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AEROSOL TRANSPORT BY TURBULENT CONTINUA

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Abstract

The stochastic transport equations, derived rigorously under the condition of continuum fluctuations in the framework of an ensemble theory, both in differential and integral form, are then verified by establishing an unambiguous connection between this stochastics and the associated deterministics.

1. Introduction

Aerosol transport through turbulent continua is characterized by the fact that aerosols can only follow movements of fluid elements if they neither fall below nor exceed a certain size and weight. This is in contrast to molecular diffusion through matter, which is often successfully accomplished by known diffusion equations. This physical process is

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fundamentally different from turbulent aerosol transport. In the former case, the diffusing molecules have an intrinsic motion between two interactions, while in the latter case, only predetermined paths are followed.

These predetermined paths must correspond to a continuum system in which fluid elements follow the collective motions of many individual molecules moving locally apparently independently. I.e., fluid elements, which in their totality represent a fluctuating continuum, and their paths are abstract quantities and not points of matter.

First, a fluid and turbulent fluid continuum is defined. According to this, a purely stochastic aerosol transport is excluded and a stochastic ensemble consideration is developed.

The stochastic transport equations, derived rigorously under the condition of continuum fluctuations in the framework of an ensemble theory, both in differential and integral form, are then verified by establishing an unambiguous connection between this stochastics and the associated deterministics.

This theory does not use any known models or hypothetical approaches to turbulence, but rather avoids such even in the awareness that turbulence is considered an unsolved problem. Thus, it starts first with a definition of the moving as well as a turbulent moving continuum. The described turbulent aerosol transport is based only on necessary conditions to corresponding continuous conditions of the aerosol movements, which are verified by deterministic equations of motion of turbulence derived from this in the most general way possible.

The equations found in this way do not yet represent a complete system of equations, so that the discussion about the pros or cons of, e.g., the Navier-Stokes equations is still missing in this article. However, it will be thoroughly made up for in a follow-up article.

2. Definition of a Moved Fluid

2.1. Definition of a fluid element

At every time, space points $(\vec{x})$ are assigned to fluid elements in a unique correspondence. As this applies to every space point $(\vec{x})$ of the fluid field, the set of fluid elements is seen as a continuum. A Continuum of fluid element points (simply called fluid elements) is considered, where a fluid environment of non infinitesimal size is uniquely allocated to every fluid element point. Two infinitesimally neighboring fluid elements differ apart from their distance by their velocities and not quite identical material distributions of their neighborhoods. The neighborhoods of two nearby fluid elements overlap. A fluid element is shifted moving the material of its neighborhood. Though the material of such a fluid element may have changed marginally after an infinitesimal time interval t_ε, it can be identified principally by its prior material status. As every molecule possesses its own identity, there has to be at least an infinitesimally greater difference of material distribution to the neighborhoods of other fluid elements.

The neighborhoods exchange material with neighborhoods of adjacent fluid elements and vary their thermodynamic state (a local thermodynamic state does not necessarily exist). Their size is not infinitesimal, because a local thermodynamic state (if physically existent) has to be detectable at least in thought experiment. The open neighborhoods have equally sized spherical shapes, generally. Near a solid border they are described by parts of spheres. Infinitesimally adjacent fluid elements possess overlapping neighborhoods. In an ε-surrounding they move in parallel. So one obtains a fluid, which is assumed to be a dense fluctuating point set, though there is no continuous matter distribution in Space-Time. That means it is possible to follow theoretically the history of every fluid element, though it has exchanged a lot of its initial material altering its local thermodynamic state.

The fluid is an abstract, dense set of fluctuating fluid elements, which do not generally correspond to material points. A continuum of moved fluid elements is considered each uniquely assigned to a neighborhood and a velocity.

$$\vec{\mathbf{v}}_{t_\varepsilon} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_\varepsilon}. \qquad (1)$$

The fluid element first determined in space point $\vec{\mathbf{x}}_1$ and t_ε-time later detected at $\vec{\mathbf{x}}_2$ is identified having at time $t_0 + t_\varepsilon$ in $\vec{\mathbf{x}}_2$ in comparison to all other points $\vec{\mathbf{x}}$ the most similar material to that of $\vec{\mathbf{x}}_1$ in t_0. In this context it is noted, that parts of the individual aerosols or molecules may be identified, too. The accuracies of the considered motion quantities are determined by t_ε-measurement processes. t_ε characterising the accuracy. According to a process $\lim\, t_\varepsilon \to 0$, the fluid elements move along trajectories that can a sufficient number of times be continuously differentiated forming a continuum as a whole. This continuum has a velocity vector field with $rot(\vec{\mathbf{v}}) \neq 0$ generally.[1] Though $rot(\vec{\mathbf{v}})$ has dimension $[1/\text{sec}]$ in the laminar case it does not refer to a rotation.

2.2. The orthogonality of $rot(\vec{\mathbf{v}})$ and $\vec{\mathbf{v}}$ is a consequence of the moved fluid continuum

A fluid continuum is characterized by

1. continuously differentiable velocities,

2. parallel velocities in an ε-surrounding of a space point $\vec{\mathbf{x}}$.

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[1]In english literature $curl(\vec{\mathbf{v}}) \neq 0$ is used but in turbulence the name *rot* is more adapted as will be seen.

Considering without loss of generality a fluid movement of velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ in a space point $\vec{\mathbf{x}}_0$ in cartesian coordinates, the velocity is described in an ε-neighborhood and parallel to the x-coordinate as follows:

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}) = \begin{pmatrix} \vec{\mathbf{v}}_x(\vec{\mathbf{x}}_0) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \left.\frac{\partial \vec{\mathbf{v}}_x}{\partial x}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_x}{\partial y}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_x}{\partial z}\right|_{\vec{\mathbf{x}}_0} \\ \left.\frac{\partial \vec{\mathbf{v}}_y}{\partial x}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_y}{\partial y}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_y}{\partial z}\right|_{\vec{\mathbf{x}}_0} \\ \left.\frac{\partial \vec{\mathbf{v}}_z}{\partial x}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_z}{\partial y}\right|_{\vec{\mathbf{x}}_0} & + & \left.\frac{\partial \vec{\mathbf{v}}_z}{\partial z}\right|_{\vec{\mathbf{x}}_0} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \dots .$$

The velocity components $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ **osculate** at the velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ spatially approaching (constant time t_0),

$$\mathbf{v}_y(x_0, y, z_0) \to \mathbf{v}_y(x_0, y_0, z_0) = 0,$$

$$\mathbf{v}_x(x_0, y, z) \to \mathbf{v}_z(x_0, y_0, z_0) = 0.$$

That means especially, that all the partial derivations by y- or z-coordinate of 1. Order of $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ disappear in the point (x_0, y_0, z_0).

$$\lim_{z \to z_0} \left.\frac{\Delta \mathbf{v}_y}{\Delta z}\right|_{\vec{\mathbf{x}}_0} = \lim_{y \to y_0} \left.\frac{\Delta \mathbf{v}_z}{\Delta y}\right|_{\vec{\mathbf{x}}_0} = 0,$$

$$\vec{\mathbf{x}}_0 = (x_0, y_0, z_0). \tag{2}$$

Applying the differential quotients in the $\vec{\nabla} \times$-operator expressed in cartesian coordinates gives for the fluid velocity

$$(\vec{\nabla}\times\vec{\mathbf{v}})_{|\vec{\mathbf{x}}_0} = \begin{pmatrix} 0 \\ \dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x} \\ \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{pmatrix}_{|\vec{\mathbf{x}}_0} , \quad \vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x,\ 0,\ 0) \tag{3}$$

$\Rightarrow$

The orthogonality of $\vec{\nabla}\times\vec{\mathbf{v}} \perp \vec{\mathbf{v}}$ is a fundamental quality[2][3] and a necessary condition for continuous fluid flow.

In this orthogonality velocity vector fields differ from deformation vector fields.

3. Definition of a Turbulent Fluid

Trying to identify the state of movement of a fluid element in turbulent fluids by a velocity $\vec{\mathbf{v}}_{t_\varepsilon}$ it should be recognized, that the state of movement is not yet determined, as the path in every space point (except in turning points) is uniquely adapted by an infinitesimal circle segment. In the infinitesimal neighborhood of a path point the velocity is identified by an instantaneous axis of rotation $\vec{\omega}_{t_\varepsilon}$ and a radius vector $\vec{\mathbf{r}}_{t_\varepsilon}$.[4]

$$\vec{\mathbf{v}}_{t_\varepsilon} = \vec{\omega}_{t_\varepsilon} \times \vec{\mathbf{r}}_{t_\varepsilon}. \tag{4}$$

In a turbulently moved fluid the fluid elements move on curved trajectories in some space time points having turning points with $\vec{\omega}_{t_\varepsilon} = 0$

[2]This relationship is not to be found in literature, although it is obvious and mathematically not very demanding.

[3]This is one reason why the known millenium prize question does not lead to a solution of the turbulence problem. However the validity problem of the Navier-Stokes-equations is more fatal. So this is not a question for mathematics at first, but for physics.

[4]That is why turbulence can not be uniquely identified by experiments of local velocity statistics.

and a curvature vector $\vec{\mathbf{b}}_{t_\varepsilon} = 0$. The considered vectorial motion quantities $\vec{\omega}_{t_\varepsilon}$ and $\vec{\mathbf{r}}_{t_\varepsilon}$ are determined by t_ε- measurement processes, which are calculated later on by a limes process lim $t_\varepsilon \to 0$. A fluid element originating from the point $\vec{\mathbf{x}}_0$ crossing $\vec{\mathbf{x}}_1$ after the time t_ε reaches $\vec{\mathbf{x}}_2$ after a further time t_ε.

$$\vec{\mathbf{x}}_0 \xrightarrow{t_\varepsilon} \vec{\mathbf{x}}_1 \xrightarrow{t_\varepsilon} \vec{\mathbf{x}}_2.$$

A segment of a circle is clearly drawn through these 3 points with radius vector $\vec{\mathbf{r}}_{t_\varepsilon}$ and velocity of rotation $\vec{\omega}_{t_\varepsilon}$ in $\vec{\mathbf{x}}_1$, unless a turning point is passed through. The local state of motion can not be described by velocity only, neither statistically nor deterministically.[5]

Thus the fluid element in the space-time-point $(\vec{\mathbf{x}}, t)$ is identified principally by the contents of the matter of its neighborhood and state of movement expressed by $\vec{\omega}_{t_\varepsilon}$ and $\vec{\mathbf{r}}_{t_\varepsilon}$. In that way defined fluid elements move on sufficiently often continuously differentiable trajectories. At each instant they lead to a new continuum of fluctuating fluid elements with several times continuously differentiable velocity field. The continuum of moving fluid elements represents the turbulent collective motion of a discontinuously spaced Matter. This is the result of the connection between deterministics and stochastics in the sense of an ensemble theory, which is presented in the following:

The field of turbulence is described by the two vector fields $\vec{\omega}_{t_\varepsilon}$ and $\vec{\mathbf{b}}_{t_\varepsilon}$,

$$\vec{\mathbf{b}}_{t_\varepsilon} = \vec{\mathbf{r}}_{t_\varepsilon} \Big/ r_{t_\varepsilon}^2 \qquad \text{-curvature vector field.} \tag{5}$$

[5]This statement contradicts that of Wilczek [7].

In addition, the results show that

$$\vec{\omega}_{t_\varepsilon} = \frac{1}{2} rot(\vec{\mathbf{v}}_{t_\varepsilon}). \tag{6}$$

$rot(\vec{\mathbf{v}})$ has the meaning of a local rotation in the frame of turbulence. An infinitesimal disturbance of stationary pipe flow leads to an change of the significance of $rot(\vec{\mathbf{v}})$, where $rot(\vec{\mathbf{v}})$ does not correspond to a rotation initially. Whether starting motions of turbulence are suppressed, depends on an existent viscosity. These decelerations are generally weak. The beginning of turbulent movements avoid Newtonian friction as well as pressure gradients by means of hereto orthogonal motions.

