubov Model of a Superfluid Liquid Helium

To describe superfluid liquid helium, we present the Bogoliubov model of a dilute Bose gas with weak interactions between the atoms. This model considers a system of N identical interacting atoms via S-wave scattering. These atoms, as spinless Bose-particles of mass m, are confined to a box of volume V. The main part of $\hat{H}$, the Hamiltonian of such a system, is expressed in the second quantization form as:

$$\hat{H} = \sum_{\vec{p}} \frac{p^2}{2m} \hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \frac{1}{2V} \sum_{\vec{p}} U_{\vec{p}} \hat{\varrho}_{\vec{p}} \hat{\varrho}^+_{\vec{p}}. \tag{1}$$

Here $\hat{a}^+_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ are, respectively, the "creation" and "annihilation" operators of free atoms with momentum $\vec{p}$; $U_{\vec{p}}$ is the Fourier transform of a S-wave pseudopotential in the momentum space:

$$U_{\vec{p}} = \frac{4\pi d\hbar^2}{m}, \tag{2}$$

where d is the scattering amplitude; and the Fourier component of the density operator is

$$\hat{\varrho}_{\vec{p}} = \sum_{\vec{p}_1} \hat{a}^+_{\vec{p}_1 - \vec{p}} \hat{a}_{\vec{p}_1}. \tag{3}$$

In Bogoliubov theory [3], it is necessary to separate the atoms in the condensate from those atoms filling states above the condensate. In this respect, the operators $\hat{a}_0$ and $\hat{a}^+_0$ are replaced by c-numbers $\hat{a}_0 = \hat{a}^+_0 = \sqrt{N_0}$ within the approximation of a macroscopic number of condensate atoms $N_0 \gg 1$. This assumption leads to a broken Bose-symmetry law for atoms in the condensate state. Thus, the density operators of atoms $\hat{\varrho}_{\vec{p}}$ and $\hat{\varrho}^+_{\vec{p}}$ in the Bogoliubov model, which describes the gas of atoms ^{4}He with weak interactions via S-wave scattering together with the approximation $\frac{N_0}{N} \sim 1$, arrive to the forms:

$$\hat{\varrho}_{\vec{p}} = \sqrt{N_0}\,(\hat{a}^+_{-\vec{p}} + \hat{a}_{\vec{p}}) \tag{4}$$

and

$$\hat{\varrho}^+_{\vec{p}} = \sqrt{N_0}\,(\hat{a}_{-\vec{p}} + \hat{a}^+_{\vec{p}}). \tag{5}$$

Consequently, the Bogoliubov Hamiltonian of system $\hat{H}$ takes the following form:

$$\hat{H} = \sum_{\vec{p}} \left(\frac{p^2}{2m} + mv^2 \right) \hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \frac{mv^2}{2} \sum_{\vec{p}} (\hat{a}^+_{-\vec{p}} \hat{a}^+_{\vec{p}} + \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}}), \tag{6}$$

where $v = \sqrt{\frac{U_{\vec{p}} N_0}{mV}} = \sqrt{\frac{4\pi d\hbar^2 N_0}{m^2 V}}$ is the velocity of sound in the Bose gas, which depends on the density atoms in the condensate $\frac{N_0}{V}$.

For the evolution of the energy level, it is necessary to diagonalize the Bogoliubov Hamiltonian $\hat{H}$. This is accomplished by the introduction of the Bose-operators $\hat{b}^+_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ by using the Bogoliubov linear transformation [3]:

$$\hat{a}_{\vec{p}} = \frac{\hat{b}_{\vec{p}} + L_{\vec{p}} \hat{b}^+_{-\vec{p}}}{\sqrt{1 - L^2_{\vec{p}}}}, \tag{7}$$

where $L_{\vec{p}}$ is the unknown real symmetrical function of a momentum $\vec{p}$.

Substitution of (7) into (6) leads to

$$\hat{H} = \sum_{\vec{p}} \varepsilon_{\vec{p}} \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}} + \frac{1}{2} \sum_{\vec{p}} \left(\varepsilon_{\vec{p}} - \frac{p^2}{2m} - mv^2 \right). \tag{8}$$

Hence we infer that $\hat{b}^+_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ are the “creation” and “annihilation” operators of

Bogoliubov quasiparticles with energy:

$$\varepsilon_{\vec{p}} = \left[\left(\frac{p^2}{2m}\right)^2 + p^2 v^2\right]^{1/2}. \tag{9}$$

In this context, the real symmetrical function $L_{\vec{p}}$ of a momentum $\vec{p}$ is found

$$L_{\vec{p}}^2 = \frac{\frac{p^2}{2m} + mv^2 - \varepsilon_{\vec{p}}}{\frac{p^2}{2m} + mv^2 + \varepsilon_{\vec{p}}}. \tag{10}$$

