OCTONION FORMULATION OF SEVEN DIMENSIONAL VECTOR SPACE

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Abstract

We have generalized the three dimensional space to seven dimensional one by using octonions and their multiplication rules. The identity $n(n - 1)(n - 3)(n - 7) = 0$ have been verified to show that there exist only $n = 0, 1, 3$ and $7$ dimensions for vector spaces. It is shown that three dimensional vector space may be extended to seven dimensional space by means of octonions under certain permutations of combinations of structure constant associated with the octonion multiplication rules.

According to celebrated Hurwitz theorem there exits [1] four-division algebra consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) [2] and $\mathbb{O}$ (octonions) [3]. All four algebras are alternative with antisymmetric associators. Real numbers and complex numbers are limited only up to two dimensions, quaternions are extended to four dimensions (one real and three imaginaries) while octonions represent eight dimensions (one scalar and seven vectors, namely, one real and seven imaginaries). There exits a lot of literature [4-12] on the applications of octonions to

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interpret wave equation, Dirac equation, and the extension of octonion non-associativity to physical theories. Accordingly, in this paper, we have extended the three dimensional vector analysis to seven dimensional one by using octonions and their multiplication rules. Thus, we have redefined the gradient, divergence and curl in seven dimensions by using octonions. The identity \( n (n - 1)(n - 3)(n - 7) = 0 \) is verified only for \( n = 0, 1, 3 \) and 7 dimensional vectors. It is shown that few vector identities loose their meaning in seven dimensions due to non-associativity of octonions. So, we have reformulated the vector analysis and it is shown that those vector identities loose their original nature are regained under certain permutations of combinations of structure constant associated with the octonion multiplication rules.

- An octonion \( x \) is expressed as a set of eight real numbers

\[
x = e_0 x_0 + \sum_{j=1}^{7} e_j x_j,
\]

where \( e_j (j = 1, 2, ..., 7) \) are imaginary octonion units and \( e_0 \) is the multiplicative unit element. Set of octets \( (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7) \) is known as the octonion basis elements and satisfies the following multiplication rules

\[
e_0 = 1; \quad e_0 e_j = e_j e_0 = e_j; \quad e_j e_k = -\delta_{jk} e_0 + f_{jkl} e_l, \quad (j, k, l = 1, 2, ..., 7).
\]

The structure constant \( f_{jkl} \) is completely antisymmetric and takes the value 1 for following combinations

\[
f_{jkl} = +1, \quad \forall \{(j, k, l) = (123), (471), (257), (165), (624), (543), (736)\}.
\]

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra \( \mathcal{O} \) is described over the algebra of real numbers having the vector space of dimension 8. Octonion conjugate is defined as

\[
\bar{x} = e_0 x_0 - \sum_{j=1}^{7} e_j x_j,
\]

where we have used the conjugates of basis elements as \( \bar{e_0} = e_0 \) and \( \bar{e_A} = -e_A \). Hence an octonion can be decomposed in terms of its scalar \( (\text{Sc}(x)) \) and vector \( (\text{Vec}(x)) \) parts as
Conjugates of product of two octonions is described as \((xy) = \overline{y}\overline{x}\) while the own conjugate of an octonion is written as \((x) = x\). The scalar product of two octonions is defined as
\[
\langle x, y \rangle = \frac{1}{2} (x \overline{y} + y \overline{x}) = \frac{1}{2} (\overline{x} y + \overline{y} x) = \sum_{\alpha=0}^{7} x_{\alpha} y_{\alpha}.
\] (6)

The norm \(N(x)\) and inverse \(x^{-1}\) (for a nonzero \(x\)) of an octonion are, respectively, defined as
\[
N(x) = x\overline{x} = \overline{x} x = \sum_{\alpha=0}^{7} x_{\alpha}^{2} \cdot e_{0}, \quad x^{-1} = \frac{\overline{x}}{N(x)} \Rightarrow xx^{-1} = x^{-1}x = 1.
\] (7)

The norm \(N(x)\) of an octonion \(x\) is zero if \(x = 0\), and is always positive otherwise. It also satisfies the following property of normed algebra
\[
N(xy) = N(x)N(y) = N(y)N(x).
\] (8)