Vortex fields in turbulence (local rotation fields will be identified with vortex fields) and radius fields may have turning points along the paths of the fluid elements, which means $\vec{\omega} = 0$ and $\vec{\mathbf{r}} = \infty$.[6] In this case the velocities are to be calculated by interpolation or extrapolation from the neighborhood. The fluid elements are accompanied by a moving frame of $\vec{\omega}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{v}}$ along their paths.

Fluid elements, at a time are infinitesimally adjacent, have later moved away from each other and represent with new neighbors a new continuum. However, since also their material environments have changed, their past and future stay is to be determined only from the knowledge of the perfect spatiotemporal movement field. To calculate these fields, a system of equations is needed that couples other independent fields, such as the acceleration field.

Independent Lagrangian turbulence calculations are not possible.

4. Definition of Markov Processes with Natural Causality

The probabilistic theory is related to random distributions of

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[6]The temporal and spatial neighborhood of a turning point does not have such singular properties.

velocities $\vec{\pi}$ moving from $(\vec{\mathbf{x}}, t)$ to $(\vec{\mathbf{x}} + \vec{\pi} t_\varepsilon, t + t_\varepsilon)$. These velocity distributions may get together of vortex and curvature vector fields

$$\vec{\pi} = \vec{\omega} \times \frac{\vec{\mathbf{b}}}{b^2}.$$

The transport from $(\vec{\mathbf{x}} - t_\varepsilon \vec{\pi}', t - t_\varepsilon)$ to $(\vec{\mathbf{x}}, t)$ is additionally controlled by transition probabilities

$$W_{t_\varepsilon} = W_{t_\varepsilon}(\vec{\mathbf{x}}, t, \vec{\pi}, \vec{\pi}'),$$

resulting in

$$f_{t_\varepsilon}(\vec{\mathbf{x}}, t, \vec{\pi}) = \int_{\vec{\pi}'} W_{t_\varepsilon}(\vec{\mathbf{x}}, t, \vec{\pi}, \vec{\pi}') f_{t_\varepsilon}(\vec{\mathbf{x}} - t_\varepsilon \vec{\pi}', t - t_\varepsilon, \vec{\pi}') d\vec{\pi}'.$$

Such a relation we call a Markov Process of natural causality. According to Sen [5] there is a so called Newtonian causality in nonrelativistic physics implying the possibility of unlimited velocities. However, Newtonian causality is restricted to Newtonian mechanics and stochastic processes of physics ending with diffusion equations when applied practically.[7] This applies not for formulations of the general or linear Boltzmann Equation. In electrodynamics the velocity of light is the limiting velocity. The Newtonian causality proves to be a limiting case of non relativistic classical physics. Subsequently a causal Markov Process is continuously used or derived. Overarching master equations can not exist, physically. The transition probabilities W_{t_ε} depend on a time quantity t_ε related to continuum fluctuations of measurement accuracy according to vectorial motion quantities. For $t_\varepsilon \to 0$ (exact motion quantities) the transition probability W_{t_ε} degenerates to a δ- function.

Simultaneous details of space and momentum are not possible in the

[7]This statement applies to the Fokker-Planck and Langevin equation. See, for example, Chandrasekhar [1].

context of quantum mechanics. The Schrödinger Equation for free "quantum particles"

$$i\hbar \frac{\partial\psi(\vec{\mathbf{x}}, t)}{\partial t} = -\frac{\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{\mathbf{x}}, t) \tag{7}$$

can be transformed into a linear homogenuous integral eqution [2] [3]

$$\psi(\vec{\mathbf{x}}, t) = i\int G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t')\psi(\vec{\mathbf{x}}', t')d\vec{\mathbf{x}}'. \tag{8}$$

The Green function

$$G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t') = \left\langle \vec{\mathbf{x}} \middle| \exp\left(-\frac{i}{\hbar}(t - t')\mathbf{H}\right) \middle| \vec{\mathbf{x}}' \right\rangle \tag{9}$$

is called Feynman kernel, too.

In the case of the diffusion equation

$$\frac{\partial\rho(\vec{\mathbf{x}}, t)}{\partial t} = D\vec{\nabla}^2\rho(\vec{\mathbf{x}}, t) \tag{10}$$

an equivalent integral equation the Green function understood as transition probability from $(\vec{\mathbf{x}}', t')$ to $(\vec{\mathbf{x}}, t)$ exists with

$$\rho(\vec{\mathbf{x}}, t) = \int_{V'} G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t')\rho(\vec{\mathbf{x}}', t')d\vec{\mathbf{x}}' \tag{11}$$

and the Green function

$$G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', t') = \left(\frac{1}{4\pi D(t - t')}\right)^{\frac{3}{2}} e^{-\frac{(\vec{\mathbf{x}} - \vec{\mathbf{x}}')^2}{4\pi D(t - t')}}. \tag{12}$$

Equations based on a "heat-kernel"-structure are not exact in classical physics (as well as the Newtonian mechanics). They are usually referred to as Markov processes.

In quantum mechanics and quantum field theory natural causality is not possible because of the uncertainty principle. In Relativity there is the maximal possible velocity, the velocity of light.

4.1. Further remarks on the difference between diffusion transport and aerosol transport

The diffusion process is a purely stochastic one, whereas the aerosol transport is due to a deterministic process. The sometimes in descriptions still assumed superimposed shear flow is supposed to shift the diffusion process spatially and temporally according to the shear flow. The assumed aerosols, however, move according to the turbulent continuum fluctuations. The developed stochastic of the aerosols takes place only in the sense of an ensemble consideration, which assumes arbitrarily many similar parallel existing continuum fluctuations in the thought experiment. From this then a distribution function $f_{t_\varepsilon} = f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r})$ arises for each space-time $(\vec{x}, t)$.

However, even in the case of particle diffusion, the diffusion equation turns out to be less than a first approximation to a sometimes most accurate equation, the linear Boltzmann equation, without necessarily involving significant numerical deficits. These relationships are a truism [6] in neutron transport in nuclear reactor physics, but hardly known outside this field of physics, not even in nuclear physics, with which the author once started.

5. Stochastic Transport of Aerosols by turbulent Continuum-Fluctuations

5.1. Introduction

The motion of passive aerosols by turbulent continuum fluctuations is examined. The aerosols are moved not affecting this field. Their trajectories correspond in every ε-neighborhood of a point to a circle segment passed with the velocity

$$\vec{\mathbf{v}}_{t_\varepsilon} = \vec{\omega}_{t_\varepsilon} \times \vec{r}_{t_\varepsilon}, \quad \vec{\omega}_{t_\varepsilon} \perp \vec{r}_{t_\varepsilon}. \tag{13}$$

The considered motion quantities $\vec{\omega}_{t_\varepsilon}$ and $\vec{r}_{t_\varepsilon}$ are determined in the thought experiment by finding the successive positions of a single aerosol moving from a point $\vec{\mathbf{x}}_0$ after a time t_ε to $\vec{\mathbf{x}}_1$ and another time t_ε to $\vec{\mathbf{x}}_2$. By these 3 points a circle segment is uniquely defined for the point $\vec{\mathbf{x}}_1$ with radius vector $\vec{r}_{t_\varepsilon}$ and a rotation speed $\vec{\omega}_{t_\varepsilon}$.

$$\vec{r}_{t_\varepsilon} = r_{t_\varepsilon} \cdot \vec{\Theta}_{t_\varepsilon},$$

$$\vec{\omega}_{t_\varepsilon} = \omega_{t_\varepsilon} \cdot \vec{\Omega}_{t_\varepsilon}. \tag{14}$$

In the special case $\vec{\omega}_{t_\varepsilon} \to 0$ and $\vec{r} \to +\infty$ the velocity $\vec{\mathbf{v}}_{t_\varepsilon}$ is revealed out of its neighborhood.[8] The aerosol density distributions are received in a thought experiment by an unlimited number of deterministic ensemble-systems. In every point $(\vec{\mathbf{x}}, t)$ a continuously differentiable aerosol density distribution of the motion quantities $\vec{\omega}_{t_\varepsilon}$ and $\vec{r}_{t_\varepsilon}$ is assigned in accordance with

$$f_{t_\varepsilon} = f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}). \tag{15}$$

The with t_ε indexed functions are automatically assumed to contain motion quantities of corresponding measurement accuracies. The indexing of the motion quantities can be omitted if the functions are indexed. After execution of a limiting process

$$\lim_{t_\varepsilon \to 0} f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}) = f(\vec{x}, t, \vec{\omega}, \vec{r}). \tag{16}$$

f and $(\vec{\omega}, \vec{r})$ are understood according to an exact measuring process. Integrating the aerosol density distribution over the motion quantities one obtains expectation values of a aerosol density not conforming with the actual aerosol density ρ.

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[8]Applying the deterministic theory this problem must be treated numerically.

$$\langle \rho t_\varepsilon(\mathbf{\vec{x}}, t)\rangle = \int_{2\pi}\int_{4\pi}\int_0^\infty\int_0^\infty f_{t_\varepsilon}\left(\vec{x}, t, \omega \cdot \vec{\Omega}, r \cdot \vec{\Theta}\right) d\omega dr d\vec{\Omega} d\vec{\Theta} \neq \rho t_\varepsilon(\mathbf{\vec{x}}, t). \quad (17)$$

A rigorously derived partial differential equation is obtained, which can be used to calculate the evolution of the spatiotemporal aerosol density distributions. The initially unbounded number of unknown coefficients is attributed to local time scaling. The abstractly formulated transition probabilities get concrete functional dependencies.

5.2. The transport as Markov process with natural causality

An aerosol at location $\vec{x}$ and time t changing its velocity from $\vec{v}' = (\vec{\omega}', \vec{r}')$ to $\vec{v} = (\vec{\omega}, \vec{r})$ is given by the transition probability

$$W_{t_\varepsilon} = W_{t_\varepsilon}(\vec{x}, t; \vec{\omega}, \vec{r}; \vec{\omega}', \vec{r}') \quad (18)$$

with

$$\int_0^\infty\int_0^\infty\int_{4\pi}\int_{2\pi} W_{t_\varepsilon}(\vec{x}, t; \vec{\omega}, \vec{r}; \vec{\omega}', \vec{r}') d\omega' dr' d\Omega' d\Theta' = 1. \quad (19)$$

$\Rightarrow$

$$f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}) = \int_0^\infty\int_0^\infty\int_{4\pi}\int_{2\pi} W_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}, \vec{\omega}', \vec{r}')$$

$$f_{t_\varepsilon}(\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon, \vec{\omega}', \vec{r}', t - t_\varepsilon) d\omega' dr' d\Omega' d\Theta'. \quad (20)$$

Continuity is required, respectively, of all variables of the transition probability W_{t_ε}. The sequence of velocities $\vec{v}'_{t_\varepsilon}$, $\vec{v}_{t_\varepsilon}$ means a motion from

$$\left(\vec{x} - \vec{\omega}'_{t_\varepsilon} \times \vec{r}'_{t_\varepsilon} \cdot t_\varepsilon, \vec{\omega}', \vec{r}', t - t_\varepsilon, \vec{\omega}'_{t_\varepsilon} \times \vec{r}'_{t_\varepsilon}\right) \quad \text{to} \quad \left(\vec{x}, t, \vec{\omega}'_{t_\varepsilon} \times \vec{r}'_{t_\varepsilon}\right). \quad (21)$$

For the limiting process $t_\varepsilon \to 0$ the transition probabilities W_{t_ε} prove to be physical realizations of test functions of the distribution theory.

$$\lim_{t_\varepsilon \to 0} W_{t_\varepsilon} = (\vec{\omega}, \vec{r}; \vec{\omega}', \vec{r}'). \quad (22)$$