3. New Transformation of the Bose System

We now attempt to obtain the diagonal form of the Bogoliubov Hamiltonian $\hat{H}$ in (6) by using of new simple transformation for the Bose system based on the Fröhlich transformation [6, 7]. In distinction from the Bogoliubov linear transformation [3] of expression (7), we introduce the Bose-operators of creation and annihilation for unknown quasiparticles $\hat{b}_{\vec{p}}^{+}$ and $\hat{b}_{\vec{p}}$ within new transformation of the Bose-operators of "creation" and "annihilation" for atoms $\hat{a}_{\vec{p}}^{+}$ and $\hat{a}_{\vec{p}}$, which replace the ones presented in (7):

$$\hat{a}_{\vec{p}} = \hat{b}_{\vec{p}}. \tag{11}$$

We now aim to prove that these unknown quasiparticles and Bogoliubov's one are the same. Hence, we note that the Bose operators $\hat{b}_{\vec{p}}^{+}$ and $\hat{b}_{\vec{p}}$ satisfy the Bose commutation relations $[\cdots]$ as:

$$[\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}'}^{+}] = \delta_{\vec{p}, \vec{p}'}, \quad [\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}'}] = 0, \quad [\hat{b}_{\vec{p}}^{+}, \hat{b}_{\vec{p}'}^{+}] = 0.$$

In these terms, the Hamiltonian $\hat{H}$ in (6) takes the new form:

$$\hat{H}_0 = \sum_{\vec{p}} \left(\frac{p^2}{2m} + mv^2\right) \hat{b}_{\vec{p}}^{+} \hat{b}_{\vec{p}} + \frac{mv^2}{2} \sum_{\vec{p}} (\hat{b}_{-\vec{p}}^{+} \hat{b}_{\vec{p}}^{+} + \hat{b}_{\vec{p}} \hat{b}_{-\vec{p}}). \tag{12}$$

This operator Hamiltonian $\hat{H}_0$ describes a non-ideal Bose gas of unknown quasiparticles.

To express the operator Hamiltonian of unknown gas-quasiparticles $\hat{H}_0$ in (12) via the operator Hamiltonian of gas-atoms $\hat{H}$ in (6), we apply the Fröhlich transformation [6, 7] which allows to do a canonical transformation for the operator $\hat{H}$ in (6):

$$\hat{H} = \exp(\hat{S}^+)\hat{H}_0 \exp(\hat{S}) \tag{13}$$

which is expanded in the following terms:

$$\begin{aligned}\hat{H} &= \exp(\hat{S}^+)\hat{H}_0 \exp(\hat{S})\\ &= \hat{H}_0 - [\hat{S}, \hat{H}_0] + \frac{1}{2!}[\hat{S}, [\hat{S}, \hat{H}_0]] - \frac{1}{3!}[\hat{S}, [\hat{S}, [\hat{S}, \hat{H}_0]]] + \cdots,\end{aligned} \tag{14}$$

where the operators represent as:

$$\hat{S}^+ = \sum_{\vec{p}} \hat{S}^+_{\vec{p}} \tag{15}$$

and

$$\hat{S} = \sum_{\vec{p}} \hat{S}_{\vec{p}} \tag{16}$$

and satisfy a condition $\hat{S}^+ = -\hat{S}$.

In this respect, we assume that

$$\hat{S}_{\vec{p}} = A_{\vec{p}}(\hat{b}_{\vec{p}}\hat{b}_{-\vec{p}} - \hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}}), \tag{17}$$

where $A_{\vec{p}}$ is the unknown real symmetrical function from a momentum $\vec{p}$.

Thus, we have

$$\hat{S}^+ = \sum_{\vec{p}} \hat{S}^+_{\vec{p}} = -\sum_{\vec{p}} A_{\vec{p}}(\hat{b}_{\vec{p}}\hat{b}_{-\vec{p}} - \hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}}). \tag{18}$$