Equation (2) directly leads to the conclusion that octonions are not associative in nature and thus do not form the group in their usual form. Non-associativity of octonion algebra \(O\) is provided by the associator \(\langle x, y, z \rangle = (xy)z - x(yz)\), \(\forall x, y, z \in O\) defined for any 3 octonions. If the associator is totally antisymmetric for exchanges of any 2 variables, i.e., \(\langle x, y, z \rangle = -(z, y, x) = -(y, x, z) = -(x, z, y)\), the algebra is called alternative. Hence, the octonion algebra is neither commutative nor associative but is alternative.

- In order to verify the identity \(n(n-1)(n-3)(n-7) = 0\) for \(n = 0, 1, 3\) and 7 dimensional vector spaces, we start with the use of octonion multiplication rules. \(n = 0, n = 1\) are limited up to scalars while \(n = 3\) dimensional structure is used for three dimensional vector space. Hurwitz theorem states that the vector identities are satisfied for the algebras of real, complex, quaternions and octonions which are, respectively, associated with \(n = 0, n = 1, n = 3\) and \(n = 7\) space-dimensions. In order to generalize the three dimensional space to seven dimensional vector space, let

\[
Sc(x) = \frac{1}{2} (x + \overline{x}); \quad Vec(x) = \frac{1}{2} (x - \overline{x}) = \sum_{j=1}^{7} e_{j} x_{j}.
\] (5)
us define a vector in seven dimensional vector space over the field of real numbers as
\[ \vec{A} = \hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3 + \hat{e}_4 A_4 + \hat{e}_5 A_5 + \hat{e}_6 A_6 + \hat{e}_7 A_7, \] (9)

where \( A_i (\forall i = 1, 2, 3, ..., 7) \in \mathbb{R}; \) and the \( \hat{e}_i \)'s \( (\forall i = 1, 2, 3, ..., 7) \) are described as the octonionic basis elements which satisfy the following vector product rule [3, 12]

\[ e_j \times e_k = \sum_{k=1}^{7} f_{jk} \hat{e}_l, \quad (\forall j, k, l = 1, 2, ..., 6, 7), \] (10)

where \( f_{jk} \) is the structure constant (3) which is a totally \( G_2 \) - invariant anti-symmetric tensor. As such, we have

\[ f_{jk} f_{lmn} = g_{jkmn} + \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} = -f_{lmn} f_{kkl}, \] (11)

where \( g_{jkmn} = \langle e_j, e_k, e_m, e_n \rangle \) is a totally \( G_2 \) - invariant anti-symmetric tensor [11] and \( g_{jkmn} = -g_{nmkj}. \) The only independent components are \( g_{1254} = g_{1267} = g_{1364} = g_{1375} = g_{2347} = g_{2365} = g_{4576} = 1. \)

- Hence, the following vector cross products are verified in seven dimensional, i.e.,

\[ \vec{A} \times \vec{A} = 0, \]
\[ (\vec{A} \times \vec{B}) \cdot \vec{B} = (\vec{A} \times \vec{B}) \cdot \vec{A} = 0, \]
\[ |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}|, (\forall (\vec{A} \cdot \vec{B} = 0), \]
\[ (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{A})(\vec{B} \cdot \vec{B}) - (\vec{A} \cdot \vec{B})^2, \]
\[ \vec{A} \times (\vec{B} \times \vec{A}) = (\vec{A} \cdot \vec{A})\vec{B} - (\vec{A} \cdot \vec{B})\vec{A}. \] (12)

- Here, we observe that the vector triple cross product identity \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \) of three dimensional vector space is not satisfied in seven dimensional vector space. So, we may write the left hand side of \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \) as

\[ \vec{A} \times (\vec{B} \times \vec{C}) = -\sum_{j=1}^{7} \sum_{pq=1}^{7} [g_{pqjk} + \delta_{pq} \delta_{qk} - \delta_{pk} \delta_{qj}] A_j B_p C_q \hat{e}_k \]
\[
\sum \sum \sum \sum g_{pqjk} A_j B_p C_q \hat{e}_k = \sum \sum A_j B_j C_k \hat{e}_k + \sum A_j B_j C_k \hat{e}_k
\]

\[= \text{irreducible term} + \hat{B} (\hat{A} \cdot \hat{C}) - \hat{C} (\hat{A} \cdot \hat{B}). \quad (13)\]