The passive scalar aerosols precisely reproduce the motions of the fluctuation field. For the aerosol density distribution $f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r})$ the following separation approach is used without loss of generality:

$$f_{t_\varepsilon}(\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon, t, \vec{\omega}', \vec{r}_\varepsilon')$$

$$= G_{t_\varepsilon}(\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon, t, \vec{\omega}', \vec{r}_\varepsilon')\bar{f}_{t_\varepsilon}\left(\vec{x} - \bar{v}\vec{\Omega}' \times \vec{\Theta}' \cdot t_\varepsilon, t\ \vec{\Omega}', \vec{\Theta}'\right) \tag{23}$$

with

$$\int_0^\infty \int_0^\infty G_{t_\varepsilon}\left(\vec{x}, t, \omega\vec{\Omega}, r\vec{\Theta}\right) d\omega dr = 1,$$

$$\int_0^\infty \int_0^\infty G_{t_\varepsilon}\left(\vec{x}, t, \omega\vec{\Omega}, r\vec{\Theta}\right) \omega r d\omega dr = \bar{v}\left(\vec{x}, t, \vec{\Omega}, \vec{\Theta}\right),$$

$$\bar{v}\left(\vec{x}, t, \vec{\Omega}, \vec{\Theta}\right) = \varpi\left(\vec{x}, t, \vec{\Omega}, \vec{\Theta}\right), \tag{24}$$

$\Rightarrow$

$$\bar{f}_{t_\varepsilon}\left(\vec{x} - \bar{v}\vec{\Omega} \times \vec{\Theta} \cdot t_\varepsilon, t, \vec{\Omega}, \vec{\Theta}\right)$$

$$= \int_0^\infty \int_0^\infty f_{t_\varepsilon}\left(\vec{x} - \vec{\omega} \times \vec{r} \cdot t_\varepsilon, t, \omega \cdot \vec{\Omega}, r \cdot \vec{\Theta}\right) d\omega dr. \tag{25}$$

One obtains a transition probability $\overline{W}_{t_\varepsilon}$ only depending on the directions by integrating W_{t_ε} over the amounts ω', r', ω, r.

$$\overline{W}_{t_\varepsilon}\left(\vec{x}, t, \vec{\Omega}, \vec{\Theta}, \vec{\Omega}', \vec{\Theta}'\right)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty W_{t_\varepsilon} G_{t_\varepsilon}(\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon, t - t_\varepsilon, \vec{\omega}', \vec{r}') d\omega' dr' d\omega dr. \tag{26}$$

The integration

$$\int_0^\infty \int_0^\infty (20) d\omega dr \tag{27}$$

gives

$$\bar{f}_{t_\varepsilon}\left(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta}\right)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_{4\pi} \int_{2\pi} \overline{W_{t_\varepsilon}} \bar{f}_{t_\varepsilon} (\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon,\, t - t_\varepsilon,\, \vec{\omega}',\, \vec{r}')$$

$$.d\omega' dr' d\omega dr d\vec{\Omega}' d\vec{\Theta}'. \qquad (28)$$

$$\Rightarrow \bar{f}_{t_\varepsilon}\left(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta}\right) = \int_{4\pi} \int_{2\pi} \overline{W_{t_\varepsilon}} \bar{f}_{t_\varepsilon} \left(\vec{x} - \bar{v}' \vec{\Omega}' \times \vec{\Theta}' \cdot t_\varepsilon,\, t - t_\varepsilon,\, \vec{\Omega}',\, \vec{\Theta}'\right) d\vec{\Omega}' d\vec{\Theta}'. \quad (29)$$

In the integrand $\bar{f}_{t_\varepsilon}$ is developed around $\vec{x}$ and t:

$$\bar{f}_{t_\varepsilon}\left(\vec{x} - \Delta\vec{x},\, t - t_\varepsilon,\, \vec{\Omega}',\, \vec{\Theta}'\right)$$

$$= \bar{f}_{t_\varepsilon}\left(\vec{x},\, t,\, \vec{\Omega}',\, \vec{\Theta}'\right) - \tau_E \cdot \varepsilon \cdot \left[\frac{\partial \bar{f}'_{t_\varepsilon}}{\partial t} + \bar{v}' \vec{\Omega}' \times \vec{\Theta}' \cdot \nabla \bar{f}'_{t_\varepsilon} + O(\varepsilon^2)\right]. \qquad (30)$$

This leads to

$$\int_{4\pi} \int_{2\pi} \overline{W_{t_\varepsilon}} \bar{f}_{t_\varepsilon} d\vec{\Omega}' d\vec{\Theta}' - \bar{f}_{t_\varepsilon} = \int_{4\pi} \int_{2\pi} \left(\vec{x},\, t,\, \vec{\Omega}',\, \vec{\Theta}',\, \vec{\Omega},\, \vec{\Theta}\right) \cdot \tau_E$$

$$\left[\cdot \frac{\partial \bar{f}'_{t_\varepsilon}}{\partial t} + \bar{v}' \vec{\Omega}' \times \vec{\Theta}' \cdot \nabla \bar{f}'_{t_\varepsilon} + O(\varepsilon^2)\right] d\vec{\Omega}' d\vec{\Theta}'. \quad (31)$$

As

$$\lim_{t_\varepsilon \to 0} \overline{W_{t_\varepsilon}} = \delta\left(\vec{\Omega},\, \vec{\Theta},\, \vec{\Omega}',\, \vec{\Theta}'\right) \qquad (32)$$

$\Rightarrow$

$$\lim_{\varepsilon \to 0} \frac{\int_{4\pi} \int_{2\pi} \overline{W_{t_\varepsilon}} \bar{f}_{t_\varepsilon} d\vec{\Omega}' d\vec{\Theta}' - \bar{f}_{t_\varepsilon}}{\varepsilon \cdot \tau_E} = \frac{\partial \bar{f}}{\partial t} + \bar{v} \vec{\Omega} \times \vec{\Theta} \cdot \nabla \bar{f}'. \qquad (33)$$

Further on

$$\lim_{\varepsilon\to 0}\frac{\int_{4\pi}\int_{2\pi}\overline{W}_{t_\varepsilon}\bar{f}_{t_\varepsilon}d\vec{\Omega}'d\vec{\Theta}' - \bar{f}_{t_\varepsilon}}{\varepsilon\cdot\tau_E} \tag{34}$$

is called **exchange-term**.

5.3. Calculation of the Exchange-Term

Exchange term dependencies of scalar products $\vec{\Omega}\cdot\vec{\Omega}'$ and $\vec{\Theta}\cdot\vec{\Theta}'$ are taken into account instead of individually depending directions $\vec{\Omega}$, $\vec{\Omega}'$ and $\vec{\Theta}$, $\vec{\Theta}'$ demanding the following relation

$$\lim_{\varepsilon\to 0}\frac{\int_{4\pi}\int_{2\pi}\overline{W}_{t_\varepsilon}\bar{f}_{t_\varepsilon}d\vec{\Omega}'d\vec{\Theta}' - \bar{f}_{t_\varepsilon}}{\varepsilon\cdot\tau_E}$$

$$= \lim_{\varepsilon\to 0}\frac{\int_{4\pi}\int_{2\pi}\widetilde{W}_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right)\bar{f}_{t_\varepsilon}d\vec{\Omega}'d\vec{\Theta}' - \bar{f}_{t_\varepsilon}}{\varepsilon\cdot\tau_E}. \tag{35}$$

The following transitions

$$\tau_E = \text{const.} \to t_E = t_E\left(\vec{x},\ t,\ \vec{\Omega},\ \vec{\Theta}\right),$$

$$W_{t_\varepsilon}\left(\vec{x},\ t,\ \vec{\Omega},\ \vec{\Theta},\ \vec{\Omega}',\ \vec{\Theta}'\right) \to \widetilde{W}_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right) \tag{36}$$

are regarded. Moreover, a separation of $\vec{\Omega}\cdot\vec{\Omega}'$ and $\vec{\Theta}\cdot\vec{\Theta}'$ is assumed:

$$\widetilde{W}_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right) = V_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}'\right)\cdot M_{t_\varepsilon}\left(\vec{\Theta}\cdot\vec{\Theta}'\right). \tag{37}$$

Functions of the unit vectors $\vec{\Omega}$ and $\vec{\Theta}$ are presented by a complete orthogonal function system representing an extension of the spherical harmonics called turbulence functions.

$$\bar{f}_{t_\varepsilon}\left(\vec{x},\ t,\ \vec{\Omega},\ \vec{\Theta}\right) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\bar{f}_{t_\varepsilon lmk}(\vec{x}\ \ t)Q_{lmk}\left(\vec{\Omega},\ \vec{\Theta}\right)$$

$$= \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \bar{f}_{t_\varepsilon lmk}(\vec{x}\ t) Q^*_{lmk}(\vec{\Omega}, \vec{\Theta}), \tag{38}$$

$$\int_{2\pi} \int_{4\pi} Q_{lmk}(\vec{\Omega}, \vec{\Theta}) Q^*_{lmk}(\vec{\Omega}', \vec{\Theta}') d\vec{\Omega}' d\vec{\Theta}'$$

$$= \begin{cases} \dfrac{8\pi^2}{2l+1} \text{ for } l = l' \text{ and } m = m', \\ 0 \text{ else} \end{cases} \tag{39}$$

with

$$Q_{lmk}(\vec{\Omega}, \vec{\Theta}) = P_{lm}(\vec{\Omega}) H_k(\vec{\Theta}'),$$

$$\int_{2\pi} H_{k'}(\vec{\Theta}) H^*_k(\vec{\Theta}) d\vec{\Theta} = \begin{cases} 2\pi \text{ for } k' = k, \\ 0 \text{ else,} \end{cases}$$

$$H_k(\vec{\Theta}) = e^{ik\theta}. \tag{40}$$

The product $\vec{\Omega} \cdot \vec{\Omega}'$ in the separated exchange function V_{t_ε} is developed by spherical harmonics.

$$V_{t_\varepsilon}(\vec{\Omega}' \cdot \vec{\Omega}) = \sum_{l=0}^{+\infty} V_{t_\varepsilon l} P_l(\cos(\alpha)) = \sum_{l=0}^{+\infty} \sum_{m=-1}^{m=+l} P_{lm}(\vec{\Omega}') P^*_{lm}(\vec{\Omega}),$$

$$\lim_{t_\varepsilon \to 0} V_{t_\varepsilon}(\vec{\Omega}' \cdot \vec{\Omega}) = \delta(\vec{\Omega}, \vec{\Omega}'). \tag{41}$$

The product $\vec{\Theta} \cdot \vec{\Theta}'$ in the separated exchange function M_{t_ε} is developed by functions H_k.

$$M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) = \sum_{k=0}^{+\infty} M_{t_\varepsilon k} \cos(k\beta)$$

$$= \frac{1}{2} \sum_{k=0}^{+\infty} M_{t_\varepsilon k} \left[H_k(\vec{\Theta}') H^*_k(\vec{\Theta}) + H_{-k}(\vec{\Theta}') H^*_{-k}(\vec{\Theta}) \right] \tag{42}$$

with

$$\cos(k\beta) = \frac{1}{2}\left[H_k(\vec{\Theta}')H_k^*(\vec{\Theta}) + H_{-k}(\vec{\Theta}')H_{-k}^*(\vec{\Theta})\right] = \frac{1}{2}\left[e^{ik(\theta'-\theta)} + e^{-ik(\theta'-\theta)}\right],$$

$$\vec{\Theta}' \cdot \vec{\Theta} = \cos(\beta) = \cos(\theta' - \theta) = \frac{1}{2}\left[H_1(\vec{\Theta}')H_1^*(\vec{\Theta}) + H_{-1}(\vec{\Theta}')H_{-1}^*(\vec{\Theta})\right]$$

$$= \frac{1}{2}\left[e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}\right],$$

$$\lim_{t_\varepsilon \to 0} M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) = \delta(\vec{\Theta}, \vec{\Theta}'), \tag{43}$$