To find $A_{\vec{p}}$, we substitute (16) with (17) and (12) into (14). Then

$$\begin{aligned}[\hat{S}, \hat{H}_0] &= \frac{1}{2}\sum_{\vec{p}} (4A_{\vec{p}})\left(\frac{p^2}{2m} + mv^2\right)(\hat{b}_{\vec{p}}\hat{b}_{-\vec{p}} + \hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}})\\ &\quad + \sum_{\vec{p}} (4A_{\vec{p}})mv^2\hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}} + \frac{1}{2}\sum_{\vec{p}} (4A_{\vec{p}})mv^2\end{aligned} \tag{19}$$

because

$$[\hat{b}_{\vec{p}_1}\hat{b}_{-\vec{p}_1} - \hat{b}^+_{-\vec{p}_1}\hat{b}^+_{\vec{p}_1}, \hat{b}^+_{\vec{p}_2}\hat{b}_{\vec{p}_2}]$$

$$= \hat{b}_{\vec{p}_1}\hat{b}_{\vec{p}_2}\delta_{\vec{p}_1,-\vec{p}_2} + \hat{b}_{-\vec{p}_1}\hat{b}_{\vec{p}_2}\delta_{\vec{p}_1,\vec{p}_2} + \hat{b}^+_{\vec{p}_2}\hat{b}^+_{\vec{p}_1}\delta_{\vec{p}_1,-\vec{p}_2} + \hat{b}^+_{\vec{p}_2}\hat{b}^+_{-\vec{p}_1}\delta_{\vec{p}_1,\vec{p}_2} \tag{20}$$

and

$$[\hat{b}_{\vec{p}_1}\hat{b}_{-\vec{p}_1} - \hat{b}^+_{-\vec{p}_1}\hat{b}^+_{\vec{p}_1}, \hat{b}_{\vec{p}_2}\hat{b}_{-\vec{p}_2} + \hat{b}^+_{-\vec{p}_2}\hat{b}^+_{\vec{p}_2}]$$

$$= [\hat{b}_{\vec{p}_1}\hat{b}_{-\vec{p}_1}, \hat{b}^+_{-\vec{p}_2}\hat{b}^+_{\vec{p}_2}] - [\hat{b}^+_{-\vec{p}_1}\hat{b}^+_{\vec{p}_1}, \hat{b}_{\vec{p}_2}\hat{b}_{-\vec{p}_2}]$$

$$= 8\hat{b}^+_{\vec{p}_1}\hat{b}_{\vec{p}_2}\delta_{\vec{p}_1,\vec{p}_2} + 4\delta_{\vec{p}_1,\vec{p}_2}. \tag{21}$$

In this context,

$$\frac{1}{2!}[\hat{S}, [\hat{S}, \hat{H}_0]]$$

$$= \frac{1}{2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\left(\frac{p^2}{2m} + mv^2\right)\hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}} + \frac{1}{2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\,\frac{mv^2}{2}(\hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}})$$

$$+ \frac{1}{2\cdot 2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\left(\frac{p^2}{2m} + mv^2\right) \tag{22}$$

and

$$\frac{1}{3!}[\hat{S}, [\hat{S}, [\hat{S}, \hat{H}_0]]] = \frac{1}{2\cdot 3!}\sum_{\vec{p}}(4A_{\vec{p}})^3\left(\frac{p^2}{2m} + mv^2\right)(\hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}})$$

$$+ \frac{1}{3!}\sum_{\vec{p}}(4A_{\vec{p}})^3 mv^2\hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}} + \frac{1}{2\cdot 3!}\sum_{\vec{p}}(4A_{\vec{p}})^3 mv^2. \tag{23}$$

Thus, the operator $\hat{H}$ in (14) takes the form:

$$\hat{H} = \sum_{\vec{p}}\left(\frac{p^2}{2m} + mv^2\right)\hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}} + \frac{mv^2}{2}\sum_{\vec{p}}(\hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}})$$

$$- \frac{1}{2}\sum_{\vec{p}}(4A_{\vec{p}})\left(\frac{p^2}{2m} + mv^2\right)(\hat{b}_{\vec{p}}\hat{b}_{-\vec{p}} + \hat{b}^+_{-\vec{p}}\hat{b}^+_{\vec{p}}) - \sum_{\vec{p}}(4A_{\vec{p}})mv^2\hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{\vec{p}}(4A_{\vec{p}})mv^2 + \frac{1}{2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\left(\frac{p^2}{2m}+mv^2\right)\hat{b}^{+}_{\vec{p}}\hat{b}_{\vec{p}} \\
&+\frac{1}{2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\,\frac{mv^2}{2}(\hat{b}^{+}_{-\vec{p}}\hat{b}^{+}_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}}) + \frac{1}{2\cdot 2!}\sum_{\vec{p}}(4A_{\vec{p}})^2\left(\frac{p^2}{2m}+mv^2\right) \\
&-\frac{1}{2\cdot 3!}\sum_{\vec{p}}(4A_{\vec{p}})^3\left(\frac{p^2}{2m}+mv^2\right)(\hat{b}^{+}_{-\vec{p}}\hat{b}^{+}_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}}) \\
&-\frac{1}{3!}\sum_{\vec{p}}(4A_{\vec{p}})^3 mv^2\hat{b}^{+}_{\vec{p}}\hat{b}_{\vec{p}} - \frac{1}{2\cdot 3!}\sum_{\vec{p}}(4A_{\vec{p}})^3 mv^2 + \cdots. \qquad (24)
\end{aligned}
$$