This identity of vector triple product in seven dimensional vector space is satisfied under the octonion multiplication rule by taking the irreducible term as a ternary product such that

\[\text{irreducible term} = \sum g_{pqjk} A_j B_p C_q \hat{e}_k = \{\hat{A}, \hat{B}, \hat{C}\}. \quad (14)\]

which gives rise to

\[\hat{A} \times (\hat{B} \times \hat{C}) = \hat{B} (\hat{A} \cdot \hat{C}) - \hat{C} (\hat{A} \cdot \hat{B}) + \{\hat{A}, \hat{B}, \hat{C}\}. \quad (15)\]

The \{\hat{A}, \hat{B}, \hat{C}\} is now be re-called as the associator for octonions. Under the condition of alternativity of octonions, equation (15) reduces to the well-known vector identity \(\hat{A} \times (\hat{B} \times \hat{C}) = \hat{B} (\hat{A} \cdot \hat{C}) - \hat{C} (\hat{A} \cdot \hat{B})\) for the various permutation values of structure constant \(f_{jkl}\). As such, the vector product in seven dimensional vector space is verified and the mutual rotation of three different vectors in seven dimensional space is possible only for the certain permutations of combination of basis vectors.

- Let us check the following identity of three dimensional vector space

\[\hat{A} \times (\hat{B} \times \hat{C}) = [\hat{A} \cdot (\hat{B} \times \hat{C})] \hat{C} - [\hat{A} \cdot (\hat{B} \times \hat{C})] \hat{D} \quad (16)\]

to our seven dimensional case. The left hand side of equation (16) now reduces to

\[\hat{A} \times \hat{B} \times \hat{C} = \sum f_{knp} \left( \sum f_{ijk} A_i B_j \left( \sum f_{lmm} C_l D_m \right) \hat{e}_p \right) \]

\[= \sum f_{knp} f_{lmm} C_l D_m \hat{e}_p \left( \sum f_{ijk} A_i B_j \right) \]

\[= \sum f_{ijmp} A_i B_j D_m C_p \hat{e}_p - \sum f_{ijlp} A_i B_j C_l D_p \hat{e}_p \]
$$- \sum_{lmkp}^7 g_{lmkp} C_l D_m (f_{ijk} A_j) \hat{e}_p$$  \hspace{1cm} (17)

while the right hand is to be written as

$$[\hat{A} \cdot (\hat{B} \times \hat{D})] \hat{C} - [\hat{A} \cdot (\hat{B} \times \hat{C})] \hat{D}$$

$$= \sum_{xyz}^7 (f_{xyz} A_x B_x D_y) C_z \hat{e}_t - \sum_{eghv}^7 (f_{egh} A_h B_e C_g) D_v \hat{e}_v.$$  \hspace{1cm} (18)

Comparing equations (17) and (18), we get

$$(\hat{A} \times \hat{B}) \times (\hat{C} \times \hat{D}) = [\hat{A} \cdot (\hat{B} \times \hat{D})] \hat{C} - [\hat{A} \cdot (\hat{B} \times \hat{C})] \hat{D} + \text{irreducible term.}$$  \hspace{1cm} (19)

The irreducible term is not anything but the octonion associator as

$$\text{irreducible term} = - \sum_{lmkp}^7 g_{lmkp} C_l D_m (f_{ijk} A_j) \hat{e}_p = \{\hat{A} \times \hat{B}, \hat{C}, \hat{D}\}.$$  \hspace{1cm} (20)

Thus, the seven dimensional analogue of equation (16) is described as

$$(\hat{A} \times \hat{B}) \times (\hat{C} \times \hat{D}) = [\hat{A} \cdot (\hat{B} \times \hat{D})] \hat{C} - [\hat{A} \cdot (\hat{B} \times \hat{C})] \hat{D} + \{\hat{A} \times \hat{B}, \hat{C}, \hat{D}\},$$  \hspace{1cm} (21)

where the associator $\{\hat{A} \times \hat{B}, \hat{C}, \hat{D}\}$ reduces to be vanishing under the octonion multiplication rule. Hence the vector-identity (16) is verified for seven dimensions. It means that the three dimensional space can be generalized to seven dimensional space only if certain combinations of basis vectors are allowed with the possible permutations of structure constant (3).

