A NON-UNITAL ALGEBRA HAS UUNP IFF ITS UNITIZATION HAS UUNP

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Abstract

Let $A$ be a non-unital Banach algebra, S. J. Bhatt and H. V. Dedania showed that $A$ has the unique uniform norm property (UUNP) if and only if its unitization has UUNP. Here we prove this result for any non-unital algebra.

1. Preliminaries

Let $A$ be a non-unital algebra and let $A_e = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$ be the unitization of $A$ with the identity denoted by $e$. For an algebra norm $\| \|$ on $A$, define $\|a + \lambda e\|_{op} = \sup\{(a + \lambda e)b : b \in A, \|b\| \leq 1\}$ and $\|a + \lambda e\| = \|a\| + |\lambda|$ for all $a + \lambda e \in A_e$. $\| \|$ is an algebra norm on $A_e$. An algebra norm $\| \|$ on $A$ is called regular if $\| \|_{op} = \| \|_1$ on $A$. A uniform norm $\| \|$ on $A$ is an algebra norm satisfying the square property $\|a^2\| = \|a\|^2$ for all...
a ∈ A; and in this case, ∥∥ is regular and \( \| \|_{op} \) is a uniform norm on \( A_e \). An algebra has the unique uniform norm property (UUNP) if it admits exactly one uniform norm.

2. The Result

**Theorem.** A non-unital algebra \( A \) has UUNP if and only if its unitization \( A_e \) has UUNP.

**Proof.** Let \( \| \| \) and \( \| \|_1 \) be two uniform norms on \( A_e \), then \( \| \| = \| \|_1 \) on \( A \) since \( A \) has UUNP, and so \( \| \|_{op} = \| \|_{op} \) on \( A_e \). By [3, Corollary 2.2(1)] and since two equivalent uniform norms are identical, it follows that \((\| \| = \| \|_{op} \) or \( \| \| \equiv \| \|_1 \)) and \((\| \| = \| \|_{op} = \| \|_{op} \) or \( \| \| \equiv \| \|_1 = \| \|_1 \))

equivalently, at least one of the following holds:

(i) \( \| \| = \| \|_{op} \) and \( \| \| \equiv \| \|_{op} = \| \|_{op} \);

(ii) \( \| \| = \| \|_{op} \) and \( \| \| \equiv \| \|_1 = \| \|_1 \);

(iii) \( \| \| \equiv \| \|_1 \) and \( \| \| \equiv \| \|_{op} = \| \|_{op} \);

(iv) \( \| \| \equiv \| \|_1 \) and \( \| \| \equiv \| \|_1 = \| \|_1 \).

If either (i) or (iv) is satisfied, then \( \| \| = \| \|_1 \). By noting that (ii) and (iii) are similar by interchanging the roles of \( \| \| \) and \( \| \|_1 \), it is enough to assume (ii). Let \((c(A), \| \|^-)\) be the completion of \((A, \| \|)\), we distinguish two cases:

1. \( c(A) \) has not an identity:

\( \| \|^- \) is regular since it is uniform. By [1, Corollary 2], \( \| \|^-_{op} \leq \| \|^- \leq 3 \| \|_{op} \) on \( c(A)_e \) (unitization of \( c(A) \)). Let \( a + \lambda e \in A_e \subset c(A)_e \), \( \| a + \lambda e \|^- = \| a \|^- + |\lambda| = \| e \| + |\lambda| = \| a + \lambda e \|_1 \) and \( \| a + \lambda e \|^-_{op} = \operatorname{sup}\{\| (a + \lambda e)b \|^- : b \in c(A), \| b \|^- \leq 1 \} \)

\( = \operatorname{sup}\{\| (a + \lambda e)b \| : b \in A, \| b \| \leq 1 \} = \| a + \lambda e \|_{op} \). Therefore \( \| \|_{op} \leq \| \|_1 \leq 3 \| \|_{op} \).

By (ii), \( \| \| \) and \( \| \|_1 \) are equivalent uniform norms, and so \( \| \| = \| \|_1 \).
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(2) $c(A)$ has an identity $e$:

Let $(c(A_e), \| \cdot \|)$ be the completion of $(A_e, \| \cdot \|)$. Since $\| \cdot \| = \| \cdot \|$ on $A$, $c(A)$ can be identified to the closure of $A$ in $(c(A_e), \| \cdot \|)$ so that $\| \cdot \|$ on $c(A)$. Let $a + \lambda e \in A_e \subset c(A)$,

$$
\|a + \lambda e\| = \|a + \lambda e\|_0 \text{ by (ii)}
$$

$$
= \sup\{\|a + \lambda e\|_0 : b \in A, \|b\| \leq 1\}
$$

$$
= \sup\{\|a + \lambda e\|_0^- : b \in c(A), \|b\|^- \leq 1\}
$$

$$
= \|a + \lambda e\|^- \text{ since } c(A) \text{ is unital}
$$

Thus $\|a\| = \|a + \lambda e\|$. Conversely, let $\| \cdot \|$ and $\| \cdot \|$ be two uniform norms on $A$, then $\| \cdot \|_0$ and $\| \cdot \|_0$ are uniform norms on $A_e$, hence $\| \cdot \|_0 = \| \cdot \|_0$ since $A_e$ has UUNP. Therefore $\| \cdot \| = \| \cdot \|_0 = \| \cdot \|_0 = \| \cdot \|$ on $A$ since $\| \cdot \|$ and $\| \cdot \|$ are regular.

References