$\Rightarrow$

$$\int_{4\pi}\int_{2\pi} \tilde{W}_{t_\varepsilon} \bar{f}'_{t_\varepsilon} d\vec{\Omega}' d\vec{\Theta}'$$

$$= \int_{4\pi}\int_{2\pi} V_{t_\varepsilon}(\vec{\Omega}' \cdot \vec{\Omega}) \cdot M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) \bar{f}'_{t_\varepsilon} d\vec{\Omega}' d\vec{\Theta}'$$

$$= \int_{4\pi}\int_{2\pi}\left[\left\{\sum_{l=0}^{+\infty} V_{t_\varepsilon l} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\Omega}')P_{lm}^*(\vec{\Omega})\right.\right.$$

$$\left. \cdot \frac{1}{2}\sum_{k=0}^{k=+\infty} M_{t_\varepsilon k}\left[H_k(\vec{\Theta}')H_k^*(\vec{\Theta}) + H_{-k}(\vec{\Theta}')H_{-k}^*(\vec{\Theta})\right]\right\}$$

$$\left. \cdot \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty} \bar{f}'_{t_\varepsilon lmk}(\vec{x}, t)P_{lm}^*(\vec{\Omega}')H_k^*(\vec{\Theta}')\right] d\vec{\Omega}' d\vec{\Theta}'$$

$$= \sum_{l=0}^{+\infty} V_{t_\varepsilon l} \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} P_{lm}^*(\vec{\Omega}) \sum_{k=0}^{+\infty} M_{t_\varepsilon k} 2\pi \bar{f}_{t_\varepsilon lmk}(\vec{x}, t) H_k^*(\vec{\Theta}). \tag{44}$$

Finally the **exchange-term** results in

$$\lim_{\varepsilon \to 0} \frac{\int_{4\pi}\int_{2\pi} \tilde{W}_{t_\varepsilon} \bar{f}'_{t_\varepsilon} d\vec{\Omega}' d\vec{\Theta}' - \bar{f}_{t_\varepsilon}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{\left(V_{t_\varepsilon l} \frac{4\pi}{2l+1} M_{t_\varepsilon k} 2\pi - 1 \right)}{\varepsilon} \bar{f}_{t_\varepsilon lmk}(\vec{x}, t) P_{lm}^{*}(\vec{\Omega}) H_k^{*}(\vec{\Theta})$$

$$= \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \Upsilon_{lk} \bar{f}_{lmk}(\vec{x}, t) P_{lm}(\vec{\Omega}) H_k(\vec{\Theta}). \quad (45)$$

With the **exchange coefficients**

$$\Upsilon_{lk} \lim_{\varepsilon \to 0} \frac{\left(V_{t_\varepsilon l} \frac{4\pi}{2l+1} M_{t_\varepsilon k} 2\pi - 1 \right)}{\varepsilon} \quad (46)$$

the transport equation

$$\frac{\partial \bar{f}}{\partial t} + \bar{v} \vec{\Omega} \times \vec{\Theta} \cdot \nabla \bar{f} = \frac{1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \Upsilon_{lk} \bar{f}_{lmk}(\vec{x}, t) P_{lm}(\vec{\Omega}) H_k(\vec{\Theta}) \quad (47)$$

is achieved. Further on it is shown that in Υ_{lk} the index k may be skipped.

5.4. Calculation of the Exchange-Coefficients Υ_l

Considering an overall closed volume range $\mathbb{V}$ the aerosol number in the entire volume remains constant if no absorption is assumed.

$$\text{total number of aerosols} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \bar{f} d\vec{\Omega} d\vec{\Theta} d\mathbb{V} = const. \quad (48)$$

$\Rightarrow$

$$\frac{d}{dt} \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \bar{f} d\vec{\Omega} d\vec{\Theta} d\mathbb{V} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \left[\frac{\partial \bar{f}}{\partial t} + \bar{v} \vec{\Omega} \times \vec{\Theta} \cdot \nabla \bar{f} \right] d\vec{\Omega} d\vec{\Theta} d\mathbb{V}$$

$$= \Upsilon_{00} \cdot V = 0 \quad (49)$$

and thus

$$\Upsilon_{00} = 0. \quad (50)$$

Getting an overview over the exchange function M_{t_ε} the essential relations are presented again with the following equations:

$$M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) = \sum_{k=0}^{+\infty} M_{t_\varepsilon k} \cos(k\beta)$$

$$= \frac{1}{2}\sum_{k=0}^{+\infty} M_{t_\varepsilon k}\left[H_k(\vec{\Theta}')H_k^*(\vec{\Theta}) + H_{-k}(\vec{\Theta}')H_{-k}^*(\vec{\Theta})\right],$$

$$\cos(k\beta) = \frac{1}{2}\left[H_k(\vec{\Theta}')H_k^*(\vec{\Theta}) + H_{-k}(\vec{\Theta}')H_{-k}^*(\vec{\Theta})\right] = \frac{1}{2}\left[e^{ik(\theta'-\theta)} + e^{-ik(\theta'-\theta)}\right],$$

$$\vec{\Theta}' \cdot \vec{\Theta} = \cos(\beta) = \cos(\theta' - \theta)$$

$$= \frac{1}{2}\left[H_1(\vec{\Theta}')H_1^*(\vec{\Theta}) + H_{-1}(\vec{\Theta}')H_{-1}^*(\vec{\Theta})\right] = \frac{1}{2}\left[e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}\right],$$

$$\lim_{\varepsilon \to 0} M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) = \delta(\vec{\Theta}, \vec{\Theta}'),$$

$$\int_{2\pi} H_{k'}(\vec{\Theta}')H_k^*(\vec{\Theta})d\vec{\Theta} = \begin{cases} 2\pi \text{ for } k' = k, \\ 0 \text{ else.} \end{cases}$$

$M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) = \sum_{k=0}^{+\infty} M_{t_\varepsilon k} \cos(k\beta)$ only takes values essentially different from 0 in an ε-neighborhood of $\beta = 0$, such that $\vec{\Theta}' \cdot \vec{\Theta} = \cos(\beta) = 1 - O(\varepsilon^2)$ is sufficient.

$\Rightarrow$

$$2\pi \cdot M_{t_\varepsilon k} = \int_{-\pi}^{+\pi} M_{t_\varepsilon} \cos(k\beta) d\beta$$

$$= \int_{-\pi}^{+\pi} M_{t_\varepsilon k}(1 - O(\varepsilon))d\beta = 2\pi \cdot M_{t_\varepsilon 0} - O(\varepsilon^2). \tag{51}$$

On the other hand

$$\int_{2\pi} M_{t_\varepsilon}(\vec{\Theta}' \cdot \vec{\Theta}) d\vec{\Theta}'$$

$$= \frac{1}{2} \int_{2\pi} \sum_{k=0}^{k=+\infty} M_{t_\varepsilon k} \left[H_k(\vec{\Theta}') H_k^*(\vec{\Theta}) + H_{-k}(\vec{\Theta}') H_{-k}^*(\vec{\Theta}) \right] d\vec{\Theta}'$$

$$= 2\pi \cdot M_{t_\varepsilon 0} = 1 \tag{52}$$

is valid. $\Rightarrow$

$$\lim_{t_\varepsilon \to 0} M_{t_\varepsilon k} = M_{t_\varepsilon 0} = \frac{1}{2\pi}. \tag{53}$$

The calculation of the **exchange coefficients** is not influenced by M_{t_ε}. The Υ - values are given by

$$\Upsilon_{lk} = \Upsilon_l = \lim_{t_\varepsilon \to 0} \frac{\left(V_{t_\varepsilon l} M_{t_\varepsilon k} \frac{8\pi^2}{2l+1} - 1 \right)}{t_\varepsilon} = \Upsilon_l = \lim_{t_\varepsilon \to 0} \frac{\left(V_{t_\varepsilon l} \frac{4\pi}{2l+1} - 1 \right)}{t_\varepsilon}. \tag{54}$$

The transition probability is outlined by Legendre-polynomials respectively spherical harmonics:

$$V_{t_\varepsilon}(\vec{\Omega} \cdot \vec{\Omega}') = \sum_{l=0}^{+\infty} V_{t_\varepsilon l} P_l(\cos(\vartheta)) = \sum_{l=0}^{+\infty} V_{t_\varepsilon l} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\Omega}) P_{lm}^*(\vec{\Omega}'),$$

$$\cos(\vartheta) = \vec{\Omega} \cdot \vec{\Omega}' = \mu. \tag{55}$$

On the other hand is

$$\lim_{t_\varepsilon \to 0} V_{t_\varepsilon}(\vec{\Omega} \cdot \vec{\Omega}') = \delta(\vec{\Omega} \cdot \vec{\Omega}'),$$

$$\delta(\vec{\Omega} \cdot \vec{\Omega}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\Omega}) P_{lm}^*(\vec{\Omega}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} P_l. \text{ see (131).} \tag{56}$$

$V_{t_\varepsilon}(\mu) \geq 0$ is only in the range $\mu \in [1 - \varepsilon, 1]$ essentially different from 0.

So the Legendre polynomials are approximated by

$$P_l(\mu) = 1 - \left.\frac{dP}{d\mu}\right|_1 \cdot \varepsilon + O(\varepsilon^2),\ \varepsilon = 1 - \mu,$$

$$\left.\frac{dP}{d\mu}\right|_1 = \frac{l(l+1)}{2} \quad \text{see (8.1) } P_0 = 1, \quad P_1 = \mu,$$

$$\Rightarrow P_l(\mu) = P_0 - (P_0 - P_1)\frac{l(l+1)}{2} + O(\varepsilon^2). \tag{57}$$

Using

$$\int_{-1}^{+1} P_l P_{l'} d\mu = \delta_{ll'} \frac{2}{2l+1} \tag{58}$$

Follows

$$\int_{-1}^{+1} V_{t_\varepsilon} P_l d\mu = 2V_{t_\varepsilon 0} - l(l+1)V_{t_\varepsilon 0} + \frac{l(l+1)}{3} V_{t_\varepsilon 1} = \frac{2}{2l+1} V_{t_\varepsilon l}. \tag{59}$$

Furthermore is

$$\int_{4\pi} V_{t_\varepsilon}\left(\vec{\Omega} \cdot \vec{\Omega}'\right) = \int_{4\pi} V_{t_\varepsilon 0} d\vec{\Omega}' = 4\pi V_{t_\varepsilon 0} = 1$$

$$\Rightarrow V_{t_\varepsilon 0} = \frac{1}{4\pi}, \tag{60}$$

as V_{t_ε} for $t_\varepsilon \to 0$ degenerates to a δ-function. That is why the $V_{t_\varepsilon l}$ are expressed by $V_{t_\varepsilon 1}$ and the determination of $V_{t_\varepsilon 1}$ remains to be calculated. We set

$$\lim_{\varepsilon \to 0} \frac{\left(V_{t_\varepsilon 1} \frac{4\pi}{3} - 1\right)}{\varepsilon} = \zeta. \tag{61}$$

Multiplying equation (59) with 2π leads to

$$\frac{4\pi}{2l+1} V_{t_\varepsilon l} = 4\pi V_{t_\varepsilon 0} - 4\pi \frac{l(l+1)}{2} V_{t_\varepsilon 0} + \frac{4\pi}{3} \frac{l(l+1)}{2} V_{t_\varepsilon 1}, \tag{62}$$

i.e.,

$$\frac{4\pi}{2l+1}V_{t_\varepsilon l} - 1 = \frac{l(l+1)}{2}\left(\frac{4\pi}{3}V_{t_\varepsilon 1} - 1\right)$$

$$= -\frac{l(l+1)}{2}\zeta + O(\varepsilon^2) = \Upsilon_l + O(\varepsilon^2) \tag{63}$$

$\Rightarrow$

$$\Upsilon_l = -\frac{l(l+1)}{2}\zeta, \qquad \zeta = const. \tag{64}$$

Now the equation of turbulent aerosol transport is written

$$\frac{\partial \bar{f}}{\partial t} + \bar{v}\vec{\Omega}\times\vec{\Theta}\cdot\nabla\bar{f} = \frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\frac{l(l+1)}{2}\bar{f}_{lmk}(\vec{x}, t)P_{lm}(\vec{\Omega})H_k(\vec{\Theta}). \tag{65}$$

The coefficient $\frac{\zeta}{t_E}$ replaced by $\frac{1}{t_E}$. A more complicated dependency of $t_E = (\vec{x}, t, \vec{\Omega}, \vec{\Theta})$ possibly remains. Maybe, physically justified simplifications lead to practical solutions.