• Let us verify the another identity of three dimensional vector space, i.e.,

$$(\hat{A} \times \hat{B}) \cdot (\hat{C} \times \hat{D}) = (\hat{A} \cdot \hat{C})(\hat{B} \cdot \hat{D}) - (\hat{A} \cdot \hat{D})(\hat{B} \cdot \hat{C}).$$  \hspace{1cm} (22)

The left hand side of this equation reduces to

$$(\hat{A} \times \hat{B}) \cdot (\hat{C} \times \hat{D}) = \sum_{ijlm}^7 f_{ijkl} f_{lmk} A_l B_j C_l D_m$$

$$= \sum_{ij}^7 A_i C_j D_j - \sum_{ij}^7 A_i D_i B_j C_j - \sum_{ijklm}^7 g_{ijkl} A_l B_j C_l D_m$$
where the irreducible term can be simplified and accordingly we may write it as the scalar product of a ternary product of three vectors, i.e.,

\[
\text{irreducible term } = -\sum_{ijlm}^7 g_{ijlm} A_i B_j C_l D_m = \{\tilde{A}, \tilde{B}, \tilde{C}\} \cdot \tilde{D}. 
\]

So, equation (23) changes to

\[
(\tilde{A} \times \tilde{B}) \cdot (\tilde{C} \times \tilde{D}) = (\tilde{A} \cdot \tilde{C})(\tilde{B} \cdot \tilde{D}) - (\tilde{A} \cdot \tilde{D})(\tilde{B} \cdot \tilde{C}) + \{\tilde{A}, \tilde{B}, \tilde{C}\} \cdot \tilde{D},
\]

where the irreducible term may be considered vanishing for the octonion associator for the allowed permutation of structure constants. Hence, the vector identity equation (22) is verified for seven dimensional vector space.

- Similarly, we can verify the following vector identities in seven dimensional space time, i.e.,

\[
\tilde{A} \times (\tilde{B} \times \tilde{C}) + \tilde{B} \times (\tilde{C} \times \tilde{A}) + \tilde{C} \times (\tilde{A} \times \tilde{B}) = 0
\]

and

\[
\tilde{A} \times [\tilde{B} \times (\tilde{C} \times \tilde{D})] = (\tilde{A} \times \tilde{C})(\tilde{B} \cdot \tilde{D}) - (\tilde{A} \times \tilde{D})(\tilde{B} \cdot \tilde{C}),
\]

which are also satisfied in seven dimensional space for certain combinations of permutations of structure constant.

- Accordingly, we may develop the vector calculus for seven dimensional vector space in terms of octonion variables. Let us define the seven dimensional differential operator Nabla as \(\tilde{\nabla} = \sum_i^7 \hat{e}_i \frac{\partial}{\partial x_i}\). So, the gradient, curl and divergence of scalar and vector quantities are described as

\[
\text{grad } u = \tilde{\nabla} u = \sum_i^7 \hat{e}_i \frac{\partial u}{\partial x_i},
\]

\[
\text{curl } \tilde{A} = \tilde{\nabla} \times \tilde{A} = \sum_{ijk}^7 f_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k,
\]
div \vec{A} = \nabla \cdot \vec{A} = \sum_1^7 \frac{\partial A_i}{\partial x_i} \tag{30}

• The divergence of a curl is shown to be vanishing as

\[
\nabla \cdot (\nabla \times \vec{A}) = \sum_1^7 \left[ \hat{e}_i \frac{\partial}{\partial x_i} \right] \left[ \sum_1^7 f_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k \right] = \sum_1^7 \sum_1^7 f_{ijk} \frac{\partial^2 A_j}{\partial x_i \partial x_j} \delta_{ik}
\]

\[
= \sum_1^7 f_{ijk} \frac{\partial^2 A_j}{\partial x_i \partial x_j} = \frac{1}{2} \sum_1^7 \left[ f_{ijk} + f_{kji} \right] \frac{\partial^2 A_j}{\partial x_j \partial x_k} = 0. \tag{31}
\]

Hence curl of a vector is also solenoidal in seven dimensional vector space.