Hence, we introduce the following hyperbolic functions:

$$sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

and

$$ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

which in turn lead to the reducing form of an operator H:

$$
\begin{aligned}
\hat{H} = &\sum_{\vec{p}}\left(\frac{p^2}{2m}+mv^2\right)ch(4A_{\vec{p}})\hat{b}^{+}_{\vec{p}}\hat{b}_{\vec{p}} + \frac{mv^2}{2}\sum_{\vec{p}}ch(4A_{\vec{p}})(\hat{b}^{+}_{-\vec{p}}\hat{b}^{+}_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}_{-\vec{p}}) \\
&-\frac{1}{2}\sum_{\vec{p}}sh(4A_{\vec{p}})\left(\frac{p^2}{2m}+mv^2\right)(\hat{b}_{\vec{p}}\hat{b}_{-\vec{p}} + \hat{b}^{+}_{-\vec{p}}\hat{b}^{+}_{\vec{p}}) \\
&-\sum_{\vec{p}}sh(4A_{\vec{p}})mv^2\hat{b}^{+}_{\vec{p}}\hat{b}_{\vec{p}} + \frac{1}{2}\sum_{\vec{p}}(ch(4A_{\vec{p}})-1)\left(\frac{p^2}{2m}+mv^2\right) \\
&-\frac{1}{2}\sum_{\vec{p}}sh(4A_{\vec{p}})mv^2. \qquad (25)
\end{aligned}
$$

Removing of the term of the interaction between the density of unknown quasiparticles claims to remove a sum of second and third terms on right side of (25). This process leads to a following equation for obtaining a quantity for $A_{\vec{p}}$:

$$\frac{mv^2 ch(4A_{\vec{p}})}{2} - \frac{1}{2} sh(4A_{\vec{p}})\left(\frac{p^2}{2m} + mv^2\right) = 0. \tag{26}$$

This equation can be rewritten as:

$$th(4A_{\vec{p}}) = \frac{mv^2}{\frac{p^2}{2m} + mv^2}. \tag{27}$$

We have now achieved a reduction of the Hamiltonian $\hat{H}$ of system (25):

$$\hat{H} = \sum_{\vec{p}} \varepsilon_{\vec{p}} \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}} + \frac{1}{2}\sum_{\vec{p}}\left(\varepsilon_{\vec{p}} - \frac{p^2}{2m} - mv^2\right), \tag{28}$$

where

$$\varepsilon_{\vec{p}} = ch(4A_{\vec{p}})\left(\frac{p^2}{2m} + mv^2 - th(4A_{\vec{p}})mv^2\right) \tag{29}$$

is the energy of unknown quasiparticles.

Consequently, we may state that $\hat{b}^+_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ are "creation" and "annihilation" Bose-operators of Bogoliubov's quasiparticles, with energies given by (9), since the energy of unknown quasiparticles and the Bogoliubov's one are the same:

$$\varepsilon_{\vec{p}} = ch(4A_{\vec{p}})\left(\frac{p^2}{2m} + mv^2 - th(4A_{\vec{p}})mv^2\right) = \left[\left(\frac{p^2}{2m}\right)^2 + p^2v^2\right]^{1/2} \tag{30}$$

because

$$ch(4A_{\vec{p}}) = \frac{1}{\sqrt{1 - th^2(4A_{\vec{p}})}} = \frac{\frac{p^2}{2m} + mv^2}{\sqrt{\left(\frac{p^2}{2m}\right)^2 + p^2v^2}},$$

$$\frac{p^2}{2m} + mv^2 - th(4A_{\vec{p}})mv^2 = \frac{\left(\frac{p^2}{2m}\right)^2 + p^2v^2}{\frac{p^2}{2m} + mv^2}.$$

4. Conclusion

Hence, we note that the Bogoliubov transformation (7) is exact and uniqueness of the diagonalisation of the Hamiltonian (6). However, there is a presence of the alternative approach which may replace the Bogoliubov’s one. Indeed, as first step, we described a non-ideal Bose gas of atoms with the Hamiltonian (6) via a non-ideal Bose gas of unknown quasiparticles with the Hamiltonian (12), due to introduction of the new simple transformation (11).

After, as second step, we diagonalised the Hamiltonian (12) of a non-ideal Bose gas of unknown quasiparticles by the Fröhlich transformation (14) with (15), which in turn leads to the foundation of kind of the unknown quasiparticles as the Bogoliubov's one because we get the Bogoliubov Hamiltonian (8).

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