The total derivative with respect to time gives

$$\frac{\partial \bar{f}}{\partial t}(\vec{x}, t, \vec{\Omega}, \vec{\Theta}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\frac{d}{dt}\bar{f}_{lmk}(\vec{x}, t)P_{lm}(\vec{\Omega})H_k(\vec{\Theta})$$

$$= \frac{1}{t_E}\sum_{l=1}^{+\infty}\gamma_l\cdot\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\bar{f}_{lmk}(\vec{x}, t)P_{lm}(\vec{\Omega})H_k(\vec{\Theta}). \tag{66}$$

The time behavior of the single modes are obtained by

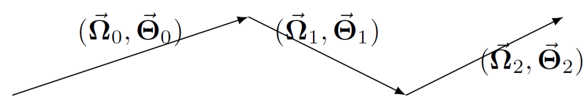
$$\frac{d}{dt}\bar{f}_{lmk}(t) = \frac{\gamma_l}{t_E}\bar{f}_{lmk}, \qquad \bar{f}_{lmk}(t) \sim \exp\left(\frac{\gamma_l}{t_E}\cdot t\right). \tag{67}$$

$\Rightarrow$

The greater the order l the more powerful is its temporal decay. The function development can be terminated with the first order, since, as shown, such an approximation approaches asymptotically the exact solution with the distance against assumed sources and the time.

5.5. Reconstruction of the transition probabilities $\overline{W}_{t_\varepsilon}$

The transition probability $\widetilde{W}_{t_\varepsilon, 0\to1\to2}$, an aerosol changing its motion pair of directions $(\vec{\Omega}, \vec{\Theta})$ at the times t_0, t_1, t_2 from $(\vec{\Omega}_0, \vec{\Theta}_0)$ via $(\vec{\Omega}_1, \vec{\Theta}_1)$ to $(\vec{\Omega}_2, \vec{\Theta}_2)$



results out of the product of the single probabilities of the pairs of directions (vortex vector and radius vector direction of motion in a circle segment). The graphical presentation is meant symbolically because such a pair of directions does not compose to an overall direction. $\vec{\Omega}_i$ is always orthogonal to $\vec{\Theta}_i$. A vectorial overall direction of $\vec{\Omega}_i$ and $\vec{\Theta}_i$ has no physical meaning in the 3 dimensional space.[9]

$$\widetilde{W}_{t_\varepsilon, 0\to1\to2} = \widetilde{W}_{\frac{t_\varepsilon}{2}}(\vec{\Omega}_0 \cdot \vec{\Omega}_1, \vec{\Theta}_0 \cdot \vec{\Theta}_1) \cdot \widetilde{W}_{\frac{t_\varepsilon}{2}}(\vec{\Omega}_1 \cdot \vec{\Omega}_2, \vec{\Theta}_1 \cdot \vec{\Theta}_2). \qquad (68)$$

The probability, that a aerosol changes its pair of directions within a time $t_\varepsilon = \varepsilon \cdot t_E$ from $(\vec{\Omega}_0, \vec{\Theta}_0)$ to $(\vec{\Omega}_2, \vec{\Theta}_2)$, is obtained by

$$\widetilde{W}_{t_\varepsilon}(\vec{\Omega}_0 \cdot \vec{\Omega}_2, \vec{\Theta}_0 \cdot \vec{\Theta}_2)$$

$$= \int_{2\pi}\int_{4\pi} \widetilde{W}_{\frac{t_\varepsilon}{2}}(\vec{\Omega}_0 \cdot \vec{\Omega}_1, \vec{\Theta}_0 \cdot \vec{\Theta}_1) \cdot \widetilde{W}_{\frac{t_\varepsilon}{2}}(\vec{\Omega}_1 \cdot \vec{\Omega}_2, \vec{\Theta}_1 \cdot \vec{\Theta}_2) d\vec{\Omega}_1 d\vec{\Theta}_1. \qquad (69)$$

---

[9] $\vec{\Omega}, \vec{\Theta}$ would make a single direction vector in a 4-dimensional space. The longitudinal fluctuations in the 4-dimensional space should accord to turbulence in the 3-dimensional space.

The evolution coefficients of the transition probability are available for sufficiently small ε

$$\widetilde{W}_{\frac{t_\varepsilon}{2}l} \approx \left\{1 + \frac{\varepsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \tag{70}$$

and therefore

$$\widetilde{W}_{\frac{t_\varepsilon}{2}}\left(\vec{\Omega}_0 \cdot \vec{\Omega}_1,\ \vec{\Theta}_0 \cdot \vec{\Theta}_1\right) \approx \sum_{l=0}^{+\infty} \left\{1 + \frac{\varepsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}\left(\vec{\Omega}_1\right) P_{lm}^*\left(\vec{\Omega}_0\right)$$

$$\cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left[H_k\left(\vec{\Theta}_1\right) H_k^*\left(\vec{\Theta}_0\right) + H_{-k}\left(\vec{\Theta}_1\right) H_{-k}^*\left(\vec{\Theta}_0\right)\right], \tag{71}$$

respectively,

$$\widetilde{W}_{\frac{t_\varepsilon}{2}}\left(\vec{\Omega}_1 \cdot \vec{\Omega}_2,\ \vec{\Theta}_1 \cdot \vec{\Theta}_2\right) \approx \sum_{l=0}^{+\infty} \left\{1 + \frac{\varepsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}$$

$$\left(\vec{\Omega}_2\right) P_{lm}^*\left(\vec{\Omega}_1\right) \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left[H_k\left(\vec{\Theta}_2\right) H_k^*\left(\vec{\Theta}_1\right) + H_{-k}\left(\vec{\Theta}_2\right) H_{-k}^*\left(\vec{\Theta}_1\right)\right]. \tag{72}$$

Integrating (69) one obtains

$$\widetilde{W}_{t_\varepsilon}\left(\vec{\Omega}_0 \cdot \vec{\Omega}_2,\ \vec{\Theta}_0 \cdot \vec{\Theta}_2\right) \approx \sum_{l=0}^{+\infty} \left\{1 + \frac{\varepsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}^*$$

$$\left(\vec{\Omega}_0\right) P_{lm}\left(\vec{\Omega}_2\right) \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left[H_k\left(\vec{\Theta}_0\right) H_k^*\left(\vec{\Theta}_2\right) + H_{-k}\left(\vec{\Theta}_0\right) H_{-k}^*\left(\vec{\Theta}_2\right)\right]. \tag{73}$$

Using n intermediate stages $\widetilde{W}_{t_\varepsilon}$ is expressed by an integral over the product of the single transition probabilities.

$$\widetilde{W}_{t_\varepsilon,\, 0\to 1\, \ldots\, \to n} = \widetilde{W}_{\frac{t_\varepsilon}{n}}\left(\vec{\Omega}_0 \cdot \vec{\Omega}_1,\ \vec{\Theta}_0 \cdot \vec{\Theta}_1\right) \cdot \widetilde{W}_{\frac{t_\varepsilon}{n}}\left(\vec{\Omega}_1 \cdot \vec{\Omega}_2,\ \vec{\Theta}_1 \cdot \vec{\Theta}_2\right)$$

$$\ldots\ \tilde{W}_{\frac{t_\varepsilon}{n}}\left(\vec{\Omega}_{n-1}\cdot\vec{\Omega}_n,\ \vec{\Theta}_{n-1}\cdot\vec{\Theta}_n\right), \tag{74}$$

$$\tilde{W}_{t_\varepsilon}\left(\vec{\Omega}_0\cdot\vec{\Omega}_n,\ \vec{\Theta}_0\cdot\vec{\Theta}_n\right) = \int_{2\pi}\int_{4\pi}\int_{2\pi}\int_{4\pi}\ \ldots\ \int_{2\pi}\int_{4\pi}\tilde{W}_{\frac{t_\varepsilon}{n}}\cdot\tilde{W}_{\frac{t_\varepsilon}{n}}$$

$$\ldots\ \tilde{W}_{\frac{t_\varepsilon}{n}}\,d\vec{\Omega}_1 d\vec{\Theta}_1\ \ldots\ d\vec{\Omega}_{n-1}d\vec{\Theta}_{n-1}, \tag{75}$$

$$\tilde{W}_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right) \approx \lim_{n\to\infty}\sum_{l=0}^{+\infty}\left\{1+\frac{\varepsilon\cdot\Upsilon_l}{2}\right\}\frac{2l+1}{4\pi}\cdot\frac{1}{2\pi}\sum_{m=-l}^{+l}P_{lm}^{*}\left(\vec{\Omega}\right)P_{lm}\left(\vec{\Omega}'\right)$$

$$\cdot\frac{1}{2}\sum_{k=-\infty}^{+\infty}\left[H_k\left(\vec{\Theta}'\right)H_k^{*}\left(\vec{\Theta}\right)+H_{-k}\left(\vec{\Theta}'\right)H_{-k}^{*}\left(\vec{\Theta}\right)\right]. \tag{76}$$

For $n\to\infty$ arises

$$\lim_{n\to\infty}\left\{1+\frac{\varepsilon\cdot\Upsilon_l}{2}\right\}^n = e^{\Upsilon_l\cdot\varepsilon} \tag{77}$$

and $\Rightarrow$

$$\tilde{W}_{t_\varepsilon}\left(\vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right) = \sum_{l=0}^{+\infty}\ e^{\Upsilon_l\cdot\varepsilon}\,\frac{2l+1}{4\pi}\cdot\frac{1}{2\pi}\cdot\sum_{m=-l}^{+l}P_{lm}^{*}\left(\vec{\Omega}\right)P_{lm}\left(\vec{\Omega}'\right)$$

$$\cdot\frac{1}{2}\sum_{k=-\infty}^{+\infty}\left[H_k\left(\vec{\Theta}'\right)H_k^{*}\left(\vec{\Theta}\right)+H_{-k}\left(\vec{\Theta}'\right)H_{-k}^{*}\left(\vec{\Theta}\right)\right]. \tag{78}$$

Choosing $\varepsilon = \dfrac{t_\varepsilon}{t_E\left(\vec{x},\ t,\ \vec{\Omega}\right)}$ the exchange function $\tilde{W}_{t_\varepsilon}$ may be understood in the dependencies

$$\tilde{W}_{t_\varepsilon} = \tilde{W}_{t_\varepsilon}\left(\vec{x},\ t,\ \vec{\Omega}\cdot\vec{\Omega}',\ \vec{\Theta}\cdot\vec{\Theta}'\right) \tag{79}$$

and

$$\overline{W_{t_\varepsilon}}\left(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Omega}',\, \vec{\Theta},\, \vec{\Theta}'\right) \approx \widetilde{W}_{t_\varepsilon}\left(\vec{x},\, t,\, \vec{\Omega}\cdot\vec{\Omega}',\, \vec{\Theta}\cdot\vec{\Theta}'\right) \tag{80}$$

is given, too. $\Rightarrow$

$$\bar{f}_{t_\varepsilon}\left(\vec{x},\, \vec{\Omega},\, \vec{\Theta},\, t\right) = \int_{2\pi}\int_{4\pi}\widetilde{W}_{t_\varepsilon}\left(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta},\, \vec{\Omega}',\, \vec{\Theta}'\right)$$

$$\times\, \bar{f}_{t_\varepsilon}\left(\vec{x} - t_\varepsilon\cdot \bar{v}'\vec{\Omega}'\times\vec{\Theta}',\, \vec{\Omega}',\, \vec{\Theta}',\, t - t_\varepsilon\right) d\vec{\Omega}' d\vec{\Theta}',$$

$$\bar{v}' = \bar{v}'\left(\vec{x},\, \vec{\Omega}',\, \vec{\Theta}',\, t\right) = \bar{\omega}'\left(\vec{x},\, \vec{\Omega}',\, \vec{\Theta}',\, t\right)\cdot \bar{r}'\left(\vec{x},\, \vec{\Omega}',\, \vec{\Theta}',\, t\right). \tag{81}$$

6. Verification that the Stochastic Aerosol Motions occur through turbulently moving Continua

6.1. The relationship between stochastic aerosol transport and known fluid dynamics

6.1.1. Introduction

In this last section, it will be shown that the stochastic aerosol transport in terms of an ensemble theory can indeed be assigned to turbulent continua characterized by fluid elements

$$\vec{v}_{t_\varepsilon} = \vec{\omega}_{t_\varepsilon}\times\vec{r}_{t_\varepsilon}.$$

Separation of vectors $\vec{\omega}$ and $\vec{r}$ in magnitude and direction corresponding to (14) is not needed now. As with aerosol transport, the term

$$\lim_{t_\varepsilon\to 0}\frac{\int_{\vec{r}}\int_{\vec{\omega}} W_{t_e} f'_{t_\varepsilon} d\vec{\omega}' d\vec{r}' - f'_{t_\varepsilon}}{t_\varepsilon} = F(\vec{x},\, t,\, \vec{\omega},\, \vec{r}) \tag{82}$$

turns out to be the key to the problem. F is the exchange term of $\vec{r}$ and $\vec{\omega}$ not integrated out with respect to vector amounts as in (34).