• We may now write the curl of a gradient in 7-dimensional vector-space by adopting the octonion multiplication rules (2) as

\[
\nabla \times (\nabla u) = \sum_1^7 f_{ijk} \frac{\partial}{\partial x_i} \left( \sum_1^7 \frac{\partial u}{\partial x_j} \hat{e}_k \right)
\]

\[
= \sum_1^7 f_{ijk} \frac{\partial^2 u}{\partial x_i \partial x_j} \hat{e}_k = \frac{1}{2} \sum_1^7 \left[ f_{ijk} + f_{kji} \right] \frac{\partial^2 u}{\partial x_j \partial x_k} \hat{e}_k = 0. \tag{32}
\]

Thus, the gradient of a curl is also irrotational in seven dimensional vector space under octonion multiplication rules.

• Let us check the following vector identity of three dimensions

\[
\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \tag{33}
\]

in seven dimensional vector-space. The left hand side of equation (33) reduces to

\[
\nabla \cdot (\vec{A} \times \vec{B}) = \sum_1^7 \hat{e}_l \frac{\partial}{\partial x_l} \left[ \sum_1^7 f_{ijk} A_j B_k \hat{e}_l \right]
\]

\[
= \sum_1^7 \sum_1^7 f_{ijk} \frac{\partial}{\partial x_l} \left[ A_j B_k \right] \hat{e}_l = \sum_1^7 f_{ijk} \left[ A_j \frac{\partial B_k}{\partial x_k} + B_j \frac{\partial A_k}{\partial x_k} \right] \tag{34}
\]

while the right hand side of equation (33) changes to
\( \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \)

\[
\begin{align*}
&= \sum_l \hat{e}_l B_l \left[ \sum_{mns} f_{mnt} \frac{\partial A_m}{\partial x_n} \hat{e}_t \right] - \sum_s \hat{e}_s A_s \left[ \sum_{pqr} f_{pqr} \frac{\partial B_q}{\partial x_p} \hat{e}_r \right] \\
&= \sum_{ijk} f_{ijk} B_j \frac{\partial A_i}{\partial x_k} - \left[ - \sum_{ijk} f_{ijk} A_i \frac{\partial B_j}{\partial x_k} \right] = \sum_{ijk} f_{ijk} \left[ A_i \frac{\partial B_j}{\partial x_k} + B_j \frac{\partial A_i}{\partial x_k} \right],
\end{align*}
\]

which is equal to the left hand side of equation (34). Hence the identity (33) is satisfied for seven dimensional space.

- Similarly, on using the octonion multiplication rules (2), we may prove the following identities in seven dimensions, i.e.,

\[
\vec{\nabla} \cdot (u\vec{A}) = u(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} u),
\]

\(\text{(36)}\)

\[
\vec{\nabla} \times (u\vec{A}) = u(\vec{\nabla} \times \vec{A}) + (\vec{\nabla} u) \times \vec{A},
\]

\(\text{(37)}\)

\[
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}.
\]

\(\text{(38)}\)

- However the vector identity of three dimensional vector space

\[
\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B})
\]

\(\text{(39)}\)

is not satisfied in seven dimension and its left hand side reduces to

\[
\begin{align*}
\vec{\nabla} \times (\vec{A} \times \vec{B}) &= \sum_{wkr} f_{wkr} \frac{\partial}{\partial x_w} \left[ \sum_{ijk} f_{ijk} A_i B_j \right] \hat{e}_t \\
&= - \sum_{wkr} \sum_{ijk} f_{ijk} f_{kwrt} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t
\end{align*}
\]