Turbulently moved one phase fluids are examined considering statistical deliberations and its deterministic counterparts. That a linking of deterministic and stochastic theory may be available and further more that out of this connection additionally important (sometimes otherwise not known) relations arise for deterministic formulations, is shown in the following.

6.1.2. The Transition: Stochastic Theory $\rightarrow$ Deterministic Theory

Every space-time-point $(\vec{x}, t)$ is assigned a continuously differentiable fluid element distribution over the motion amounts $\vec{\omega}_{t_\varepsilon}$ and $\vec{r}_{t_\varepsilon}$ according to

$$f_{t_\varepsilon} = f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}). \tag{83}$$

For indexed functions with t_ε, it is automatically assumed that the dependent motion quantities $(\vec{\omega}, \vec{r})$ are assigned to a t_ε- measurement accuracy. The indexing of the motion quantities may be omitted in the functions if the functions are accordingly indexed.

After an execution of a lim $t_\varepsilon \rightarrow 0$ process, such as

$$\lim_{t_\varepsilon \rightarrow 0} f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}) = f(\vec{x}, t, \vec{\omega}, \vec{r}) \tag{84}$$

f and $(\vec{\omega}, \vec{r})$ are understood as results of an exact measuring process.

The change of motion quantities in point $(\vec{x}, t)$

$$(\vec{\omega}'_{t_\varepsilon}(\vec{x} - \Delta\vec{x}, t - t_\varepsilon), \vec{r}'_{t_\varepsilon}(\vec{x} - \Delta\vec{x}, t - t_\varepsilon)) \rightarrow (\vec{\omega}_{t_\varepsilon}(\vec{x}, t), \vec{r}_{t_\varepsilon}(\vec{x}, t))$$

is controlled by the transition probability density $W_{t_\varepsilon} = W_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}, \vec{\omega}', \vec{r}')$ [10] with

$$\lim_{t_\varepsilon \rightarrow 0} W_{t_\varepsilon} = \delta(\vec{\omega}, \vec{r}, \vec{\omega}', \vec{r}'),$$

[10]The test functions otherwise used in distribution theory have an immediate physical meaning in this context with the formulation of the transition probability density.

$$f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}) = \int_{\vec{r}}\int_{\vec{\omega}} W_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}, \vec{\omega}', \vec{r}') \cdot f_{t_\varepsilon}(\vec{x} - \Delta\vec{x}, t - t_\varepsilon, \vec{\omega}', \vec{r}')d\vec{\omega}'d\vec{r}',$$

$$\Delta\vec{x} = t_\varepsilon \cdot \vec{\omega}' \times \vec{r}'. \quad (85)$$

These equations characterize stochastic turbulence of the continuum in the frame of an ensemble theory and represent a Markov Process with natural causality. (This is a definition of the author.)

f_{t_ε} is developed in (85) until the 1st order around $(\vec{x}, t) \Rightarrow$

$$f_{t_\varepsilon}(\vec{x} - \Delta\vec{x}, t - t_\varepsilon, \vec{\omega}', \vec{r}') = f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}', \vec{r}') - \frac{\partial f'_{t_\varepsilon}}{\partial t} \cdot t_\varepsilon - \Delta\vec{x} \cdot \vec{\nabla} f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}', \vec{r}') + O({t_\varepsilon}^2) \quad (86)$$

with $f'_{t_\varepsilon} = f_{t_\varepsilon}(\vec{x}, t, \vec{\omega}', \vec{r}')$ and one obtains

$$\int_{\vec{r}}\int_{\vec{\omega}} \left[\frac{\partial f'_{t_\varepsilon}}{\partial t} + \vec{\omega}' \times \vec{r}' \cdot \vec{\nabla} f'_{t_\varepsilon}\right] d\vec{\omega}'d\vec{r}' + O({t_\varepsilon}^2) = \frac{\int_{\vec{r}}\int_{\vec{\omega}} W_{t_\varepsilon} f'_{t_\varepsilon} d\vec{\omega}'d\vec{r}' - f_{t_\varepsilon}}{t_\varepsilon}. \quad (87)$$

$\lim\limits_{t_\varepsilon \to 0}$ applied to (87) leads to

$$\frac{\partial f}{\partial t} + \vec{\omega} \times \vec{r} \cdot \vec{\nabla} f = \lim_{t_\varepsilon \to 0} \frac{\int_{\vec{r}}\int_{\vec{\omega}} W_{t_\varepsilon} f'_{t_\varepsilon} d\vec{\omega}'d\vec{r}' - f_{t_\varepsilon}}{t_\varepsilon}. \quad (88)$$

The right side must contain the characteristics of the turbulent fluid.

$$\lim_{t_\varepsilon \to 0} \frac{\int_{\vec{r}}\int_{\vec{\omega}} W_{t_\varepsilon} f'_{t_\varepsilon} d\vec{\omega}'d\vec{r}' - f_{t_\varepsilon}}{t_\varepsilon} F(\vec{x}, t, \vec{\omega}, \vec{r}). \quad (89)$$

F has to be chosen such, that the deterministic vortex equations result under the influence of the assumed acceleration field. Thus one obtains

$$\frac{\partial f}{\partial t} + \vec{\omega} \times \vec{r} \cdot \vec{\nabla} f = F(\vec{x}, t, \vec{\omega}, \vec{r}). \tag{90}$$

If we restrict ourselves to a single system of the ensemble with the identifier ν in a space-time point $(\vec{x}, t)$, which has exactly these movement sizes $\vec{\omega}_{(\vec{x}, t; \nu)}$ and $\vec{r}_{(\vec{x}, t; \nu)}$ in this system, then the distribution function f degenerates to a δ-function with respect to these movement sizes. $\vec{\omega}_{(\vec{x}, t; \nu)}$ and $\vec{r}_{(\vec{x}, t; \nu)}$ are not vector functions but constant vectors in $(\vec{x}, t)$, whereas $\vec{\omega}(\vec{x}, t)$ and $\vec{r}(\vec{x}, t)$ represent spatiotemporal fields in dependence of $(\vec{x}, t)$.

$$f(\vec{x}, t, \vec{\omega}, \vec{r}) \to \delta(\vec{\omega}_{(\vec{x}, t; \nu)}, \vec{r}_{(\vec{x}, t; \nu)}; \vec{\omega}, \vec{r}) \tag{91}$$

and

$$F(\vec{x}, t, \vec{\omega}, \vec{r}) \to \frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x}, t; \nu)}}{\omega^2{}_{(\vec{x}, t; \nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x}, t; \nu)}\right]\delta(\vec{\omega}_{(\vec{x}, t; \nu)}, \vec{r}_{(\vec{x}, t; \nu)}; \vec{\omega}, \vec{r}), \tag{92}$$

as will be shown in the following.

The equation for stochastic propagation in terms of an ensemble theory thus degenerates to the following equation, from now on called **key equation**.

$$\left(\frac{\partial}{\partial t} + \vec{\omega}_{(\vec{x}, t; \nu)}, \vec{r}_{(\vec{x}, t; \nu)} \cdot \vec{\nabla} - \frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x}, t; \nu)}}{\omega^2{}_{(\vec{x}, t; \nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x}, t; \nu)}\right]\right)\delta = 0. \tag{93}$$

For this δ-function applies

$$\int_{\vec{r}}\int_{\vec{\omega}} \delta(\vec{\omega}_{(\vec{x}, t; \nu)}, \vec{r}_{(\vec{x}, t; \nu)}; \vec{\omega}, \vec{r})\, d\vec{\omega}\, d\vec{r} = 1,$$

$$\int_{\vec{r}}\int_{\vec{\omega}} \delta(\vec{\omega}_{(\vec{x},t;\nu)}, \vec{r}_{(\vec{x},t;\nu)}; \vec{\omega}, \vec{r})\vec{\omega}d\vec{\omega}d\vec{r} = \vec{\omega}_{(\vec{x},t;\nu)},$$

$$\int_{\vec{r}}\int_{\vec{\omega}} \delta(\vec{\omega}_{(\vec{x},t;\nu)}, \vec{r}_{(\vec{x},t;\nu)}; \vec{\omega}, \vec{r})\vec{r}d\vec{\omega}d\vec{r} = \vec{r}_{(\vec{x},t;\nu)},$$

$$\int_{\vec{r}}\int_{\vec{\omega}} \delta(\vec{\omega}_{(\vec{x},t;\nu)}, \vec{r}_{(\vec{x},t;\nu)}; \vec{\omega}, \vec{r})\vec{\omega}^2\vec{r}d\vec{\omega}d\vec{r} = \vec{\omega}^2{}_{(\vec{x},t;\nu)}\vec{r}_{(\vec{x},t;\nu)},$$

$$\int_{\vec{r}}\int_{\vec{\omega}} \delta(\vec{\omega}_{(\vec{x},t;\nu)}, \vec{r}_{(\vec{x},t;\nu)}; \vec{\omega}, \vec{r})\frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2{}_{(\vec{x},t;\nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]d\vec{\omega}d\vec{r}$$

$$= \frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2{}_{(\vec{x},t;\nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]. \tag{94}$$

Definition of the operator $\Xi[...]$

From the vector $\mathbf{A}_{(\vec{x},t;\nu)}$ respectively the scalar function value $f_{(\vec{x},t;\nu)}$ which is defined in the space-time point $(\vec{x}, t)$ of the system ν a vector function or a scalar function is obtained by the operator Ξ, if a corresponding field exists around the point $(\vec{x}, t)$

$$\Xi\left[\mathbf{A}_{(\vec{x},t;\nu)}\right] = \mathbf{A}(\vec{x}, t), \qquad \Xi\left[f_{(\vec{x},t;\nu)}\right] = f(\vec{x}, t). \tag{95}$$

The Operator $\Xi[...]$ brings this functionality to "life".

Accordingly the following relationships are noted:

$$\Xi\left[\vec{\omega}_{(\vec{x},t;\nu)}\right] = \vec{\omega}(\vec{x}, t),$$

$$\Xi\left[\vec{r}_{(\vec{x},t;\nu)}\right] = \vec{r}(\vec{x}, t),$$

$$\Xi\left[\vec{\omega}^2{}_{(\vec{x},t;\nu)}\vec{r}_{(\vec{x},t;\nu)}\right] = \vec{\omega}^2(\vec{x}, t)\vec{r}(\vec{x}, t),$$

$$\Xi\left(\frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2{}_{(\vec{x},t;\nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]\right) = \frac{1}{2}\frac{\vec{\omega}(\vec{x}, t)}{\vec{\omega}^2(\vec{x}, t)} \cdot \vec{\nabla} \times \vec{q}(\vec{x}, t). \tag{96}$$

6.1.3. Deterministic Equations of Turbulence

From the general momentum equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{q}, \tag{97}$$

the vortex equation[11] may be developed using the $\vec{\nabla} \times$ - operator

$$\frac{\partial}{\partial t}\vec{\omega} - \vec{\nabla} \times (\vec{v} \times \vec{\omega}) - \frac{1}{2}\vec{\nabla} \times \vec{q} = 0. \tag{98}$$

The relations between deterministic and stochastic description is established when the known deterministic vortex equation can be reconstructed from an associated stochastic equation of the ensemble theory. In the following the method is presented developing the dual pair of deterministic vector equations from the key equation (93).

$$\left(\frac{\partial}{\partial t} + \vec{\omega}_{(\vec{x},t;\nu)} \times \vec{r}_{(\vec{x},t;\nu)} \cdot \vec{\nabla} - \frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot (\vec{\nabla} \times \vec{q})_{(\vec{x},t;\nu)}\right]\right)\delta = 0.$$

In this situation the vectors may be pushed before and after the differential operators. The Term

$$\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot (\vec{\nabla} \times \vec{q})_{(\vec{x},t;\nu)}\right]\delta \tag{99}$$

guarantees the finding of equation (98) and its dual one. It is

$$\vec{v} \perp \vec{\omega} \perp \vec{r} \tag{100}$$

and setting

$$\vec{a} = \vec{v} \times \vec{\omega} \tag{101}$$

this results in

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[11]By vortex is in this paper always the swirl $\omega = \frac{1}{2}\vec{\nabla} \times \vec{v}$ meant.

$$\vec{r}||\vec{a}. \tag{102}$$

Such $\vec{a}$ and $\vec{r}$ are linked as follows[12]

$$\vec{r} = \frac{\vec{a}}{\omega^2}. \tag{103}$$

$\Rightarrow$

with $\delta = \delta(\vec{\omega}_{(\vec{x},t;\nu)}, \vec{r}_{(\vec{x},t;\nu)}; \vec{\omega}, \vec{r})$.

$$\vec{\omega}_{(\vec{x},t;\nu)} \times \vec{r}_{(\vec{x},t;\nu)} \cdot \vec{\nabla}\delta = -\vec{r}_{(\vec{x},t;\nu)} \times \vec{\omega}_{(\vec{x},t;\nu)} \cdot \vec{\nabla}\delta$$

$$= \vec{\omega}_{(\vec{x},t;\nu)} \cdot \vec{\nabla} \times \vec{r}_{(\vec{x},t;\nu)}\delta = -\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot \vec{\nabla} \times \vec{a}_{(\vec{x},t;\nu)}\delta.$$

Inserting in (93) gives

$$\frac{\partial}{\partial t}\left(\frac{\vec{\omega}_{(\vec{x},t;\nu)} \cdot \vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}}\delta\right) - \frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot \vec{\nabla} \times (\vec{a}_{(\vec{x},t;\nu)}\delta)$$

$$-\frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]\delta = 0$$

$$\Rightarrow \frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}} \cdot \left(\frac{\partial}{\partial t}(\vec{\omega}_{(\vec{x},t;\nu)}\delta) - \vec{\nabla} \times (\vec{a}_{(\vec{x},t;\nu)}\delta) - \frac{1}{2}\left[\cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]\delta\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\omega}_{(\vec{x},t;\nu)}\delta) - \vec{\nabla} \times (\vec{a}_{(\vec{x},t;\nu)}\delta) - \frac{1}{2}\left[\cdot \left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]\delta = 0 \tag{104}$$

and

$$\Xi\left[\int_{\vec{r}}\int_{\vec{\omega}}\left[\frac{\partial}{\partial t}(\vec{\omega}_{(\vec{x},t;\nu)}\delta) - \vec{\nabla} \times (\vec{a}_{(\vec{x},t;\nu)}\delta)\right] - \frac{1}{2}\left[\left(\vec{\nabla} \times \vec{q}\right)_{(\vec{x},t;\nu)}\right]\delta\right]d\vec{\omega}d\vec{r} = 0 \tag{105}$$

is obtained and since integration and differentiation are interchangeable

[12]Symbols as ω, r, a, v etc. always mean amounts of the corresponding vectors.

in order, it follows that

$$\frac{\partial}{\partial t}\,\Xi\left[\vec{\omega}_{(\vec{x},\,t;\,\nu)}\right] - \vec{\nabla}\times\Xi\left[\vec{a}_{(\vec{x},\,t;\,\nu)}\right] - \frac{1}{2}\,\vec{\nabla}\times\Xi\left[\vec{q}_{(\vec{x},\,t;\,\nu)}\right] = 0. \tag{106}$$

Now we have the first of the dual turbulence equations

$$\frac{\partial}{\partial t}\,\vec{\omega} - \vec{\nabla}\times\vec{a} - \frac{1}{2}\,\vec{\nabla}\times\vec{q} = 0, \tag{107}$$

accordingly

$$\frac{\partial}{\partial t}\,\vec{\omega} - \vec{\nabla}\times(\vec{v}\times\vec{\omega}) - \frac{1}{2}\,\vec{\nabla}\times\vec{q} = 0.$$

Hereby the connection of stochastics and deterministics is achieved. From the key-equation above a second equation, the dual one, may be derived.

Back to the initial equation (93)

$$\left(\frac{\partial}{\partial t} + \vec{\omega}_{(\vec{x},\,t;\,\nu)}\times\vec{r}_{(\vec{x},\,t;\,\nu)}\cdot\vec{\nabla} - \frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},\,t;\,\nu)}}{\omega^2_{(\vec{x},\,t;\,\nu)}}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},\,t;\,\nu)}\right]\right)\delta = 0.$$

Simple conversions give

$$\frac{\partial}{\partial t}\left(\vec{r}_{(\vec{x},\,t;\,\nu)}\cdot\frac{\vec{r}_{(\vec{x},\,t;\,\nu)}}{r^2_{(\vec{x},\,t;\,\nu)}}\,\delta\right) + \vec{r}_{(\vec{x},\,t;\,\nu)}\cdot\vec{\nabla}(\vec{\omega}_{(\vec{x},\,t;\,\nu)})\delta$$

$$- \frac{\vec{r}_{(\vec{x},\,t;\,\nu)}\cdot\vec{r}_{(\vec{x},\,t;\,\nu)}}{r^2_{(\vec{x},\,t;\,\nu)}}\,\frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},\,t;\,\nu)}}{\omega^2_{(\vec{x},\,t;\,\nu)}}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},\,t;\,\nu)}\right]\delta = 0$$

$$\rightarrow\ \vec{r}_{(\vec{x},\,t;\,\nu)}\left[\frac{\partial}{\partial t}\,\frac{\vec{r}_{(\vec{x},\,t;\,\nu)}}{r^2_{(\vec{x},\,t;\,\nu)}}\,\delta + \vec{\nabla}\times(\vec{\omega}_{(\vec{x},\,t;\,\nu)})\delta\right.$$

$$\left. - \frac{\vec{r}_{(\vec{x},\,t;\,\nu)}}{r^2_{(\vec{x},\,t;\,\nu)}}\,\frac{1}{2}\left[\frac{\vec{\omega}_{(\vec{x},\,t;\,\nu)}}{\omega^2_{(\vec{x},\,t;\,\nu)}}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},\,t;\,\nu)}\right]\delta\right] = 0. \tag{108}$$

Using the curvature vector field of the fluid trajectories $\vec{b} = \frac{\vec{r}}{r^2}$ the equation is written

$$\frac{\partial}{\partial t}(\vec{b}_{(\vec{x},t;\nu)})\delta + \vec{\nabla}\times(\vec{\omega}_{(\vec{x},t;\nu)})\delta - \frac{1}{2}\vec{b}_{(\vec{x},t;\nu)}\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},t;\nu)}\delta = 0 \quad (109)$$

and applying the operators Ξ arises

$$\Xi\int_{\vec{r}}\int_{\vec{\omega}}\left[\frac{\partial}{\partial t}(\vec{b}_{(\vec{x},t;\nu)})\delta + \vec{\nabla}\times(\vec{\omega}_{(\vec{x},t;\nu)})\delta - \frac{1}{2}\vec{b}_{(\vec{x},t;\nu)}\frac{\vec{\omega}_{(\vec{x},t;\nu)}}{\omega^2_{(\vec{x},t;\nu)}}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},t;\nu)}\delta\right]d\vec{\omega}d\vec{r} = 0 \quad (110)$$

respectively

$$\frac{\partial}{\partial t}\left[\vec{b}_{(\vec{x},t;\nu)}\right] + \vec{\nabla}\times\Xi\left[\vec{\omega}_{(\vec{x},t;\nu)}\right] - \frac{1}{2}\Xi\left[\vec{b}\cdot\left(\frac{\vec{\omega}}{\omega^2}\cdot\left(\vec{\nabla}\times\vec{q}\right)_{(\vec{x},t;\nu)}\right)\right] = 0. \quad (111)$$

Thus, the second of the dual turbulence equations is obtained

$$\frac{\partial}{\partial t}\vec{b} + \vec{\nabla}\times\vec{\omega} - \frac{1}{2}\vec{b}\left[\frac{\vec{\omega}}{\omega^2}\cdot\vec{\nabla}\times\vec{q}\right] = 0. \quad (112)$$

Overall, this results in the dual system of equations

$$\frac{\partial}{\partial t}\vec{\omega} - \vec{\nabla}\times\vec{a} - \frac{1}{2}\vec{\nabla}\times\vec{q} = 0,$$

$$\frac{\partial}{\partial t}\vec{b} + \vec{\nabla}\times\vec{\omega} - \frac{1}{2}\vec{b}\left[\frac{\vec{\omega}}{\omega^2}\cdot\vec{\nabla}\times\vec{q}\right] = 0,$$

$$\vec{v} = \vec{\omega}\times\frac{\vec{b}}{b^2}, \qquad \vec{a} = \vec{v}\times\vec{\omega}. \quad (113)$$

The term

$$-\frac{1}{2}\vec{b}\left[\frac{\vec{\omega}}{\omega^2}\cdot\vec{\nabla}\times\vec{q}\right]$$

leads to removable singularities in space-time-points $(\vec{x},\, t)$ if $\vec{\omega} = 0$ and $\vec{b} = 0$ occur in the fluid-element trajectories.

In this case the whole term is calculated from its surroundings. The same shall apply for the calculation of the velocity $\bar{v}$.

These matters are to be discussed in connection with a statement of a complete system of equations of deterministic turbulence, which will be done in another paper.