\(\text{(40)}\)

whereas the right hand expression of equation (39) becomes

\[
(\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B})
\]

\[
= \sum_{ij} \frac{\partial A_i}{\partial x_j} B_j \hat{e}_i - \sum_{lm} A_l \frac{\partial B_m}{\partial x_l} \hat{e}_m - \sum_{ij} \frac{\partial A_i}{\partial x_j} B_j \hat{e}_j + \sum_{ij} \frac{\partial B_j}{\partial x_j} \hat{e}_i.
\]

\(\text{(41)}\)
Comparing equations (40) and (41), we get
\[ \vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B}) + \text{irreducible term}, \quad (42) \]
where the irreducible term takes the expression
\[
\text{irreducible term} = -\sum_{wt}^{7} \sum_{ij}^{7} g_{ijw} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t
\]
\[ = -\sum_{wt}^{7} \sum_{ij}^{7} g_{ijw} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t
\]
\[ = \sum_{t}^{7} \hat{e}_t [\vec{\nabla} \cdot \{\hat{e}_t, \vec{A}, \vec{B} \}]. \quad (43) \]

Hence, we get
\[ \vec{\nabla} \times (\vec{A} \times \vec{B}) + \sum_{t}^{7} \hat{e}_t [\vec{\nabla} \cdot \{\hat{e}_t, \vec{A}, \vec{B} \}]
\[ = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B}). \quad (44) \]
Thus we see that the identity (39) is satisfied only when we have the reducible term vanishing and that can be obtained for different values of permutations of structure constant given by equation (3).

- Similarly, the other vector identity
\[ \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B}(\vec{\nabla} \cdot \vec{A}) = \vec{\nabla}(\vec{A} \cdot \vec{B}) \quad (45) \]
is not satisfied in general in seven dimensional and takes the following expression
\[ \vec{A} \times (\vec{\nabla} \times \vec{B}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A}
\[ = \vec{\nabla}(\vec{A} \cdot \vec{B}) + \text{irreducible term}, \quad (46) \]
where the irreducible term is described as
\[
\text{irreducible term} = -\sum_{ikst}^{7} g_{ikst} A_i \frac{\partial B_t}{\partial x_s} \hat{e}_k + \sum_{ikst}^{7} g_{ikst} B_i \frac{\partial A_t}{\partial x_s} \hat{e}_k
\]
So using the alternativity of octonions for allowed permutations of octonions, we get

\[
\tilde{A} \times (\tilde{\nabla} \times \tilde{B}) + \tilde{B} \times (\tilde{\nabla} \times \tilde{A}) + (\tilde{A} \cdot \tilde{\nabla})\tilde{B} + \tilde{B}(\tilde{\nabla} \cdot \tilde{A}) \\
= \tilde{\nabla}(\tilde{A} \cdot \tilde{B}) - \sum_k \hat{e}_k [\tilde{A} \cdot \{\hat{e}_k, \tilde{\nabla}, \tilde{B}\}] + \sum_k \hat{e}_k [\tilde{B} \cdot \{\hat{e}_k, \tilde{\nabla}, \tilde{A}\}].
\]  

where the irreducible term has been terminated for the different allowed permutations of structure constant discussed above.

From the foregoing analysis, we may conclude that the generalization of three dimensional vector space to seven dimensional vector space needs some modifications. The scalar product of three different vectors retain their form in seven dimensional space but the vector triple product of three different vectors looses its usual form of three dimensional space in seven dimensional vector space. It regains its usual form only for certain combination of allowed value of permutations of structure constant \( f_{ijk} \) of octonion units. It means when three different vectors are mutually rotated in seven dimensional space their rotation is only possible in certain directions as allowed by the combinations of basis vectors. Similarly, the mutual rotation of four vectors also allowed for certain combination of basis vectors. Consequently, the physical variables (scalar or vector) which are solenoidal and irrotational in three dimensional vector space are also solenoidal and irrotational in seven dimensional vector space. On the other hand, there are vector identities which changes their forms in seven dimensional space and regain their usual form under allowed permutations of structure constant.

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