7. Summary and Outlook

Aerosol motions in a turbulently moving continuum are studied assuming that they accurately describe the trajectories of individual fluid elements due to their size and weight. These movements, which are actually deterministic, were considered stochastically in the sense of an ensemble theory. After the consequent derivation of the aerosol transport equations, two coefficients $\bar{v}(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta})$ and $t_E(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta})$, which are still very complicated in their dependencies, remain. Simplifying model assumptions can lead to correspondingly simplified coefficients. Furthermore, the function development can be terminated with the first order, since, as shown, such an approximation approaches asymptotically the exact solution with the distance against assumed sources and the time.

$$\frac{\partial \bar{f}}{\partial t} + \bar{v}\vec{\Omega} \times \vec{\Theta} \cdot \nabla \bar{f} = \frac{1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2} \bar{f}_{lmk}(\vec{x},\, t) P_{lm}(\vec{\Omega}) H_k(\vec{\Theta}),$$

$$\bar{v} = \bar{v}(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta}), \quad t_E = t_E(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta})$$

$$\Updownarrow$$

$$\bar{f}_{t_\varepsilon}(\vec{x},\, \vec{\Omega},\, \vec{\Theta},\, t) = \int_{4\pi} \int_{2\pi} \widetilde{W}_{t_\varepsilon}(\vec{x},\, t,\, \vec{\Omega},\, \vec{\Theta},\, \vec{\Omega}',\, \vec{\Theta}')$$

$$\bar{f}_{t_\varepsilon}\left(\vec{x} - t_\varepsilon \cdot \vec{v}\vec{\Omega} \times \vec{\Theta}', \vec{\Omega}', \vec{\Theta}', t - t_\varepsilon\right) d\vec{\Omega}' d\vec{\Theta}',$$

$$\widetilde{W}_{t_\varepsilon}\left(\vec{\Omega} \cdot \vec{\Omega}', \vec{\Theta} \cdot \vec{\Theta}'\right) = \sum_{l=0}^{+\infty} e^{\Upsilon_l \cdot \varepsilon} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \cdot \sum_{m=-l}^{+l} P_{lm}^{*}\left(\vec{\Omega}\right) P_{lm}\left(\vec{\Omega}'\right)$$

$$\cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left[H_k\left(\vec{\Theta}'\right) H_k^{*}\left(\vec{\Theta}\right) + H_{-k}\left(\vec{\Theta}'\right) H_{-k}^{*}\left(\vec{\Theta}\right)\right],$$

$$\Upsilon_l = -\frac{l(l+1)}{2}\zeta, \qquad \zeta = const.$$

The integral form of the transport equation with its explicitly formulated transition probability indicates the possibility of using Monte Carlo methods for its numerical evaluation.

It is probably relatively difficult to experimentally confirm the relationships presented. Therefore, the connection between such a stochastics in the sense of an ensemble theory and a deterministic fluid dynamics is established.

$$\bar{f}_{t_\varepsilon}\left(\vec{x}, t, \vec{\Omega}, \vec{\Theta}\right) = \int_{4\pi}\int_{2\pi} W_{t_\varepsilon}(\vec{x}, t, \vec{\omega}, \vec{r}, \vec{\omega}', \vec{r}')\bar{f}_{t_\varepsilon}(\vec{x} - \vec{\omega}' \times \vec{r}' \cdot t_\varepsilon, \vec{\omega}', \vec{r}', t - t_\varepsilon) d\vec{\omega}' d\vec{r}',$$

$$\Downarrow$$

$$\frac{d}{dt}\vec{\omega} - \vec{\nabla} \times \vec{a} - \frac{1}{2}\vec{\nabla} \times \vec{q} = 0,$$

$$\frac{d}{dt}\vec{b} + \vec{\nabla} \times \vec{\omega} - \frac{1}{2}\vec{b}\left[\frac{\vec{\omega}}{\omega^2} \cdot \vec{\nabla} \times \vec{q}\right] = 0,$$

$$\vec{v} = \vec{\omega} \times \frac{\vec{b}}{b^2}, \qquad \vec{a} = \vec{v} \times \vec{\omega}, \qquad \vec{v} \perp \vec{\omega} \perp \vec{r}. \tag{114}$$

The result is a dual pair of deterministic equations of turbulence. In this respect, the desired goal is achieved. However, this pair of equations is not yet complete. The completion happens in a further paper, where then the whole system of equations represents a geometrodynamics of turbulence. I.e. the whole system of equations consists of vector fields of **velocities**, **vortex rotations**, **their curvature vector fields and non-conservative accelerations**.

8. Appendix

8.1. Legendre-Polynomials

The Legendre-polynomials are defined within the interval $[-1, +1]$ by

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n \in N. \tag{115}$$

They represent a complete orthogonal function system with

$$\int_{-1}^{+1} P_n(x) P_m(x) dx = \begin{cases} \frac{2}{2m+1} & \text{for } m = n, \\ 0 & \text{else.} \end{cases} \quad f(x) \quad [-1, +1] \tag{116}$$

Every continuously differentiable function $f(x)$ defined within $[-1, +1]$ can be developed by Legendre-polynomials according to

$$f(x) = \sum_{l=0}^{\infty} f_l P_l(x). \tag{117}$$

The f_l are the evolution coefficients. A presentation of the δ- function by Legendre-polynomials is obtained by

$$\delta(x, x') = \sum_{l=0}^{\infty} \frac{2m+1}{2} P_l(x) P_l(x'). \tag{118}$$

Important recurrence equations are

$$(n+1) P_{n+1} = (2n+1) x P_n(x) - n P_{n-1}(x),$$

$$P'_{n+1}(x) - x P'_n(x) = (n+1) P_n(x), \; n = 0, 1, 2, \ldots,$$

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x). \tag{119}$$

An integral representation of the Legendre-polynomials is obtained by

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2-1}\cos(\varphi))^n d\varphi. \tag{120}$$

Owing to $\left| x + \sqrt{x^2-1}\cos(\theta)\right| = |\cos\theta + i\sin(\theta)\cos(\theta)| \leq 1,$

$$|P_n(x)| \leq 1 \tag{121}$$

follows. These polynomials have their maximum for $x = 1,$ particularly

$$P_n(1) = 1. \tag{122}$$

$$\frac{dP_l(x)}{dx}\Big|_1 = \frac{l(l+1)}{2} \tag{123}$$

is proved by complete induction.

Proof .

1. $P_0'(1) = 0.$

Assumption:

2. $P_n'(1) = \dfrac{n(n+1)}{2}$

$\Rightarrow$

3. $P_{n+1}'(1) = \dfrac{(n+2)(n+1)}{2}$ wegen (119) $P_{n+1}'(1) - P_n'(1) = (n+1)P_n(1)$

q.e.d.

8.2. Spherical Harmonics

The Spherical harmonics [[6] page 224] represent a complete orthogonal, complex function system on the spherical surface

$$P_{lm}(\vec{\Omega}) = e^{im\varphi}\frac{-(\sin(\vartheta))^{m}}{l!2^{l}}\cdot\left(\frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}}\frac{d^{l+m}(\cos^{2}\vartheta-1)^{l}}{(d\cos\vartheta)^{l+m}}$$

$$= e^{im\varphi}\frac{(\sin(\vartheta))^{-m}}{l!2^{l}}\cdot\left(\frac{(l+m)!}{(l-m)!}\right)^{\frac{1}{2}}\frac{d^{l-m}(\cos^{2}\vartheta-1)^{l}}{(d\cos\vartheta)^{l-m}} \tag{124}$$

with

$$P_{l,\,-m}(\vec{\Omega}) = (-)^{m}P^{*}_{lm}(\vec{\Omega}) \tag{125}$$

and

$$\int_{4\pi} d\vec{\Omega}P_{l'm'}(\vec{\Omega})P^{*}_{lm}(\vec{\Omega}) = \delta_{l'l}\delta_{m'm}\frac{4\pi}{2l+1}. \tag{126}$$

All continuously differentiable functions on the spherical surface $f(\Omega) = f(\theta, \phi)$ can be developed according to

$$f(\vec{\Omega}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P_{lm}(\vec{\Omega}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P^{*}_{lm}(\vec{\Omega}) \tag{127}$$

the f_{lm} representing the evolution coefficients. The $P^{*}_{lm}(\vec{\Omega})$ being complex to $P_{lm}(\vec{\Omega})$. $f(\vec{\Omega})$ can be alternatively considered.

The complex version of spherical harmonics is chosen because it is successfully used in quantum mechanics as well as in nuclear reactor physics. The formulations appear more compact in this way.

The spherical harmonics for $l = 0, 1$ are

$$P_{00} = P^{*}_{00} = 1,$$

$$P_{-1,1}(\vec{\Omega}) = 2^{-\frac{1}{2}}e^{-i\varphi}\sin(\varphi), \quad P^{*}_{-1,1} = 2^{-\frac{1}{2}}e^{-i\varphi}\sin(\varphi),$$

$$P_{1,0}(\vec{\Omega}) = P^{*}_{1,0}(\vec{\Omega}) = \cos(\varphi) = P_{1}(\vec{\Omega}),$$

$$P_{1,1}(\vec{\Omega}) = -2^{-\frac{1}{2}} e^{i\varphi} \sin(\varphi), \;\; P^*_{1,1}(\vec{\Omega}) = -2^{-\frac{1}{2}} e^{-i\varphi} \sin(\varphi). \quad (128)$$

The connection of spherical harmonics and Legendre-polynomials is obtained by

$$P_{l0} = P^*_{l0} = P_l. \quad (129)$$

Furthermore the addition theorem

$$P_l(\cos(\varphi)) = \sum_{m=-l}^{m=+l} P_{lm}(\vec{\Omega}')P^*_{lm}(\vec{\Omega}), \qquad \cos(\varphi) = \vec{\Omega}' \cdot \vec{\Omega}. \quad (130)$$

The δ- function depending on the spherical harmonics may be stated by

$$\delta(\vec{\Omega}, \vec{\Omega}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sum_{m=-1}^{m=+l} P_{lm}(\vec{\Omega})P^*_{lm}(\vec{\Omega}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\vec{\Omega} \cdot \vec{\Omega}'). \quad (131)$$

8.3. Turbulence-Functions

Functions of the unit direction vectors $\vec{\Omega} \perp \vec{\Theta}$ are represented by a complete orthogonal function system meaning an extension of the spherical harmonics. We call them turbulence functions.

$$Q_{lmk}(\vec{\Omega}, \vec{\Theta}) = P_{lm}(\vec{\Omega})H_k(\vec{\Theta}),$$

$P_{lm}(\vec{\Omega})$ spherical harmonics,

$$\int_{2\pi} H_{k'}(\vec{\Theta})H^*_k(\vec{\Theta})d\vec{\Theta} = \begin{cases} 2\pi \text{ for } k' = k, \\ 0 \text{ else,} \end{cases}$$

$$H_k(\vec{\Theta}) = e^{ik\theta}, \quad (132)$$

$$\cos(\varphi) = \vec{\Omega}' \cdot \vec{\Omega} \quad (133)$$

with

$$\int_{2\pi}\int_{4\pi} Q_{lmk}(\vec{\Omega}, \vec{\Theta}) Q^*_{lmk}(\vec{\Omega}', \vec{\Theta}') d\vec{\Omega}' d\vec{\Theta}'$$

$$= \begin{cases} \frac{8\pi^2}{2l = 1} \text{ for } l = l'; \; m = m'; \; k = k', \\ 0 \text{ else.} \end{cases} \tag{134}$$

Such, suitable distribution functions are described by

$$\bar{f}_{t_\varepsilon}(\vec{x}, t, \vec{\Omega}, \vec{\Theta}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{x}, t) Q_{lmk}(\vec{\Omega}, \vec{\Theta}),$$

$$\bar{f}_{t_\varepsilon}(\vec{x}, t, \vec{\Omega}, \vec{\Theta}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} P_{lm}(\vec{\Omega}) \sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{x}, t) H_k(\vec{\Theta}). \tag{135}$$

Die δ- function depending on the turbulence functions is expressed

$$\delta(\vec{\Omega}, \vec{\Omega}'; \vec{\Theta}, \vec{\Theta}')$$

$$= \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\Omega}) P^*_{lm}(\vec{\Omega}') \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi} H_k(\vec{\Theta}) H^*_k(\vec{\Omega}') \tag{136}$$

and such

$$\delta(\vec{\Omega}, \vec{\Omega}'; \vec{\Theta}, \vec{\Theta}') = \frac{1}{8\pi^2} \sum_{l=0}^{\infty} (2l+1) P_l(\vec{\Omega} \cdot \vec{\Omega}') \sum_{k=-\infty}^{+\infty} \exp(ik(\vec{\Theta} - \vec{\Theta}')). \tag{137}$$

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